Final - Fall 2003-2004 (January 28) (3) (Makdisi) (C)

PART I. QUICK ANSWERS, NO JUSTIFICATION REQUIRED, NO PARTIAL CREDI

1 (2 pts/part, total 10 pts). Which of the following series converge and which diverge? Circle your answer.

1a) $\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{n-2}}$ Converges Diverges

1b) $\sum_{n=1}^{\infty} (-1)^n$ Converges Diverges

1c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Converges Diverges

1d) $\sum_{n=1}^{\infty} \frac{\sin(n^2)}{n^2}$ Converges Diverges

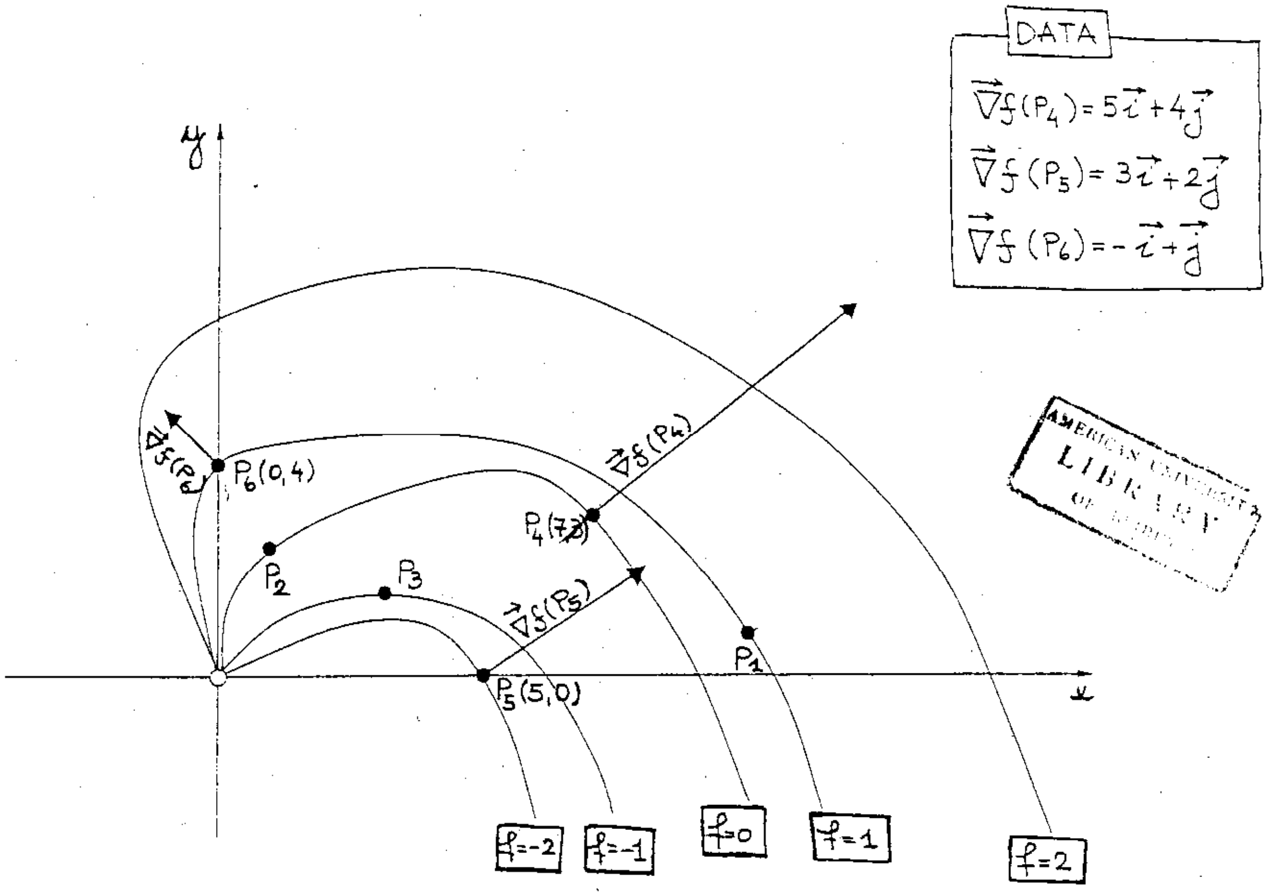
1e) $\sum_{n=1}^{\infty} \frac{(1+1/n)^n}{n}$ Converges Diverges

2 (2pts). Fill in the blank: the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n!\,(n+1)!}{(2n)!} x^n$ is $R =$ __________.

