American University of Beirut
Math 201, Fall 2005–06, sections 5–8
Final exam, February 1, 2006, 2.5 hours

Remarks and instructions:

Remember to write your **name, AUB ID number, and SECTION** on your exam booklet. The sections are: Section 5, Tu 11:00, with Ms. Jaber — Section 6, Tu 12:30, with Ms. Jaber — Section 7, Tu 2:00, with Ms. Jaber — Section 8, Tu 3:30, with Prof. Makdisi.

The exam is open book and notes. Calculators are **not** allowed. Please make it clear in your exam booklet which problem you are solving on each page. Remember to justify your work carefully.

The problems are listed in the order of the material in the book, **not** in order of increasing difficulty. **Take a few minutes to look over the exam to decide which problems you wish to work on first.** Remember to budget your time wisely. **The total number of points on the exam is 157.**

Good luck!

Question 1 (15 pts = 3 pts for each part). Which of the following series converge or diverge, and why?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{n^3+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{4+\sin(n^2)}{n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \cos\left[\frac{n^2}{2^n}\right]$ (e) $\displaystyle\sum_{n=1}^{\infty} \sin\left[\frac{n^2}{2^n}\right]$

Question 2 (12 pts = 6 pts for each part).

(a) Express $\displaystyle\int_{x=0}^{0.1} \ln(1+x^2)\,dx$ as a series.

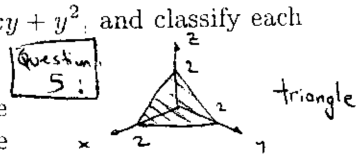
(b) Find a specific partial sum s_N of the above series such that $|s_N - L| < 10^{-10}$. You do not have to check the hypotheses of the theorem that you use to estimate the error.

Question 3 (12 pts = 6 pts for each part).

(a) Find the third-order Taylor approximation P_3 to $f(x) = e^{2x}$ **at the center** $x = 1$. Your answer should have the form $P_3(x) = c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^3$ for specific numbers $c_0, \ldots, c_3$.

(b) As usual, we write $f(x) = P_3(x) + R_3(x)$, where R_3 is the error in the Taylor approximation. **We assume that** $0.9 \le x \le 1.1$. Find an explicit constant A for which $|R_3(x)| \le A$. (You do not need to simplify your expression for A).

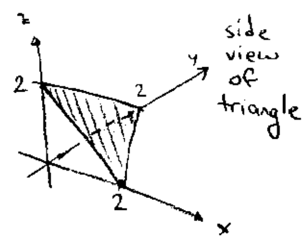
Question 4 (12 pts). Find all the critical points of $f(x,y) = x^4 + 2xy + y^2$, and classify each critical point as a local maximum, a local minimum, or a saddle point.



Question 5 (12 pts total). We wish to minimize and maximize the function $f(x,y,z) = xy^2z^3$ on the triangle given by the part of the plane $x+y+z=2$ lying in the first octant.

(a) (2 pts) Use common sense to explain why the minimum value of f on the triangle is 0, which is attained on the edges of the triangle.

(b) (10 pts) Use Lagrange multipliers to find the maximum value of f on the triangle. Note that by part (a) you can assume that $x, y, z \neq 0$.



Question 6 (15 pts = 5 pts for each part).

(a) Given the function $f(x,y) = \sqrt{y + x^2 y}$. Find $\vec{\nabla} f$ and $\vec{\nabla} f\Big|_{(1,2)}$.

(b) Use the linear approximation (i.e., the increment theorem) to obtain an approximate value of $f(1.02, 2.01)$.

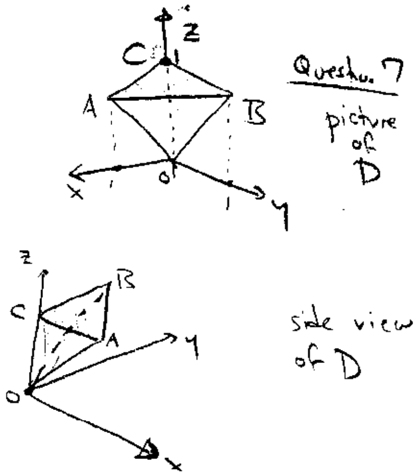
(c) Given a moving point $P(t) = (t^2 + 4t + 4, t^2 - t)$, find the value of $\dfrac{d}{dt}[f(P(t))]$ at the instant when the moving point passes through $(1,2)$.

