6.1

6.1 The velocity in a certain two-dimensional flow field is given by the equation

$$\mathbf{V} = 2xt\hat{\mathbf{i}} - 2yt\hat{\mathbf{j}}$$

where the velocity is in ft/s when x, y, and t are in feet and seconds, respectively. Determine expressions for the local and convective components of acceleration in the x and y directions. What is the magnitude and direction of the velocity and the acceleration at the point $x = y = 2$ ft at the time $t = 0$?

From expression for velocity, $u = 2xt$ and $v = -2yt$.

Since

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

then

$$a_x\,(local) = \frac{\partial u}{\partial t} = \underline{\underline{2x}}$$

and

$$a_x\,(conv) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (2xt)(2t) + (-2yt)(0)$$

$$= \underline{\underline{4xt^2}}$$

Similarly,

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

and

$$a_y\,(local) = \frac{\partial v}{\partial t} = \underline{\underline{-2y}}$$

$$a_y\,(conv.) = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (2xt)(0) + (-2yt)(-2t)$$

$$= \underline{\underline{4yt^2}}$$

At $x = y = 2$ ft and $t = 0$

$$u = 2(2)(0) = 0 \qquad v = -2(2)(0) = 0$$

So that $\underline{\underline{\vec{V} = 0}}$

and

$$a_x = 2x + 4xt^2 = 2(2) + 4(2)(0) = 4\ ft/s^2$$

$$a_y = -2y + 4yt^2 = -2(2) + 4(2)(0) = -4\ ft/s^2$$

Thus,

$$\vec{a} = \underline{\underline{4\hat{i} - 4\hat{j}\ ft/s^2}} \quad \text{with } |\vec{a}| = \sqrt{(4)^2 + (-4)^2} = \underline{\underline{5.66\ ft/s^2}}$$

6.2

6.2 Repeat Problem 6.1 if the flow field is described by the equation

$$\mathbf{V} = (3x^2 + 1)\hat{\mathbf{i}} - 6xy\hat{\mathbf{j}}$$

where the velocity is in ft/s when x and y are in feet.

From expression for velocity, $u = 3x^2+1$ and $v = -6xy$.

Since

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

then

$$a_x(\text{local}) = \frac{\partial u}{\partial t} = \underline{\underline{0}}$$

and

$$a_x(\text{conv}) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (3x^2+1)(6x) - (6xy)(0)$$

$$= 6x(3x^2+1) = \underline{\underline{18x^3 + 6x}}$$

Similarly,

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

and

$$a_y(\text{local}) = \frac{\partial v}{\partial t} = \underline{\underline{0}}$$

$$a_y(\text{conv}) = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (3x^2+1)(-6y) + (-6xy)(-6x)$$

$$= \underline{\underline{18x^2y - 6y}}$$

At $x = y = 1$ ft and $t = 0$

$$u = 3(1)^2 + 1 = 4 \qquad v = -6(1)(1) = -6$$

so that

$$\vec{V} = 4\hat{i} - 6\hat{j}\ \frac{ft}{s} \quad \text{and} \quad |\vec{V}| = \sqrt{4^2 + (-6)^2} = \underline{\underline{7.21\ ft/s}}$$

and

$$a_x = 6x(3x^2+1) = 6(1)[3(1)^2+1] = 24\ ft/s^2$$

$$a_y = 18x^2y - 6y = 18(1)^2(1) - 6(1) = 12\ ft/s^2$$

Thus,

$$\vec{a} = 24\hat{i} + 12\hat{j}\ ft/s^2$$

with

$$|\vec{a}| = \sqrt{(24)^2 + (12)^2} = \underline{\underline{26.8\ ft/s^2}}$$

6.3

6.3 The velocity in a certain flow field is given by the equation

$$\mathbf{V} = x\hat{\mathbf{i}} + x^2z\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

Determine the expressions for the three rectangular components of acceleration.

