1) (20%) The solution of the insulated $1-d$ heat flow problem:

PDE: $u_t = \alpha^2 u_{xx}$ $\quad 0<x<1, \quad 0<t<\infty$

BCs: $u(0,t)=0$ $\quad 0<t<\infty$

$u(1,t)=0$ $\quad 0<t<\infty$

IC: $u(x,0)=\phi(x)$ $\quad 0\le x\le 1$

is given by $u(x,t)=\sum_{1}^{\infty} A_n e^{-(n\pi\alpha)^2 t}.\sin n\pi x$, where $A_n = 2\int_0^1 \phi(x)\sin n\pi x\,dx$.

Find the solution of the following $1-d$ heat flow problem with lateral heat loss:

PDE: $u_t = \alpha^2 u_{xx} - \beta u$ $\quad 0<x<1, \quad 0<t<\infty$

BCs: $u(0,t)=0$ $\quad 0<t<\infty$

$u(1,t)=0$ $\quad 0<t<\infty$

IC: $u(x,0)=\psi(x)$ $\quad 0\le x\le 1$

Hint: Consider the transformation $u(x,t)=e^{-\beta t}.w(x,t)$ to rewrite IBVP in terms of w, then use ~~(a)~~. (α, β are positive constants)
the given (1st part)

$$\underline{u(x,t) = e^{-\beta t}.w ;}$$

$$u_t = -\beta e^{-\beta t} w + e^{-\beta t} w_t$$

$$u_x = e^{-\beta t} w_x$$

$$u_{xx} = e^{-\beta t} w_{xx}$$

$$u = w e^{-\beta t}$$

Substitute in $u_t = \alpha^2 u_{xx} - \beta u$:

$$-\beta \cancel{e^{-\beta t}} w + \cancel{e^{-\beta t}} w_t = \alpha^2 \cancel{e^{-\beta t}} w_{xx} - \beta w \cancel{e^{-\beta t}}$$

$$\cancel{-\beta w} + w_t = \alpha^2 w_{xx} \cancel{-\beta w}$$

PDE: $\boxed{w_t = \alpha^2 w_{xx}}$

BCs: $u(0,t) = e^{-\beta t}.w(0,t) = 0 \Rightarrow \boxed{w(0,t)=0}$

$u(1,t) = e^{-\beta t}.w(1,t) = 0 \Rightarrow \boxed{w(1,t)=0}$ $\Big\}$ B.Cs

$\underline{I.C}$: $u(x,0) = \psi(x) = w(x,0)$

$\Rightarrow \boxed{w(x,0) = \psi(x)}$ } I.C.

$$\boxed{\begin{array}{l} \underline{PDE}: \; w_t = \alpha^2 w_{xx} \\ \underline{B.C_s}: \begin{cases} w(0,t) = 0 \\ w(1,t) = 0 \end{cases} \\ \underline{I.C.}: w(x,0) = \psi(x) \end{array}}$$

like the given;

~~Given: $w(x,t) = \sum A_n e^{-(n\pi\alpha)^2 t} \cdot \sin n\pi x$; $A_n = 2\int_0^1 \psi(x) \sin n\pi x \, dx$~~

~~$u(x,t) = e^{-\beta t} \cdot w(x,t)$~~

~~$= e^{-\beta t}$~~

Separation of variables:

let $w(x,t) = X(x)\,T(t)$

$X T' = \alpha^2 X'' T$

$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = k \Rightarrow T' - k\alpha^2 T = 0$

$T' + (\lambda\alpha)^2 T = 0$

$\Rightarrow T(t) = c e^{-(\lambda\alpha)^2 t}$

(let $k = -\lambda^2$ for T not to blow up if $t \to \infty$)

$X'' - kX = 0$

$X'' + \lambda^2 X = 0$

$X(x) = c_1 \cos\lambda x + c_2 \sin\lambda x$

$\Rightarrow w(x,t) = e^{-(\lambda\alpha)^2 t}\left[c_1 \cos\lambda x + c_2 \sin\lambda x \right]$

$$\boxed{\begin{array}{l} \Rightarrow u(x,t) = e^{-\beta t} \sum A_n e^{-(n\pi\alpha)^2 t} \cdot \sin n\pi x \\ \text{where } A_n = 2\int_0^1 \psi(x) \sin(n\pi x)\, dx \end{array}}$$

2) **(30%)** Solve the following nonhomogeneous IBVP:

PDE: $u_t = \alpha^2 u_{xx} + \sin 3\pi x \qquad 0 < x < 1, \qquad 0 < t < \infty$

BCs: $u(0,t) = 0 \qquad 0 < t < \infty$

$u(1,t) = 0 \qquad 0 < t < \infty$

IC: $u(x,0) = \sin \pi x \qquad 0 \le x \le 1$

Show all details.