3 (1pt/part, total 4pts. Note 1 blank = 1 part.). Fill in the blanks for the first few coefficients in the Maclaurin series for the following integral:

$$\int_{t=0}^{x} \frac{\sin t}{t}\,dt = \underline{\qquad\quad} + \underline{\qquad\quad}\,x + \underline{\qquad\quad}\,x^2 + \underline{\qquad\quad}\,x^3 + \ldots$$

4 (2pts each for parts a–d, 4pts each for parts e–f, total 16pts). The following picture shows level curves for a function $f(x, y)$, and the value of the gradient $\vec{\nabla} f$ at some points.



4a) Draw a vector on the figure at P_1 pointing in the direction of maximum increase of f.

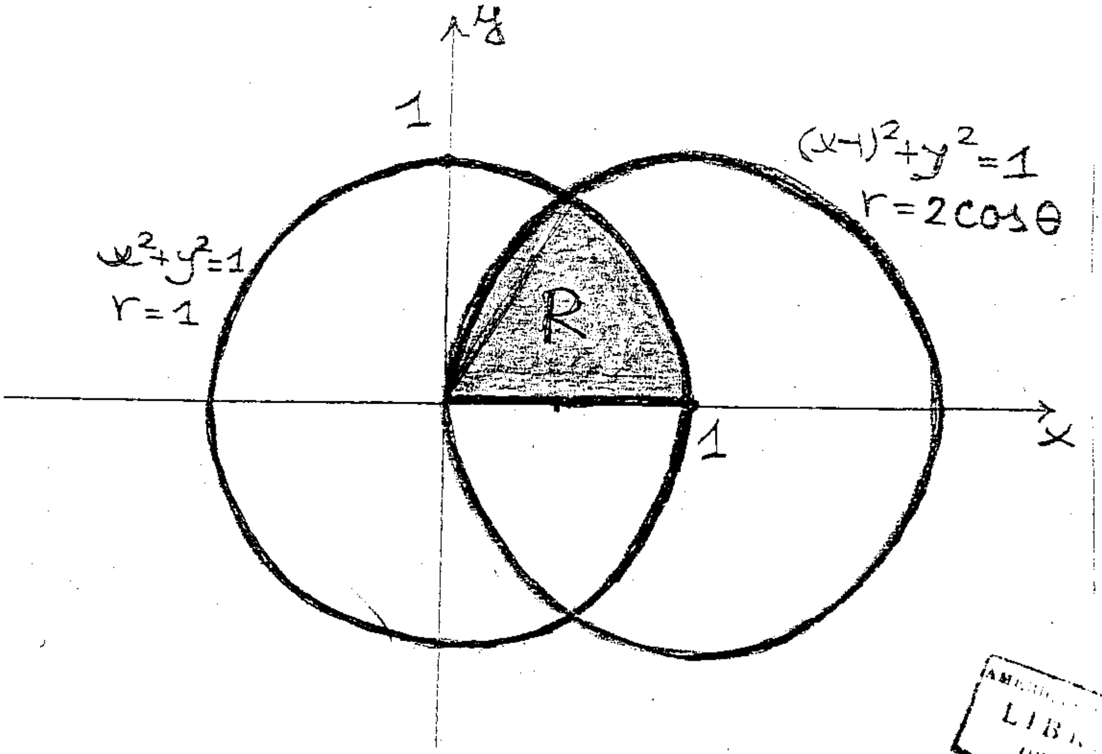
Circle the correct answer for 4b, 4c, 4d:

4b)	$\left.\frac{\partial f}{\partial x}\right\|_{P_2}$	is	positive	zero	negative	does not exist.
4c)	$\left.\frac{\partial f}{\partial x}\right\|_{P_3}$	is	positive	zero	negative	does not exist.
4d)	$\lim_{(x,y)\to(0,0)} f(x, y)$	is	positive	zero	negative	does not exist.

4e) Fill in the blank: The equation of the tangent line to the level curve passing through $P_4(7, 3)$ is ____________________.

4f) Fill in the blank: We start from the point $P_5(5, 0)$, and move a distance of $ds = \frac{1}{100}$ units in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$. Then the change in the value of $f(x, y)$ is approximately _______________.