From expression for velocity, $u = x$ $\quad v = x^2z$ $\quad w = yz$

Since

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

then

$$a_x = 0 + (x)(1) + (x^2z)(0) + (yz)(0)$$

$$= \underline{\underline{x}}$$

Similarly,

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

and

$$a_y = 0 + (x)(2xz) + (x^2z)(0) + (yz)(x^2)$$

$$= \underline{\underline{2x^2z + x^2yz}}$$

Also,

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

so that

$$a_z = 0 + (x)(0) + (x^2z)(z) + (yz)(y)$$

$$= \underline{\underline{x^2z^2 + y^2z}}$$

6.4

6.4 The three components of velocity in a flow field are given by

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z^2$$
$$w = -3xz - z^2/2 + 4$$

(a) Determine the volumetric dilatation rate, and interpret the results. **(b)** Determine an expression for the rotation vector. Is this an irrotational flow field?

(a) Volumetric dilatation rate $= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ (Eq. 6.9)

Thus, for velocity components given

volumetric dilatation rate $= 2x + (x+z) + (-3x-z) = \underline{\underline{0}}$

This result indicates that there is no change in the volume of a fluid element as it moves from one location to another.

(b) From Eqs. 6.12, 6.13, and 6.14 with the velocity components given:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(y - 2y\right) = -\frac{y}{2}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left[0 - (y+2z)\right] = -\left(\frac{y}{2}+z\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left[2z - (-3z)\right] = \frac{5z}{2}$$

Thus,

$$\underline{\underline{\vec{\omega} = -\left(\frac{y}{2}+z\right)\hat{i} + \frac{5z}{2}\hat{j} - \frac{y}{2}\hat{k}}}$$

Since $\vec{\omega}$ is not zero everywhere the flow field is not irrotational. $\underline{\underline{\text{No}}}$.

6.5

6.5 Determine an expression for the vorticity of the flow field described by

$$\mathbf{V} = -xy^3\,\hat{\mathbf{i}} + y^4\hat{\mathbf{j}}$$

Is the flow irrotational?

$$\vec{\zeta} = 2\vec{\omega} \qquad \text{(Eq. 6.17)}$$

From expression for velocity, $u = -xy^3$, $v = y^4$, and $w = 0$, and with

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \text{(Eq. 6.13)}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \text{(Eq. 6.14)}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.12)}$$

it follows that

$\omega_x = 0$, $\omega_y = 0$, and $\omega_z = \frac{1}{2}\left[0 - (-3xy^2)\right] = \frac{3}{2}xy^2$

Thus,

$$\vec{\zeta} = 2\left(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}\right)$$

$$= 2\left[(0)\hat{i} + (0)\hat{j} + \left(\tfrac{3}{2}xy^2\right)\hat{k}\right]$$

$$= \underline{\underline{3xy^2\,\hat{k}}}$$

Since $\vec{\zeta}$ is not zero everywhere the flow is not irrotational. <u>No.</u>

6.6

6.6 A one-dimensional flow is described by the velocity field

$$u = ay + by^2$$
$$v = w = 0$$

where a and b are constants. Is the flow irrotational? For what combination of constants (if any) will the rate of angular deformation as given by Eq. 6.18 be zero?

For irrotational flow $\vec{\omega} = 0$, and for the velocity distribution given:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\left(\frac{a}{2} + by\right)$$

Thus, $\vec{\omega}$ is not zero everywhere and the flow is not irrotational. No.

Since (from Eq. 6.18)

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

it follows for the velocity distribution given that

$$\dot{\gamma} = a + 2by$$

Thus, there are no values of a and b (except both equal to zero) that will give $\dot{\gamma} = 0$ for all values of y. None.

6.7

6.7 For a certain incompressible, two-dimensional flow field the velocity component in the y direction is given by the equation

$$v = 3xy - x^2y$$

Determine the velocity component in the x direction so that the continuity equation is satisfied.