Let $u(x,t) = \sum T_n(t) \sin(n\pi x)$

$\sum T_n'(t) \sin(n\pi x) = -\alpha^2 \sum \left(T_n(t) \sin(n\pi x)\right)(n\pi)^2 + \sin(3\pi x)$

$\sum T_n'(t) \underline{\sin(n\pi x)} + \sum \alpha^2 (n\pi)^2 T_n(t) \underline{\sin(n\pi x)} = \sin(3\pi x)$

$\sum \left[T_n'(t) + \alpha^2 (n\pi)^2 T_n(t)\right] \sin(n\pi x) = \sin(3\pi x)$

<u>if $n=3$:</u>

$T_3'(t) + \alpha^2 (3\pi)^2 T_3(t) = 1 \qquad ; T_3 = c e^{-(\alpha 3\pi)^2 t} + \dfrac{1}{(3\alpha\pi)^2}$

<u>if $n \neq 3$:</u>

$T_n'(t) + \alpha^2 (n\pi)^2 T_n(t) = 0 \qquad ; T_n = c_1 e^{-(\alpha n\pi)^2 t}$

<u>I.C.</u>: $u(x,0) = \sum T_n(0) \sin n\pi x = \sin \pi x$

<u>if $n=1$:</u> $T_1(0) = 1$

<u>if $n \neq 1$:</u> $T_n(0) = 0$

$* \quad T_3 = c e^{-(\alpha 3\pi)^2 t} + \frac{1}{(3\alpha\pi)^2}$; $T_1(0) = 1$; $T_n(0) = 0$

$\Rightarrow c = -\frac{1}{(3\alpha\pi)^2}$

$\Rightarrow \boxed{T_3 = \frac{1}{(3\alpha\pi)^2}\left[1 - e^{-(\alpha 3\pi)^2 t}\right]}$

$* \quad T_n = c_1 e^{-(\alpha n\pi)^2 t}$

$\Rightarrow c_1 = 0 \Rightarrow \boxed{T_n = 0}$

$*$ if $n = 1$:

$T_1'(t) + \alpha^2(\pi)^2 T_1(t) = 0$

$T_1(0) = 1$

$\Rightarrow T_1 = c_2 e^{-(\alpha^2\pi^2 t)}$

$\Rightarrow T_1(0) = c_2 = 1$

$\Rightarrow \boxed{T_1 = e^{-(\alpha\pi)^2 t}}$

$\Rightarrow u(x,t) = \sum T_n(t) \sin n\pi x$

$\Rightarrow \boxed{u(x,t) = e^{-(\alpha\pi)^2 t} \sin(\pi x) + \frac{\sin(3\pi x)}{(3\alpha\pi)^2}\left[1 - e^{-(\alpha 3\pi)^2 t}\right]}$

3) (25%)

a) Use the Fourier transform to solve the IVP:

PDE: $u_t = \alpha^2 u_{xx} - \beta u \qquad -\infty < x < \infty, \;\; 0 < t < \infty$

IC: $u(x,0) = \phi(x) \qquad -\infty < x < \infty$

Show all details.

b) What is the solution in the special case $\phi(x) \equiv 1$?

a) Let $U(\xi,t) = \mathcal{F}(u(x,t))(\xi) \overset{\text{def.}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\xi x} dx$

$U_t(\xi,t) = -\alpha^2 \xi^2 U(\xi,t) - \beta U(\xi,t)$

$U_t(\xi,t) + \alpha^2 \xi^2 U(\xi,t) + \beta U(\xi,t) = 0$

$U_t(\xi,t) + (\alpha^2\xi^2 + \beta) U(\xi,t) = 0$

$\Rightarrow U(\xi,t) = c e^{-(\alpha^2\xi^2+\beta)t}$

F.C.: $U(\xi,0) = \Phi(\xi)$ (where $\Phi = \mathcal{F}(\phi(x))$)

$= ce^0 = \Phi(\xi)$

$\Rightarrow U(\xi,t) = \Phi(\xi) e^{-(\alpha^2\xi^2+\beta)t}$

$u(x,t) = \mathcal{F}^{-1}\left(\Phi(\xi) e^{-(\alpha^2\xi^2+\beta)t}\right)(x)$

$= \mathcal{F}^{-1}(\Phi(\xi))(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2\xi^2+\beta)t}\right)(x)$

$= \phi(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2\xi^2+\beta)t}\right)(x)$

$= e^{-\beta t}\left[\phi(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2\xi^2)t}\right)(x)\right]$

$= e^{-\beta t}\left[\phi(x) * \frac{\sqrt{2}}{2\alpha t} e^{-x^2/4\alpha^2 t}\right]$