5 (0.5 pt/part, total 10.5pts. Note 1 blank = 1 part.). Consider the shaded region R below (R is common to the two circles shown and lies above the x-axis.) Fill in the blanks for the following integrals in rectangular and polar coordinates:



5a) $$\iint_R 1\,dA = \int_{y=\underline{\qquad}}^{\underline{\qquad}} \int_{x=\underline{\qquad}}^{\underline{\qquad}} dx\,dy.$$

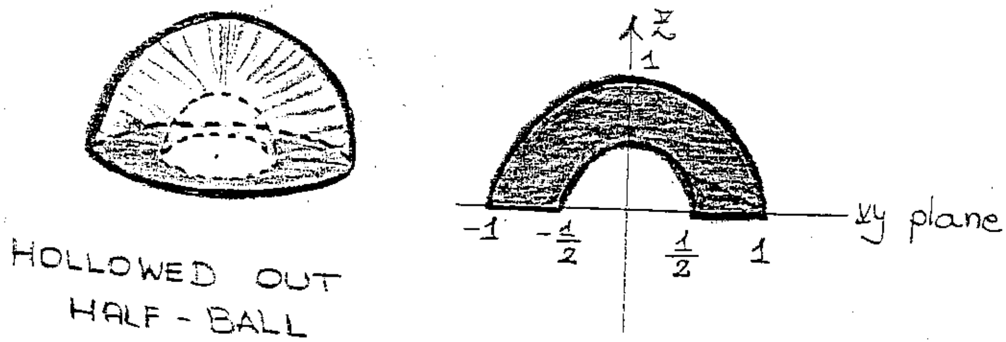
5b) $$\iint_R 1\,dA = \int_{x=\underline{\qquad}}^{\underline{\qquad}} \int_{y=\underline{\qquad}}^{\underline{\qquad}} dy\,dx$$

$$+ \int_{x=\underline{\qquad}}^{\underline{\qquad}} \int_{y=\underline{\qquad}}^{\underline{\qquad}} dy\,dx.$$

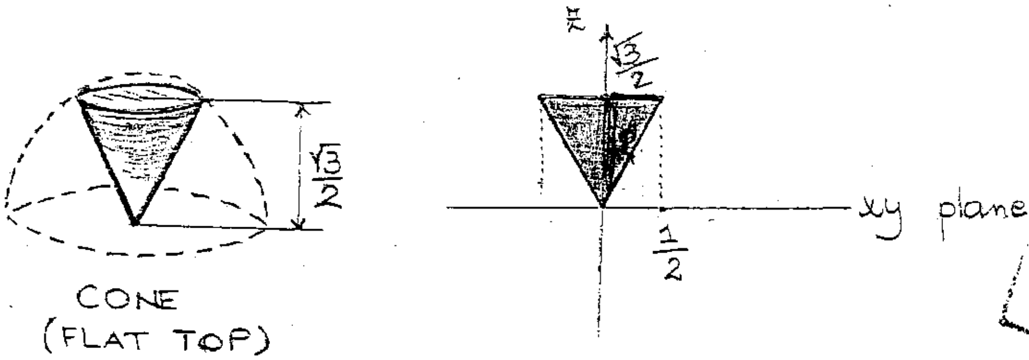
5c) $$\iint_R 1\,dA = \int_{\theta=\underline{\qquad}}^{\underline{\qquad}} \int_{r=\underline{\qquad}}^{\underline{\qquad}} \underline{\qquad}\; dr\,d\theta$$

$$+ \int_{\theta=\underline{\qquad}}^{\underline{\qquad}} \int_{r=\underline{\qquad}}^{\underline{\qquad}} [same]\; dr\,d\theta.$$