To satisfy the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Since

$$\frac{\partial v}{\partial y} = 3x - x^2$$

Then from Eq. (1)

$$\frac{\partial u}{\partial x} = -3x + x^2 \qquad (2)$$

Equation (2) can be integrated with respect to x to obtain

$$\int du = -\int 3x\,dx + \int x^2 dx + f(y)$$

or

$$u = -\frac{3}{2}x^2 + \frac{x^3}{3} + f(y)$$

where $f(y)$ is an undetermined function of y.

6.8

6.8 An incompressible viscous fluid is placed between two large parallel plates as shown in Fig. P6.8. The bottom plate is fixed and the upper plate moves with a constant velocity, U. For these conditions the velocity distribution between the plates is linear, and can be expressed as

$$u = U\frac{y}{b}$$

Determine: **(a)** the volumetric dilatation rate, **(b)** the rotation vector, **(c)** the vorticity, and **(d)** the rate of angular deformation.

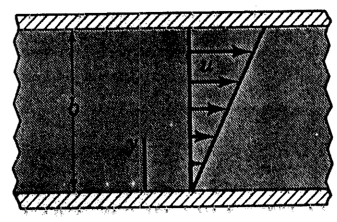


FIGURE P6.8

(a) Volumetric dilatation rate $= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \underline{\underline{0}}$

(b) For velocity distribution given,

$$\vec{\omega} = \omega_z \hat{k}$$

and

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{U}{2b}$$

Thus,

$$\vec{\omega} = \underline{\underline{-\frac{U}{2b}\hat{k}}}$$

(c) $$\vec{\zeta} = 2\vec{\omega} = \underline{\underline{-\frac{U}{b}\hat{k}}}$$

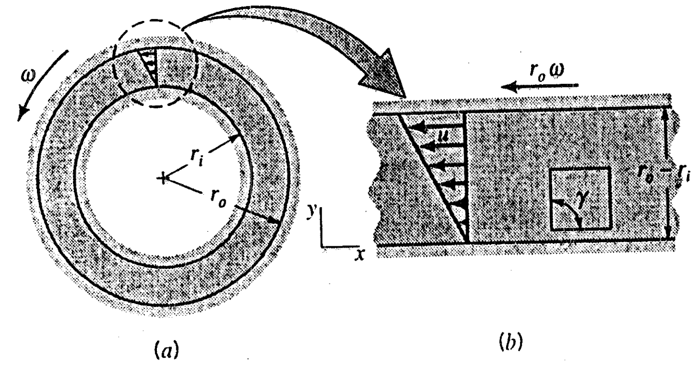
(d) $$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \text{(Eq. 6.18)}$$

Thus,

$$\dot{\gamma} = \underline{\underline{\frac{U}{b}}}$$

6.9

6.9 A viscous fluid is contained in the space between concentric cylinders. The inner wall is fixed, and the outer wall rotates with an angular velocity ω. (See Fig. P6.9a and Video V6.1.) Assume that the velocity distribution in the gap is linear as illustrated in Fig. P6.9b. For the small rectangular element shown in Fig. P6.9b, determine the rate of change of the right angle γ due to the fluid motion. Express your answer in terms of r_0, r_i, and ω.



■ FIGURE P6.9

From Eq. 6.18

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

For the linear distribution

$$u = -\frac{r_o \omega}{r_o - r_i}\, y$$

So that

$$\frac{\partial u}{\partial y} = -\frac{r_o \omega}{r_o - r_i}$$

and since $v = 0$

$$\dot{\gamma} = -\frac{r_o \omega}{r_o - r_i}$$

The negative sign indicates that the original right angle is increasing.

6.10

6.10 For a certain incompressible flow field it is suggested that the velocity components are given by the equations

$$u = 2xy \qquad v = -x^2y \qquad w = 0$$

Is this a physically possible flow field? Explain.