Question 7 (20 pts total). Let D be the solid upside-down pyramid with vertices $O(0,0,0)$, $A(1,0,1)$, $B(0,1,1)$, and $C(0,0,1)$. Note that the equation of the plane OAB is $z = x+y$. The **density** of D is given by $\delta(x,y,z) = x$.

(a) (9 pts for this part) Using xyz-coordinates, set up (5 pts) and **evaluate** (4 pts) an integral in the order $\int_z \int_x \int_y$ (i.e., $dy\,dx\,dz$) that computes the total mass of D.

(b) (5 pts) Set up **but do not evaluate** the same integral in the order $\int_x \int_y \int_z$.

(c) (6 pts) Set up **but do not evaluate** the same integral in cylindrical coordinates (any order is fine, but I recommend $\int_\theta \int_r \int_z$).

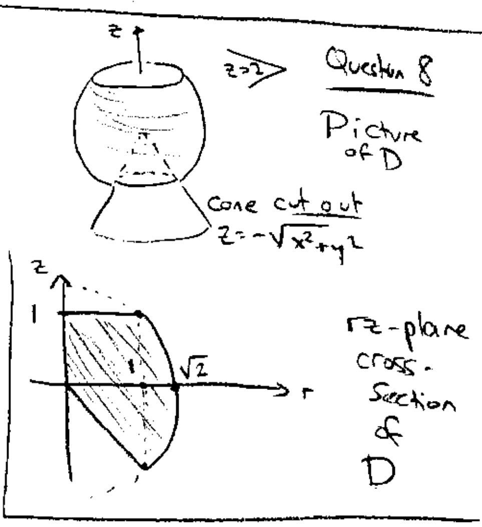


Question 8 (16 pts = 8 pts for each part). Let D be the part of the solid sphere $x^2+y^2+z^2 \le 2$ which is **below** the plane $z=1$ and **above** the cone $z = -\sqrt{x^2+y^2}$.

(a) Set up **but do not evaluate** an integral for the volume of D in cylindrical coordinates.

(b) Set up **but do not evaluate** an integral for the volume of D in spherical coordinates.

Remark: for both (a) and (b), you will need to split your integral into two parts.



Question 9 (12 pts = 6 pts for each part). (a) Find a potential function $f(x,y,z)$ for the conservative vector field $\vec{\mathbf{F}} = (y+1, x+yz^2, y^2z+z^2)$.

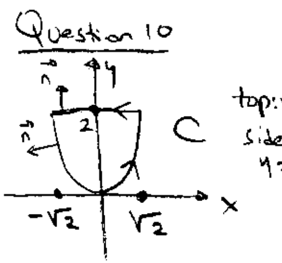
(b) Show that the vector field $\vec{\mathbf{G}} = (2xz, z^2+y^2z^3, x^2+z^2y^3)$ is **not** conservative (i.e., does not have a potential function).

Question 10 (16 pts total). Given the vector field $\vec{\mathbf{F}} = (0, 3x)$ and the closed curve C given by parts of the parabola $y = x^2$ and the line $y = 2$. We orient C counterclockwise and use an outward-pointing normal.

(a) (7 pts) Directly calculate the work (i.e., circulation) integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$, by parametrizing C.

(b) (7 pts) Use Green's theorem to re-calculate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds$.

(c) (2 pts) Use Green's theorem to calculate the flux integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\,ds$.



Question 11 (15 pts). Given the vector field $\vec{\mathbf{F}} = (0, y, 0)$ and the surface S shaped like a cup, given by the part of the paraboloid $y = x^2+z^2-3$ cut off by the xz-plane. We orient S using the normal vector shown, which points generally away from the origin.

Set up **but do not evaluate** the **surface flux integral** $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\,d\sigma$ in terms of an explicit integral that involves only x and z.