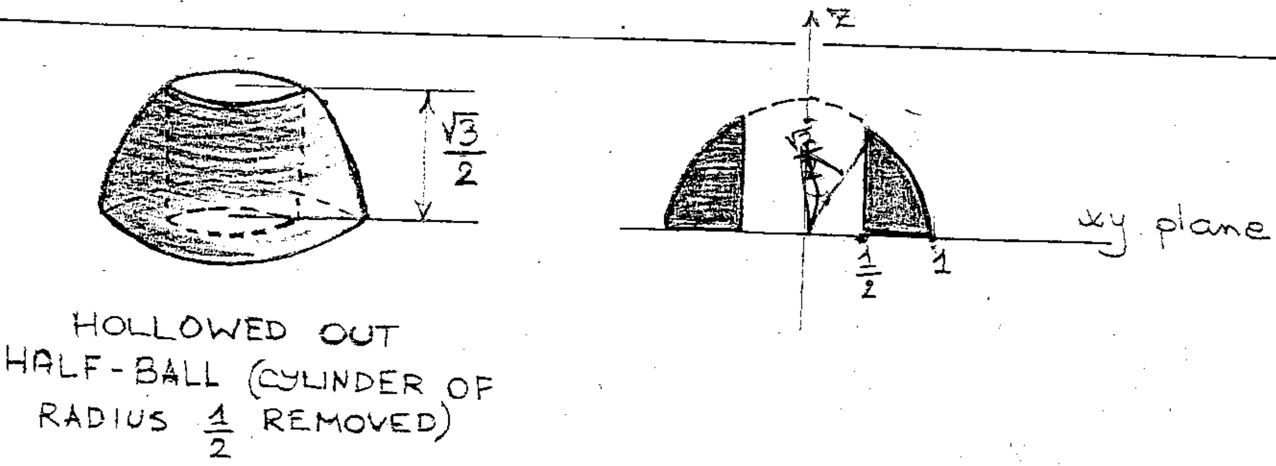
6 (0.5 pts/part, total 9.5 pts. Note 1 blank = 1 part.). In each of the pictures below, we give a 3-dimensional picture and a cross-section of a region D (which is always part of the half-ball $x^2+y^2+z^2 \leq 1, z \geq 0$). In each case, fill in the blanks for integration in spherical coordinates:



6a) $\iiint_D 1\,dV = \int_{\theta=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\rho=\_\_\_\_\_}^{\_\_\_\_\_} \_\_\_\_\_\_ \, d\rho\, d\varphi\, d\theta.$

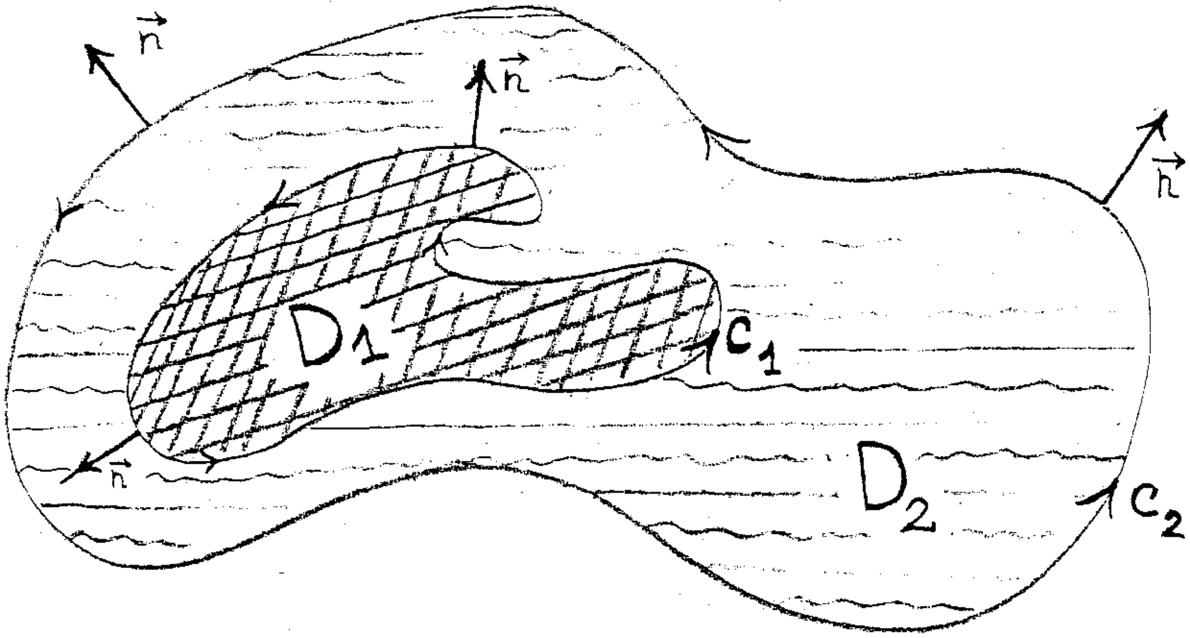


6b) $\iiint_D 1\,dV = \int_{\theta=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\rho=\_\_\_\_\_}^{\_\_\_\_\_}$ [same] $d\rho\, d\varphi\, d\theta.$