Any physically possible incompressible flow field must satisfy conservation of mass as expressed by the relationship

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

For the velocity distribution given,

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = -x^2 \qquad \frac{\partial w}{\partial z} = 0$$

Substitution into Eq. (1) shows that

$$2y - x^2 + 0 \neq 0$$

Thus, this is not a physically possible flow field. No.

6.11

6.11 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

$$u = y^2 - x(1 + x)$$
$$v = y(2x + 1)$$

Show that the flow is irrotational and satisfies conservation of mass.

If the two-dimensional flow is irrotational,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

For the velocity distribution given,

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2y$$

Thus,

$$\omega_z = \frac{1}{2}\left(2y - 2y\right) = 0$$

and the <u>flow is irrotational.</u>

To satisfy conservation of mass,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since,

$$\frac{\partial u}{\partial x} = -1 - 2x \qquad \frac{\partial v}{\partial y} = 2x + 1$$

then

$$-1 - 2x + 2x + 1 = 0$$

and

<u>Conservation of mass is satisfied</u>.

6.12

6.12 For each of the following stream functions, with units of m^2/s, determine the magnitude and the angle the velocity vector makes with the x-axis at $x = 1$ m, $y = 2$ m. Locate any stagnation points in the flow field.
(a) $\psi = xy$
(b) $\psi = -2x^2 + y$

From the definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Eqs. 6.37)}$$

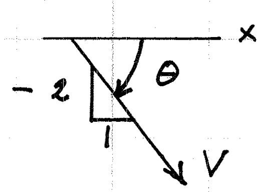
(a) For $\psi = xy$,

$$u = \frac{\partial \psi}{\partial y} = x \qquad v = -\frac{\partial \psi}{\partial x} = -y$$

At $x = 1m$, $y = 2m$, it follows that $u = 1\frac{m}{s}$ and $v = -2\frac{m}{s}$

Thus,

$$|V| = \sqrt{u^2+v^2} = \sqrt{(1m)^2+(-2m)^2} = \underline{\underline{2.24\ \frac{m}{s}}}$$



$$\tan\theta = \frac{-2}{1} \qquad \underline{\underline{\theta = -63.4°}}$$

Since $u = 0$ at $x = 0$ and $v = 0$ at $y = 0$, a $\underline{\underline{\text{stagnation point occurs at } x = y = 0.}}$

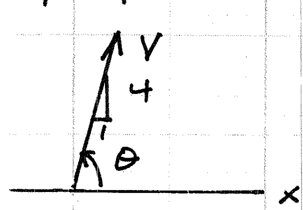
(b) For $\psi = -2x^2 + y$,

$$u = \frac{\partial \psi}{\partial y} = 1\ \frac{m}{s} \qquad v = -\frac{\partial \psi}{\partial x} = 4x$$

At $x = 1m$, $y = 2m$, it follows that $u = 1\frac{m}{s}$ and $v = 4\frac{m}{s}$

Thus,

$$|V| = \sqrt{u^2+v^2} = \sqrt{(1\tfrac{m}{s})^2+(4\tfrac{m}{s})^2} = \underline{\underline{4.12\ \frac{m}{s}}}$$



$$\tan\theta = \frac{4}{1} \qquad \underline{\underline{\theta = 76.0°}}$$

Since $u \neq 0$, there are $\underline{\underline{\text{no stagnation points}}}$.

6.13

6.13 The stream function for an incompressible, two-dimensional flow field is

$$\psi = ay - by^3$$

where a and b are constants. Is this an irrotational flow? Explain.

For the flow to be irrotational,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.12)}$$

and for the stream function given,

$$u = \frac{\partial \psi}{\partial y} = a - 3by^2$$

$$v = -\frac{\partial \psi}{\partial x} = 0$$

Thus,

$$\frac{\partial u}{\partial y} = -6by \qquad \frac{\partial v}{\partial x} = 0$$

so that

$$\omega_z = \frac{1}{2}\left[0 - (-6by)\right] = 3by$$

Since $\omega_z \neq 0$ flow is not irrotational (unless $b=0$). $\underline{\underline{\text{No}}}$.

