Monday April 14, exam I,

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(1) (35%) Consider the IBVP

PDE: $u_t = u_{xx} - u + x \qquad 0 < x < 1, \qquad 0 < t < \infty$

BCs: $u(0,t) = 0 \qquad 0 < t < \infty$

$u(1,t) = \cancel{1}\,0 \qquad 0 < t < \infty$

IC: $u(x,0) = 0 \qquad 0 \le x \le 1$

a) Let $u(x,t) = x + e^{-t} \cdot w(x,t)$. Reformulate the above IBVP in terms of w.
b) Use the method of separation of variables to find w. Show all details.

a) $u(x,t) = x + e^{-t} w(x,t)$

$-e^{-t} w + e^{-t} w_t = e^{-t} w_{xx} - x - e^{-t} w + x$

$\therefore \; w_t = w_{xx}$

$u(0,t) = 0 + e^{-t} w(0,t) = 0 \qquad w(0,t) = 0$

$u(1,t) = \cancel{1} + e^{-t} w(1,t) = \cancel{0} \qquad w(1,t) = \cancel{0}$

$u(x,0) = x + w(x,0) = 0.$

$w(x,0) = -x.$

$$\begin{cases} w_t = w_{xx} \\ w(0,t) = 0 \\ w(1,t) = \cancel{0} \\ w(x,0) = -x. \end{cases}$$

b) $w(x,t) = X(x)T(t).$

$XT' = TX''$

$\frac{T'}{T} = \frac{X''}{X} = K$ (because x and t are independent)

$T' - KT = 0 \qquad T = ce^{Kt}$ K should be negative or else T and u will blow up when $t \to \infty$

$X'' - KX = 0 \qquad T = ce^{-\lambda^2 t}$

$$w(x,t) = e^{-\lambda^2 t}\left[A\sin\lambda x + B\cos\lambda x\right]$$

$$w(0,t) = 0 \therefore Be^{-\lambda^2 t} = 0$$

$$B = 0.$$

THE DEBATE CLUB

$$w(x,t) = Ae^{-\lambda^2 t}\sin\lambda x$$

$$w(1,t) = 0 \therefore Ae^{-\lambda^2 t}\sin\lambda = 0$$

$$\sin\lambda = 0$$

$$\lambda = \lambda_n = n\pi$$

$$w(x,t) = A_n e^{(n\pi)^2 t}\sin n\pi x.$$

IC $w(x,0) = A_n \sin n\pi x = -x$

$$A_n = f(x) \longrightarrow \longleftarrow$$

By superposition principle, as PDE and BCs are linear and homogenous $\therefore$ $w(x,t) = \sum A_n e^{(n\pi)^2 t}\sin n\pi x \quad n \geq 1$

IC $w(x,0) = \sum A_n \sin n\pi x = -x$.

To get A_n we multiply by $\sin n\pi x$, integrate from 0 to 1 and we use the orthogonality property of sin.

$$A_n = 2\int_0^1 -x\sin n\pi x\,dx.$$

x	$\sin n\pi x$
1	$\frac{-1}{n\pi}\cos n\pi x$
0	$\frac{-1}{(n\pi)^2}\sin n\pi x$

$$= -2\int_0^1 x\sin n\pi x\,dx$$

$$= -2\left[\frac{-x}{n\pi}\cos n\pi x + \frac{1}{(n\pi)^2}\sin n\pi x\right]_0^1$$

$$= -2\left[\frac{-1}{(n\pi)}(-1)^n\right] = \frac{2(-1)^{n+2}}{n\pi} = \begin{cases} 2/n\pi & n \text{ even} \\ -2/n\pi & n \text{ odd} \end{cases}$$

$$w(x,t) = \sum A_n e^{(n\pi)^2 t}\sin n\pi x \quad \text{with } A_n = \begin{cases} 2/n\pi & n \text{ even} \\ -2/n\pi & n \text{ odd} \end{cases}$$

(2) (35%)

a) Use the Fourier transform to solve the IVP

PDE: $u_t = \alpha^2 u_{xx} - \beta u \qquad -\infty < x < \infty, \quad 0 < t < \infty$

IC: $u(x,0) = \phi(x) \qquad -\infty < x < \infty$

Show all details.

b) What is the solution in the special case $\phi(x) \equiv 1$?

a) $u_t = \alpha^2 u_{xx} - \beta u$.

Let $U(\xi,t) = \mathcal{F}(u(x,t))(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx$.

$\mathcal{F}(PDE) \Rightarrow U_t(\xi,t) = -\alpha^2\xi^2 U(\xi,t) - \beta U(\xi,t)$

$U_t(\xi,t) + (\alpha^2\xi^2 + \beta)U(\xi,t) = 0$.