6c) $\iiint_D 1\,dV = \int_{\theta=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\rho=\_\_\_\_\_}^{\_\_\_\_\_}$ [same] $d\rho\, d\varphi\, d\theta.$

7 (2 pts/part, total 8 pts. Note 1 blank = 1 part.). The following picture shows two curves C_1 and C_2 in the plane. D_1 is the region inside C_1. D_2 is the region that is inside C_2 AND outside C_1. We go around each of C_1 and C_2 counterclockwise, and the normal vector points outwards of C_1 and C_2 in each case, as drawn on the figure.



State Green's theorem for $\vec{F} = x\vec{i} + xy\vec{j} = (x, xy)$:

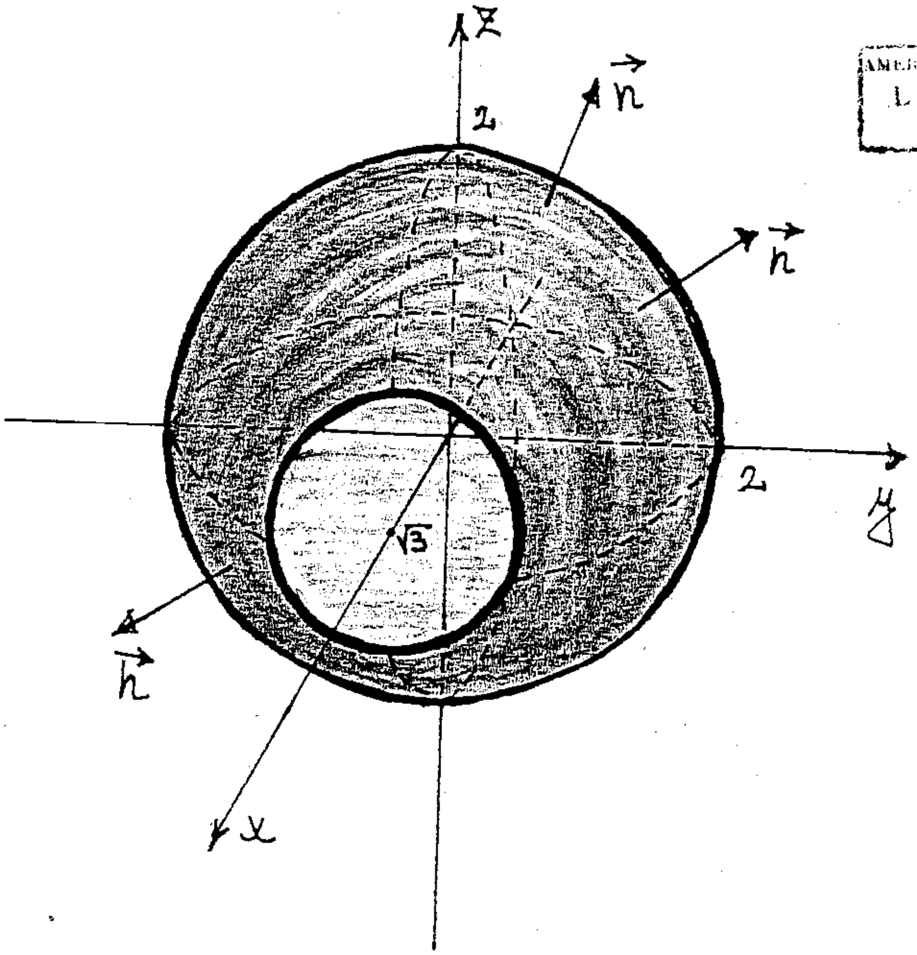
7a) $\iint_{D_1}$ ____________ $dA = \int_{C_1} \vec{F} \cdot \vec{T}\, ds.$

7b) $\iint_{D_2}$ ____________ $dA = \frac{}{(+ \text{ or } -?)} \int_{C_1} \vec{F} \cdot \vec{n}\, ds + \frac{}{(+ \text{ or } -?)} \int_{C_2} \vec{F} \cdot \vec{n}\, ds.$

8 (1pt/part, 4 pts total). Recall that Stokes' Theorem says:

$$\iint_S (\text{curl}\,\vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, d\sigma = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}},$$

where the curve C is the boundary of the surface S, and we go around C in the direction compatible with $\vec{\mathbf{n}}$, according to the right hand rule. Below is a picture where S is the part of the sphere of radius 2 ($x^2 + y^2 + z^2 = 4$) with $x \leq \sqrt{3}$; the normal vector $\vec{\mathbf{n}}$ points away from the origin. C is the circle which is the boundary of S.