6.14

6.14 The stream function for an incompressible, two-dimensional flow field is

$$\psi = ay^2 - bx$$

where a and b are constants. Is this an irrotational flow? Explain.

For the flow to be irrotational (see Eq. 6.12),

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

and for the stream function given,

$$u = \frac{\partial \psi}{\partial y} = 2ay$$

$$v = -\frac{\partial \psi}{\partial x} = b$$

Thus,

$$\frac{\partial u}{\partial y} = 2a \qquad \frac{\partial v}{\partial x} = 0$$

so that

$$\omega_z = \frac{1}{2}\left[0 - (2a)\right] = -a$$

Since $\omega_z \neq 0$ flow is not irrotational (unless $a = 0$). No.

6.15

6.15 The velocity components for an incompressible, plane flow are

$$v_r = Ar^{-1} + Br^{-2}\cos\theta$$

$$v_\theta = Br^{-2}\sin\theta$$

where A and B are constants. Determine the corresponding stream function.

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = -\frac{\partial\psi}{\partial r} \qquad \text{(Eq. 6.42)}$$

so that for the velocity distribution given,

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = Ar^{-1} + Br^{-2}\cos\theta \qquad (1)$$

$$\frac{\partial\psi}{\partial r} = -Br^{-2}\sin\theta \qquad (2)$$

Integrate Eq. (1) with respect to θ to obtain

$$\int d\psi = \int (A + Br^{-1}\cos\theta)\,d\theta + f_1(r)$$

or

$$\psi = A\theta + Br^{-1}\sin\theta + f_1(r) \qquad (3)$$

Similarly, integrate Eq. (2) with respect to r to obtain

$$\int d\psi = -\int Br^{-2}\sin\theta\,dr + f_2(\theta)$$

or

$$\psi = Br^{-1}\sin\theta + f_2(\theta) \qquad (4)$$

Thus, to satisfy both Eqs. (3) and (4)

$$\psi = \underline{\underline{A\theta + Br^{-1}\sin\theta + C}}$$

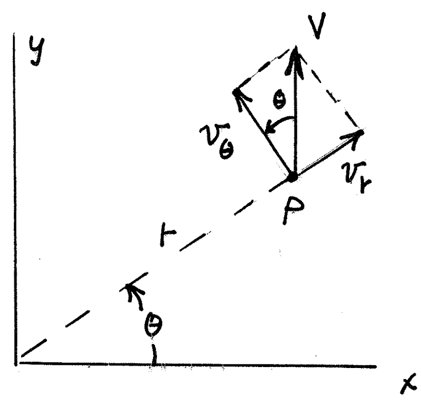
where C is an arbitrary constant.

6.16

6.16 For a certain two-dimensional flow field

$$u = 0$$
$$v = V$$

(a) What are the corresponding radial and tangential velocity components? **(b)** Determine the corresponding stream function expressed in Cartesian coordinates and in cylindrical polar coordinates.



(a) At an arbitrary point P (see figure)

$$\underline{\underline{v_r = V \sin\theta}}$$

$$\underline{\underline{v_\theta = V \cos\theta}}$$

(b) Since

$$u = \frac{\partial \psi}{\partial y} = 0 \qquad v = -\frac{\partial \psi}{\partial x} = V$$

it follows that ψ is not a function of y and

$$\underline{\underline{\psi = -Vx + C}}$$

where C is an arbitrary constant.

Also, with $x = r\cos\theta$

$$\underline{\underline{\psi = -Vr\cos\theta + C}}$$

6.17

6.17 Make use of the control volume shown in Fig. P6.17 to derive the continuity equation in cylindrical coordinates (Eq. 6.33 in text).

