NDU

MAT 335

Partial Differential Equations

Exam# 1

Thursday November 18, 2004

Duration: 55 minutes

Name: Mikel Abi

ID#: 2002 1007

Section: A

Grade: 98

1) **(30%)** Solve the following heat problem with insulated boundaries:

PDE: $u_t = u_{xx}$ $\quad 0 < x < 1, \quad 0 < t < \infty$

BC: $\begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases}$ $\quad 0 < t < \infty$

IC: $u(x,0) = x$ $\quad 0 \le x \le 1$

Let $u(x,t) = X(x)\,T(t)$

PDE $\Rightarrow XT' = X''T \Rightarrow \dfrac{T'}{T} = \dfrac{X''}{X} = K$ (constant since each side is independent of the other variable

$* \; T' = KT \Rightarrow T' - KT = 0 \Rightarrow T = c\,e^{+Kt}$ $\quad$ K should be negative or else T blows up (physically imp.).

$\Rightarrow$ let $K = -\lambda^2 \Rightarrow T = c\,e^{-\lambda^2 t}$

$* \; X'' = -\lambda^2 X \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = a\sin\lambda x + b\cos\lambda x$

$\Rightarrow u(x,t) = c\,e^{-\lambda^2 t}[a\sin\lambda x + b\cos\lambda x] = e^{-\lambda^2 t}[A\sin\lambda x + B\cos\lambda x]$

$\underline{BC_1} \Rightarrow u_x(0,t) = 0 \Rightarrow$ ~~$B e^{-\lambda^2 t} = 0 \Rightarrow B = 0 \Rightarrow u(x,t) = A e^{-\lambda^2 t}\sin\lambda x$~~

$u_x(x,t) = e^{-\lambda^2 t}[\lambda A\cos\lambda x - \lambda B\sin\lambda x]$.

at $x = 0 \Rightarrow A\lambda e^{-\lambda^2 t} = 0 \Rightarrow A = 0$ $\quad \lambda \neq 0$ or else T independent of t

$u_x(1,t) = 0 \Rightarrow -B\lambda e^{-\lambda^2 t}\sin\lambda = 0$ $\quad B \neq 0$ or else Trivial solution

$\Rightarrow \sin\lambda = 0 \Rightarrow \lambda_n = n\pi \quad n = 0, 1, 2, 3 \ldots$

$\Rightarrow$ ~~$u(x,t) = Be^{\lambda}$~~ Since PDE and BC are L.H. $\Rightarrow$ Superposition principle

$\Rightarrow u(x,t) = \sum B_n e^{-(n\pi)^2 t}\cos n\pi x$

$\underline{IC} \Rightarrow u(x,0) = x$

$\Rightarrow \sum B_n \cos n\pi x = x$

$$\Rightarrow B_0 = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

$$B_n = 2\int_0^1 x \cos n\pi x \, dx.$$

$$\begin{array}{ccc} x & \xrightarrow{+} & \cos n\pi x \\ 1 & \xrightarrow{-} & \frac{\sin n\pi x}{n\pi} \\ 0 & & -\frac{\cos n\pi x}{(n\pi)^2} \end{array}$$

$$= 2\left[\frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\right]_0^1$$

$$= 2\left[\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right] = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow B_n = \begin{cases} 0 & n \text{ is even} \\ \frac{-4}{(n\pi)^2} & n \text{ odd.} \end{cases}$$

$$\Rightarrow u(x,t) = \sum B_n e^{-(n\pi)^2 t} \cos n\pi x$$

for $B_0 = \frac{1}{2}$ and $B_n = \begin{cases} 0 & n \text{ even} \\ \frac{-4}{(n\pi)^2}, & n \text{ odd} \end{cases}$ for $n = 1, 2, 3, \ldots$

~~The first one terms~~

The first 3 terms are:

$$u(x,t) = \frac{1}{2} + \frac{4}{\pi^2} e^{-\pi^2 t} \cos \pi x + -\frac{4}{(3\pi)^2} e^{-(3\pi)^2 t} \cos 3\pi x + \ldots$$

30

2) (30%)

a) Use the Fourier transform to solve the IVP

PDE: $u_t = \alpha^2 u_{xx} - \beta u \qquad -\infty < x < \infty, \quad 0 < t < \infty \qquad (\alpha, \beta > 0 \text{ constants})$