$= \frac{e^{-\beta t} \cdot \sqrt{2}}{2\alpha t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x-\zeta) e^{-\zeta^2/4\alpha^2 t} d\zeta$

use formula (6):

$\alpha^2 t = \frac{1}{4a^2}$

$4a^2\alpha^2 t = 1$

$a^2 = \frac{1}{4\alpha^2 t}$

$a = \frac{1}{2\alpha\sqrt{t}}$

$$= \frac{e^{-\beta t} \cancel{\sqrt{2}}}{2\alpha t} \cdot \frac{1}{\sqrt{\cancel{2}\pi}} \int_{-\infty}^{\infty} \phi(x-\xi)\, e^{-\xi^2/4\alpha^2 t}\, d\xi$$

$$\boxed{u(x,t) = \frac{e^{-\beta t}}{2\alpha\sqrt{t}\sqrt{\pi}} \int_{-\infty}^{\infty} \phi(x-\xi)\, e^{-\xi^2/4\alpha^2 t}\, d\xi}$$

b) $\underline{\phi(x) \equiv 1 :}$

$\Rightarrow \phi(x-\xi) = 1$

Let $u^2 = \frac{\xi^2}{4\alpha^2 t} \Rightarrow u = \frac{\xi}{2\alpha t}$

$du = \frac{1}{2\alpha t} d\xi$

$$\Rightarrow u(x,t) = \frac{e^{-\beta t}}{\cancel{2\alpha} t\sqrt{\pi}} \left[\int_{-\infty}^{\infty} e^{-u^2} du \right] \cdot \cancel{2\alpha t}$$

use formula (4):

$$= \frac{e^{-\beta t}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\cancel{2}} \times \cancel{2}$$

$$\Rightarrow \boxed{u(x,t) = e^{-\beta t}}$$

4) **(10%)** Let $u = u(x, y, z)$ be a function of 3 independent variables possessing derivatives of all orders. Solve the (possibly easiest) 3rd order PDE:

$u_{xyz} = 0, \qquad x, y, z \in \mathbb{R}.$

$$\int u_{xyz}\, \partial z = \int 0\, \partial z$$

$$u_{xy} = f(x, y)$$

$$\int u_{xy}\, \partial y = \int f(x, y)\, \partial y$$

$$\therefore u_x = F(x, y) + g(x, z)$$

$$\int u_x\, \partial x = \int \big(F(x, y) + g(x, z)\big)\, \partial x$$

$$u = F_1(x, y) + G(x, z) + h(y, z)$$

$$\Rightarrow \boxed{u(x, y, z) = F_1(x, y) + G(x, z) + h(y, z)}$$

where F_1, G, & h are arbitrary fts. possessing partial derivatives of 3rd order.

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5) **(15%)** Solve the following <u>nonlinear</u> IVP:

PDE: $u_t + uu_x = 0$ $\qquad -\infty < x < \infty, \quad 0 < t < \infty$

IC: $u(x,0) = x + 1$ $\qquad -\infty < x < \infty$

Hint: a) Try a solution of the form $u(x,t) = X(x)T(t)$.

b) The general solution of the 1st order nonlinear ODE

$T' + kT^2 = 0$ is $\quad T(t) = \dfrac{1}{kt+b}$

$u_t + uu_x = 0$

Let $u(x,t) = X(x)T(t)$

$XT' + XTX'T = 0$

$\cancel{X}T' = -\cancel{X}X'T^2 = k$

$\dfrac{T'}{T^2} = -X' = k$

$-\dfrac{T'}{T^2} = X' = k$

$-T' - kT^2 = 0$

$T' + $

$-\dfrac{T'}{T^2} = X' = k$

$-T' - kT^2 = 0$

$T' + kT^2 = 0$

$\Rightarrow \boxed{T(t) = \dfrac{1}{kt+b}}$

$\dfrac{dX}{dx} = k$

$\int dX = \int k\,dx$

$X = kx + c$

; $X' = k \Rightarrow \dfrac{dX}{dx} = k$

$\Rightarrow \int dX = \int k\,dx$

$\boxed{X(x) = kx + c}$

$u(x,t) = X(x)T(t) = \dfrac{kx+c}{kt+b}$

$= \dfrac{x + c/k}{t + b/k} = \dfrac{x + C}{t + B}$; I.C.: $u(x,0) = \dfrac{x+C}{B} = x+1$

$\Rightarrow \dfrac{x}{B} + \dfrac{C}{B} = x + 1$

$\Rightarrow \dfrac{1}{B} = 1 \Rightarrow B = 1$ & $\dfrac{C}{B} = 1 \Rightarrow C = 1$ $\Rightarrow \boxed{u(x,t) = \dfrac{x+1}{t+1}}$