8a) On the figure, indicate the direction that one has to go around C for Stokes' Theorem.

8b) Parametrize C in the orientation that you have described. You need to enter a constant number in each of the blanks below:

$(x(t), y(t), z(t)) = ($ √3 ________, ________ $\cos t$, ________ $\sin t$), $\quad 0 \leq t \leq 2\pi$.

PART II. FULL SOLUTIONS REQUIRED, PARTIAL CREDIT AVAILABLE.

9 (10 pts). 9a) Find the second degree Taylor approximation, centered at $a = 2$, to the function $f(x) = \ln x$. (Your answer should have the form $P_2(x) = c_0 + c_1(x-2) + c_2(x-2)^2$ for appropriate c_0, c_1, c_2.)

9b) Use the Remainder Theorem for Taylor series to show that:

$$\text{if } 1.9 \leq x \leq 2.1, \text{ then } \left|\ln(x) - P_2(x)\right| \leq \frac{1}{18000}.$$

[Useful information so you can avoid doing long calculations by hand :
$3 < (1.9)^2 < 4$, $6 < (1.9)^3 < 7$, $4 < (2.1)^2 < 5$, $9 < (2.1)^3 < 10$.]

10 (10 pts). Evaluate the following limit:

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{5/2} - 1 - \frac{5}{2n}}{e^{1/n^2} - 1}.$$

11 (10 pts). We are given a function $f(x, y)$ such that:

$$\vec{\nabla} f \Big|_{(7,3)} = 5\vec{\imath} + 4\vec{\jmath}, \qquad \vec{\nabla} f \Big|_{(5,0)} = 3\vec{\imath} + 2\vec{\jmath}, \qquad \vec{\nabla} f \Big|_{(0,4)} = -\vec{\imath} + \vec{\jmath}.$$

Assume that $x = x(s,t) = s^2 + t^2$ and $y = y(s,t) = st^2 - 4$. Find the value of

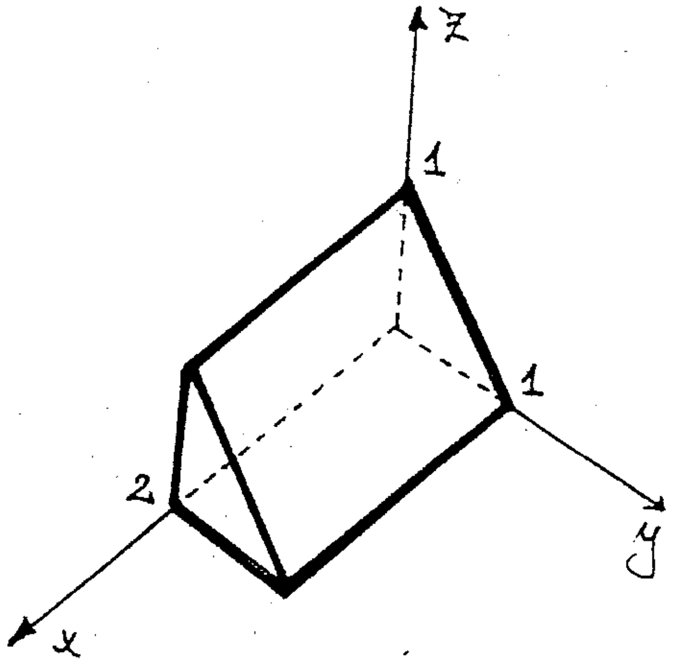
$$\frac{\partial f(x(s,t), y(s,t))}{\partial s} \Big|_{(s,t)=(1,2)}.$$

12 (15 pts). We are given the function $f(x, y, z) = z^2 - 2z + xy$, and we **restrict** (x, y, z) to **lie in the solid cylinder** $D : x^2 + y^2 \leq 2$ (and z arbitrary). At what point(s) of D does f attain its **minimum** value?