IC: $u(x,0) = \phi(x) \qquad -\infty < x < \infty$

Show all details.

b) What is the solution in the special case $\phi(x) \equiv 1$?

a) Let $\mathcal{U}(\xi, t) = \mathcal{F}(u(x,t))(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) \cdot e^{-i\xi x} dx.$

$$\mathcal{U}_t = -(\alpha\xi)^2 \mathcal{U} - \beta \mathcal{U}$$

$$\Rightarrow \mathcal{U}_t + ((\alpha\xi)^2 + \beta)\mathcal{U} = 0 \Rightarrow \mathcal{U} = c\, e^{-((\alpha\xi)^2+\beta)t}$$

$\underset{\Rightarrow}{IC}$ $\mathcal{U}(\xi, 0) = \Phi(\xi)$ when $\Phi(\xi) = \mathcal{F}(\phi(x))$

$$\Rightarrow \mathcal{U}(\xi, t) = \Phi(\xi) \cdot e^{-((\alpha\xi)^2+\beta)t}$$

$$\mathcal{F}^{-1}\left(e^{-\alpha^2 t \xi^2} \cdot e^{-\beta t}\right) \qquad \frac{1}{4a^2} = \alpha^2 t \Rightarrow a^2 = \frac{1}{4\alpha^2 t} \Rightarrow a = \frac{1}{2\alpha\sqrt{t}}$$

$$\frac{e^{-\beta t}}{\alpha\sqrt{2t}} \mathcal{F}^{-1}\left(\alpha\sqrt{2t} \cdot e^{-\alpha^2 t \xi^2}\right) = \frac{e^{-\beta t}}{\alpha\sqrt{2t}} \cdot e^{-\frac{x^2}{\alpha\sqrt{2t}}} \qquad \text{But } \frac{1}{a\sqrt{2}} = \frac{2\alpha\sqrt{t}}{\sqrt{2}} = \alpha\sqrt{2t}$$

$$\Rightarrow u(x,t) = \varphi(x) * \frac{e^{-\beta t}}{\alpha\sqrt{2t}} \cdot e^{-\frac{x^2}{4\alpha^2 t}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x-\xi) \cdot \frac{e^{-\beta t}}{\alpha\sqrt{2t}} \cdot e^{-\frac{\xi^2}{4\alpha^2 t}} d\xi$$

$$= \frac{e^{-\beta t}}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(x-\xi)\, e^{-\frac{\xi^2}{4\alpha^2 t}} d\xi$$

b) For $\varphi(x) = 1 \Rightarrow \varphi(x-\xi) = 1$

$$\Rightarrow u(x,t) = \frac{e^{-\beta t}}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} 1 \cdot e^{-\frac{\xi^2}{4\alpha^2 t}} \, d\xi$$

Let $\xi = \frac{\zeta}{2\alpha\sqrt{t}} \Rightarrow d\xi = \frac{d\zeta}{2\alpha\sqrt{t}}$ replace.

$$\Rightarrow u(x,t) = \frac{e^{-\beta t}}{2\alpha\sqrt{\pi t}} \cdot 2\alpha\sqrt{t} \int_{-\infty}^{\infty} e^{-\xi^2} \, d\xi$$

Let $\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$

$$\Rightarrow u(x,t) = \frac{e^{-\beta t}}{\sqrt{\pi}} \cdot \sqrt{\pi} = e^{-\beta t}$$

$$\Rightarrow u(x,t) = e^{-\beta t}$$

30

3) **(30%)** Solve by means of the Laplace transform the IBVP:

PDE: $u_t = u_{xx}$ $\quad 0 < x < \infty, \quad 0 < t < \infty$
BC: $u(0,t) = \sin t$ $\quad 0 < t < \infty$
IC: $u(x,0) = 0$ $\quad 0 \le x < \infty$
u is bounded as $x \to +\infty$ for all $0 < t < \infty$.