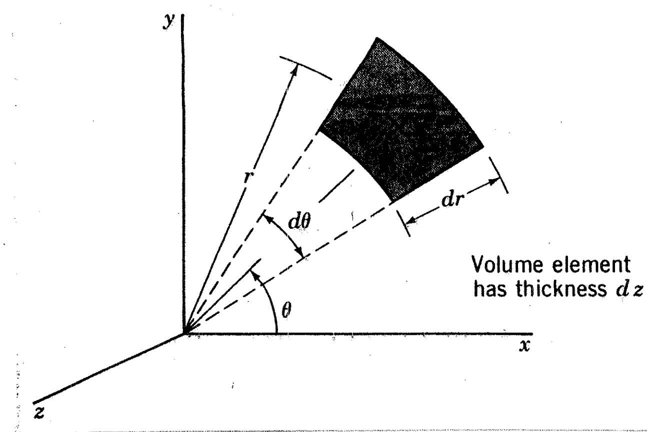


FIGURE P6.17

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V}\cdot\hat{n}\, dA = 0 \qquad \text{(Eq. 6.19)}$$

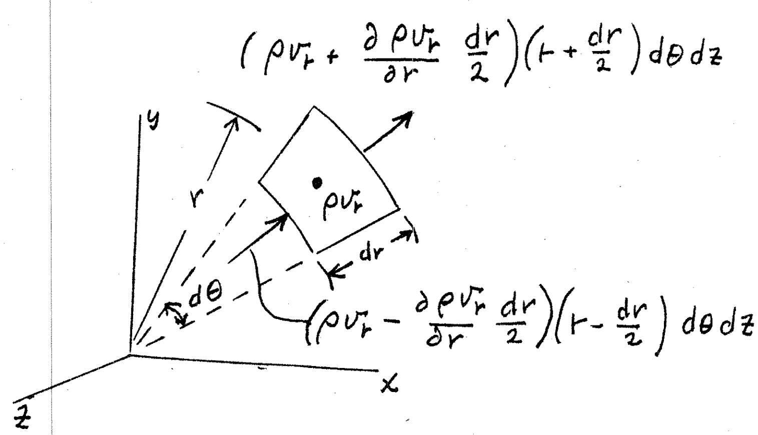
For the differential control volume shown

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall \approx \frac{\partial \rho}{\partial t}\, r\, d\theta\, dr\, dz \qquad (1)$$

and

$$\int_{cs} \rho \vec{V}\cdot\hat{n}\, dA = \text{net rate of mass outflow through surfaces of control volume}$$

From figure at right:
Net rate of mass outflow in r-direction =



$$\left(\rho v_r + \frac{\partial \rho v_r}{\partial r}\frac{dr}{2}\right)\left(r + \frac{dr}{2}\right) d\theta\, dz - \left(\rho v_r - \frac{\partial \rho v_r}{\partial r}\frac{dr}{2}\right)\left(r - \frac{dr}{2}\right) d\theta\, dz$$

$$= \frac{\partial \rho v_r}{\partial r}\, r\, dr\, d\theta\, dz + \rho v_r\, dr\, d\theta\, dz \qquad (2)$$

(con't)

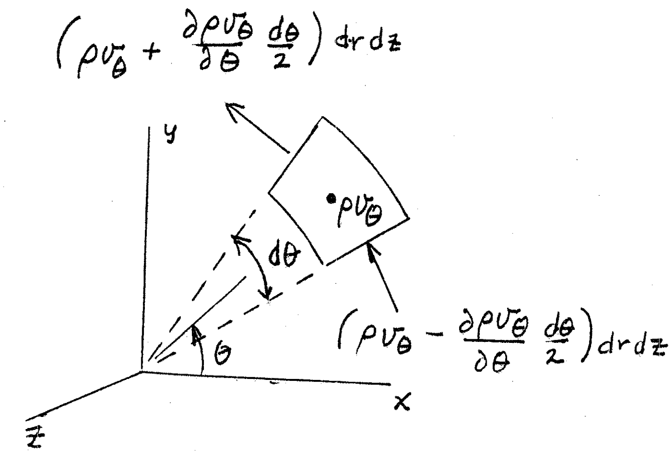
6.17 (Con't)