(Hints: (i) remember to check the interior and the boundary of D, (ii) remember that we are in three dimensions, (iii) do **not** try to do the second derivative test — we didn't cover it in class for three dimensions!)

13 (8 pts). Let R be the region in the first quadrant underneath the parabola $y = 3 - 3x^2$. Find the average value of $f(x, y) = 2x$ on the region R.

12 (8 pts).



Let D be the region in the **first octant** cut out by the planes $y + z = 1$ and $x = 2$. (See the figure.) The density of D is given by $\delta(x, y, z) = x^2$. Find the total mass of D.

15 (12 pts). Given the two vector fields in the plane:

$$\vec{F} = x\vec{j} = (0, x), \qquad \vec{G} = (2xy + 1)\vec{i} + (x^2 + e^y)\vec{j} = (2xy + 1, x^2 + e^y).$$

15a) For each of $\vec{F}$ and $\vec{G}$, **either** show that the vector field is not conservative, or show that the field is conservative by finding a potential function.

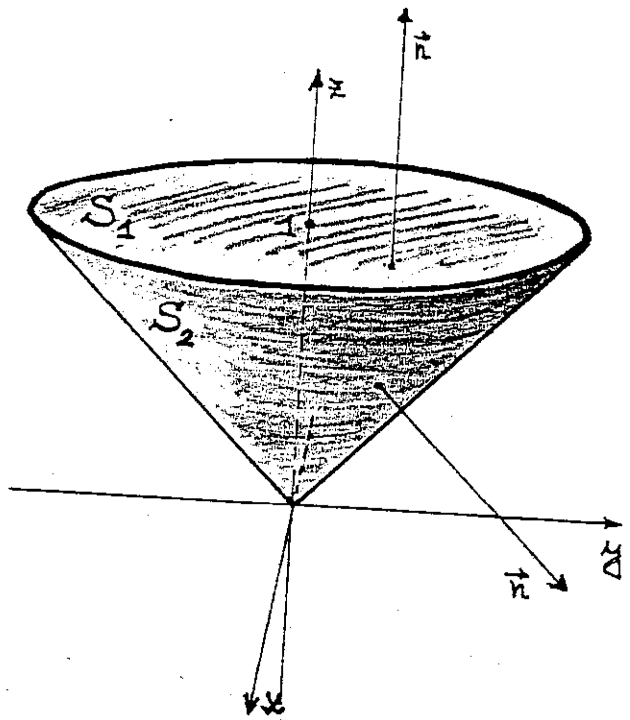
15b) Let C be the curve in the plane parametrized by

$$\big(x(t), y(t)\big) = \vec{r}(t) = (t, t^2), \qquad 0 \leq t \leq 2.$$

Find the work integrals $\int_C \vec{F} \cdot d\vec{r}$, $\int_C \vec{G} \cdot d\vec{r}$.

16 (12 pts). Let D be the solid cone with side $z = \sqrt{x^2 + y^2}$, top $z = 1$, and bottom $z = 0$. Let S be the surface of D (S has two parts: a flat circular lid S_1 and the conical side S_2). The normal vector on S points outwards as shown in the figure. We are also given the vector field

$$\bar{F} = xz\bar{i} = (xz, 0, 0).$$



16a) Explain why $\iint_{S_1} \bar{F} \cdot \bar{n}\, d\sigma = 0$. (This can be done without detailed calculations.)

16b) Find $\iint_{S_2} \bar{F} \cdot \bar{n}\, d\sigma$ by directly computing the flux integral.

16c) (Recall that the divergence theorem tells us that

$$\iint_{S_1} \bar{F} \cdot \bar{n}\, d\sigma + \iint_{S_2} \bar{F} \cdot \bar{n}\, d\sigma = \iiint_D (\operatorname{div} \bar{F})\, dV.$$

Here if $\bar{F} = M\bar{i} + N\bar{j} + P\bar{k}$, then $\operatorname{div} \bar{F} = M_x + N_y + P_z$.)

Find $\operatorname{div} \bar{F}$ and integrate $\iiint_D (\operatorname{div} \bar{F})\, dV$ directly, in order to verify that you get the same result as the sum of the answers in parts a and b above.

PLEASE START YOUR SOLUTION ON THE FOLLOWING PAGE