Let $\mathcal{U}(x,s) = \mathcal{L}(u(x,t))(s) = \int_0^\infty u(x,t) e^{-st}\, dt$

PDE $\Rightarrow$ $s\,\mathcal{U} - u(x,0) = \mathcal{U}_{xx}$

$\Rightarrow \mathcal{U}_{xx} - s\,\mathcal{U} = 0 \Rightarrow \mathcal{U}(x,s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x}$

BC $\Rightarrow$ $\mathcal{U}(0,s) = \mathcal{L}(\sin t) = \dfrac{1}{s^2+1}$

at $x \to \infty$ $\quad \mathcal{U} = c_2 e^{\sqrt{s}x}$ $\therefore$ c_2 should be zero since $\mathcal{U}$ should be bounded.

for $x=0$ $\Rightarrow$ $\mathcal{U}(0,s) = c_1 = \dfrac{1}{s^2+1}$

$$\Rightarrow \mathcal{U}(x,s) = \frac{e^{-\sqrt{s}x}}{s^2+1} = \frac{e^{-\sqrt{s}x}}{s} \cdot \frac{s}{s^2+1}$$

$\Rightarrow$ $\mathcal{L}^{-1}\left(\dfrac{e^{-\sqrt{s}x}}{s}\right) = \operatorname{erfc}\left(\dfrac{x}{2\sqrt{t}}\right)$

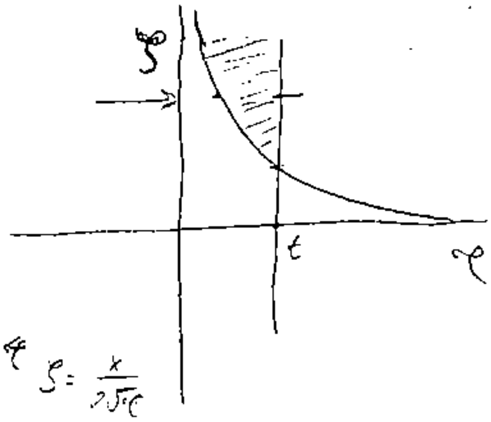
$\mathcal{L}^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t$

$\Rightarrow u(x,t) = \cos t * \operatorname{erfc}\left(\dfrac{x}{2\sqrt{t}}\right)$

But $\cos t * \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) = \int_0^t \cos(t-\tau)\cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{\tau}}\right) d\tau$

$$\Rightarrow \int_0^t \cos(t-\tau)\left[\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tau}}}^{\infty} e^{-\xi^2}\,d\xi\right]d\tau$$

$$= \frac{2}{\sqrt{\pi}}\int_0^t\int_{\frac{x}{2\sqrt{\tau}}}^{\infty}\cos(t-\tau)\cdot e^{-\xi^2}\,d\xi\,d\tau$$



$\xi = \frac{x}{2\sqrt{\tau}}$

$\therefore \xi\sqrt{\tau} = \frac{x}{2\xi} \therefore \tau = \frac{x^2}{4\xi^2}$

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tau}}}^{\infty}\int_{\frac{x^2}{4\xi^2}}^{t}\cos(t-\tau)\,d\tau\cdot e^{-\xi^2}\,d\xi$$

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tau}}}^{\infty} e^{-\xi^2}\,d\xi\int_{\frac{x^2}{4\xi^2}}^{t}\cos(t-\tau)\,d\tau$$

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tau}}}^{\infty} e^{-\xi^2}\left[-\sin(t-\tau)\Big|_{\frac{x^2}{4\xi^2}}^{t}\right]d\xi = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\tau}}}^{\infty} e^{-\xi^2}\left[-\sin\left(t-\frac{x^2}{4\xi^2}\right)\right]d\xi$$

$$\Rightarrow M(x,t) = \cancel{\frac{2}{\sqrt{\pi}}}\int_{\cancel{\frac{x}{2\sqrt{\tau}}}}^{\infty} e^{-\xi^2}\sin\left(t-\frac{x^2}{4\xi^2}\right)d\xi$$

cannot simplify any further.

4) **(10%)** Let $u = u(x, y, z)$ be a function of 3 independent variables possessing derivatives of all orders. Solve the (possibly easiest) 3rd order PDE:

$$u_{xyz} = 0, \qquad x, y, z \in \mathbb{R}.$$

$$u_{xyz} = 0$$

$$\Rightarrow \int (u_{xy})_z \, dz = \int 0 \, dz$$

$$\Rightarrow u_{xy} = f(x,y)$$

$$\int (u_x)_y \, dy = \int f(x,y) \, dy$$

$$\Rightarrow u_x = f'(x,y) + g(x,z)$$

$$\int u_x \, dx = \int [f'(x,y) + g(x,z)] \, dx$$

$$= F(x,y) + G(x,z) + W(y,z)$$

$$\Rightarrow u = F(x,y) + G(x,z) + W(y,z)$$

10

where F, G and W are some functions.