1st order linear homogeneous PDE with constant coefficients

$U(\xi,t) = Ce^{-(\alpha^2\xi^2+\beta)t}$

IC: $u(x,0) = \phi(x) \qquad U(\xi,0) = \Phi(\xi) \qquad \Phi(\xi) = \mathcal{F}(\phi(x))$

$\therefore U(\xi,0) = C = \Phi(\xi)$

$\therefore U(\xi,t) = \Phi(\xi)e^{-(\alpha^2\xi^2+\beta)t}$

$u(x,t) = \mathcal{F}^{-1}(U(\xi,t)) = \mathcal{F}^{-1}(\Phi(\xi)) * \mathcal{F}^{-1}\left(e^{-(\alpha^2\xi^2+\beta)t}\right)$

$= \phi(x) * \mathcal{F}^{-1}\left(e^{-(\alpha^2\xi^2+\beta)t}\right)$

$= \Phi(x) * \mathcal{F}^{-1}\left(e^{-\alpha^2t\xi^2} \cdot e^{-\beta t}\right)$

$= \Phi(x) * e^{-\beta t}\,\mathcal{F}^{-1}\left(e^{-\alpha^2t\xi^2}\right)$

~~$u(x,t) = \ldots$~~

$\mathcal{F}^{-1}\left(e^{-\frac{\xi^2}{4a^2}}\right) = a\sqrt{2}\,e^{-a^2x^2}$

(3) (30%) Use the Laplace transform to solve the IVBP

PDE: $u_{xx} - u_{tt} = -\sin \pi x$ $\quad 0 < x < 1, \quad 0 < t < \infty$

BCs: $u(0,t) = u(1,t) = 0$ $\quad 0 < t < \infty$

ICs: $u(x,0) = u_t(x,0) = 0$ $\quad 0 \le x \le 1, \quad 0 < t < \infty$

THE DEBATE CLUB

Let $U(x,s) = \mathcal{L}(u(x,t))(s) = \int_0^{\infty} u(x,t)\, e^{-st}\, dt$ $\qquad \mathcal{L}(-\sin \pi x) = -\sin \pi x\, \mathcal{L}(1) = \dfrac{-\sin \pi x}{s}$

$$\mathcal{L}(PDE) \Rightarrow U_{xx}(x,s) - \left[s^2 U(x,s) - s\,u(x,0) - u_t(x,0)\right] = \frac{-\sin \pi x}{s}$$

$$U_{xx}(x,s) - s^2 U(x,s) + 0 = \frac{-\sin \pi x}{s}$$

$$U_{xx}(x,s) - s^2 U(x,s) = \frac{-\sin \pi x}{s}$$

2nd order linear nonhomogeneous ODE with variable coefficients

$$U(x,s) = A e^{-sx} + B e^{sx} + c_1(x) e^{-sx} + c_2(x) e^{sx}$$

$$c_1(x) = \frac{-1}{2s} \int \frac{-\sin \pi x}{s} e^{sx}\, dx = \frac{1}{2s^2} \int \sin \pi x\, e^{sx}\, dx$$

$$= \frac{1}{2s^2} \left[\frac{e^{sx}}{s^2+\pi^2}(s \sin \pi x - \pi \cos \pi x)\right]$$

$$c_2(x) = \frac{1}{2s} \int \frac{-\sin \pi x}{s} e^{-sx}\, dx = \frac{-1}{2s^2} \left[\frac{e^{-sx}}{s^2+\pi^2}\left(-s \sin \pi x - \pi \cos \pi x\right)\right]$$

$$= \frac{1 e^{-sx}}{2s^2(s^2+\pi^2)} \left[s \sin \pi x + \pi \cos \pi x\right]$$

$$U(x,s) = A e^{-sx} + B e^{sx} + \frac{1}{2s^2(s^2+\pi^2)}(s \sin \pi x - \pi \cos \pi x) + \frac{1}{2s^2(s^2+\pi^2)}(s \sin \pi x + \pi$$

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$$U(x,s) = A e^{-sx} + B e^{sx} - \frac{s \sin \pi x}{[illegible]} = A e^{-sx} + B e^{sx} - \frac{\sin \pi x}{[illegible]}$$

$\cancel{U(x,s)} =$

$u(0,t) = 0 \Rightarrow U(0,s) = 0 \Rightarrow \cancel{A + B + \sin\pi} \Rightarrow A + B + \dfrac{\cancel{\sin\pi x}^{\,0}}{s(s^2+\pi^2)} = 0$

$\boxed{A = -B}$

$u(1,t) = 0 \Rightarrow U(1,s) = 0 \Rightarrow Ae^{-s} + Be^{s} + \dfrac{\cancel{\sin\pi}^{\,0}}{s(s^2+\pi^2)} = 0$

$Ae^{-s} = -Be^{s}$

$\cancel{Be^{-s} = Be^{s}}$

$A = -Be^{2s} \Rightarrow e^{2s} = 1$

$A = -B \quad \boxed{-B = -Be^{2s}}$ $s = 0 \rightarrow$ (

For this to be satisfied for any s $\sqrt{B = \underline{t}}$

$\boxed{A = 0}$

$U(x,s) = \dfrac{\sin\pi x}{s(s^2+\pi^2)}$

$u(x,t) = \mathcal{L}^{-1}\left(\dfrac{\sin\pi x}{s(s^2+\pi^2)}\right) = \sin\pi x\ \mathcal{L}^{-1}\left(\dfrac{1}{s(s^2+\pi^2)}\right)$

$= \sin\pi x\ \mathcal{L}^{-1}\left(\dfrac{1}{s}\ \dfrac{1}{s^2+\pi^2}\right)$ multiply up and down by

$= \sin\pi x\ \mathcal{L}^{-1}\left(\dfrac{1}{\pi s}\ \dfrac{\pi}{s^2+\pi^2}\right)$

$= \sin\pi x\ \mathcal{L}^{-1}\left(\dfrac{1}{\pi s}\right) * \mathcal{L}^{-1}\left(\dfrac{\pi}{s^2+\pi^2}\right)$

$= \sin\pi x \left(\dfrac{1}{\pi} * \sin\pi t\right) \quad f(t-\tau) = $

$= \sin\pi x \displaystyle\int_0^t \frac{1}{\pi}\sin(\pi\tau)\,d\tau$

$u(x,t) = \sin\pi x\ \dfrac{-1}{\pi^2}\cos\pi\tau\Big|_0^t$

$u(x,t) = \dfrac{-\sin\pi x}{\pi^2}(\cos\pi t - 1) = \dfrac{\sin\pi x}{\pi^2}(1-\cos$

(25%) Let $u(x,t)$ denote the concentration (at time t, at position x) in a moving medium (moving from left to right with speed $v=1$) where the concentration at the ends of the medium are kept at 0 (by some filtering device), and the initial concentration is $e^{x/2}$. The corresponding IBVP is:

PDE: $u_t = u_{xx} - u_x \qquad 0 < x < 1, \qquad 0 < t < \infty$

BCs: $u(0,t) = 0 \qquad 0 < t < \infty$

$u(1,t) = 0 \qquad 0 < t < \infty$

IC: $u(x,0) = e^{x/2} \qquad 0 \le x \le 1$

a) Transform the above problem in u to a new problem in w, where $u(x,t) = e^{\frac{x}{2} - \frac{t}{4}} \cdot w(x,t)$.

b) Solve the resulting IBVP in w by the method of separation of variables. Show all details.

c) Deduce the solution u of the given IBVP, and find the steady-state solution $u(x,\infty)$.

a) $u_t = -\frac{1}{4} e^{\frac{x}{2} - \frac{t}{4}} \cdot w(x,t) + w_t e^{\frac{x}{2} - \frac{t}{4}}$

$u_x = \frac{1}{2} e^{\frac{x}{2} - \frac{t}{4}} w + w_x e^{\frac{x}{2} - \frac{t}{4}}$

$u_{xx} = \frac{1}{4} e^{\frac{x}{2} - \frac{t}{4}} w + \frac{1}{2} e^{\frac{x}{2} - \frac{t}{4}} w_x + \frac{1}{2} e^{\frac{x}{2} - \frac{t}{4}} w_x$
$+ w_{xx} e^{\frac{x}{2} - \frac{t}{4}}$

$u_t = u_{xx} - u_x$

$-\frac{1}{4} e^{\frac{x}{2} - \frac{t}{4}} w + e^{\frac{x}{2} - \frac{t}{4}} w_t = \frac{1}{4} e^{\frac{x}{2} - \frac{t}{4}} w + \cancel{1} e^{\frac{x}{2} - \frac{t}{4}} w_x$
$+ e^{\frac{x}{2} - \frac{t}{4}} w_{xx} - \frac{1}{2} e^{\frac{x}{2} - t/4} w - e^{\frac{x}{2}}$

$-\cancel{\frac{1}{4} w} + w_t = \cancel{\frac{1}{4} w} + \cancel{1}\, w_x + w_{xx} - \cancel{\frac{1}{2} w} - w$

$w_t = w_{xx}$ is the new PDE

$u(0,t) = e^{-\frac{t}{4}} w(0,t) = 0$ or $e^{-t/4} \neq 0$

$\Rightarrow w(0,t) = 0$

$u(1,t) = e^{-t/4 + \frac{1}{2}} w(1,t) = 0$ or $e^{-t/4 + \frac{1}{2}} \neq$

$= \quad = \quad =$