From figure at right:

Net rate of mass outflow in θ-direction =

$$\left(\rho v_\theta + \frac{\partial \rho v_\theta}{\partial \theta}\frac{d\theta}{2}\right) dr\,dz - \left(\rho v_\theta - \frac{\partial \rho v_\theta}{\partial \theta}\frac{d\theta}{2}\right) dr\,dz$$

$$= \frac{\partial \rho v_\theta}{\partial \theta}\, dr\, d\theta\, dz \qquad (3)$$

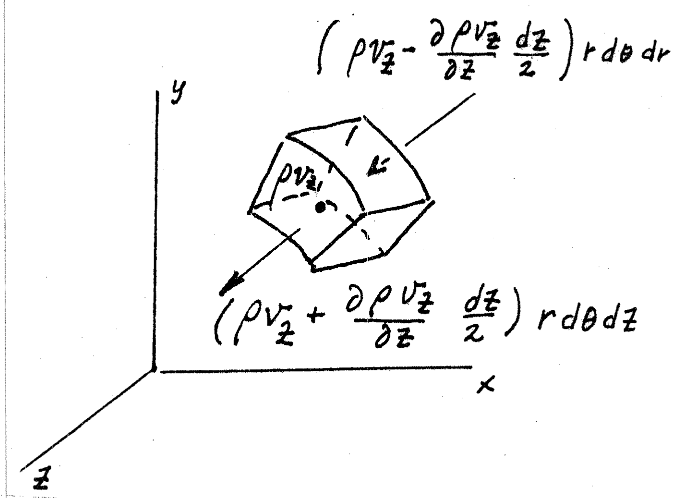


From figure at right:

Net rate of mass outflow in z-direction =

$$\left(\rho v_z + \frac{\partial \rho v_z}{\partial z}\frac{dz}{2}\right) r\,d\theta\,dr - \left(\rho v_z - \frac{\partial \rho v_z}{\partial z}\frac{dz}{2}\right) r\,d\theta\,dr$$

$$= \frac{\partial \rho v_z}{\partial z}\, r\,dr\,d\theta\,dz \qquad (4)$$



Substitution of Eqs. (1) thru (4) into Eq. 6.19 yields

$$\frac{\partial \rho}{\partial t} r\,dr\,d\theta\,dz + \frac{\partial \rho v_r}{\partial r} r\,dr\,d\theta\,dz + \rho v_r\, dr\,d\theta\,dz + \frac{\partial \rho v_\theta}{\partial \theta} dr\,d\theta\,dz + \frac{\partial \rho v_z}{\partial z} r\,dr\,d\theta\,dz = 0$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_r}{\partial r} + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial \rho v_\theta}{\partial \theta} + \frac{\partial \rho v_z}{\partial z} = 0 \qquad (5)$$

Since

$$\frac{\partial \rho v_r}{\partial r} + \frac{\rho v_r}{r} = \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)$$

Eq. (5) can be written as

$$\underline{\underline{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0}}$$

which is Eq. 6.33.

6.18

6.18 A two-dimensional, incompressible flow is given by $u = -y$ and $v = x$. Show that the streamline passing through the point $x = 10$ and $y = 0$ is a circle centered at the origin.

For two-dimensional flow along a streamline

$$\frac{dy}{dx} = \frac{v}{u}$$

so that for the velocity components given

$$\frac{dy}{dx} = \frac{x}{-y}$$

and

$$-\int y\,dy = \int x\,dx$$

Thus,

$$-\frac{y^2}{2} = \frac{x^2}{2} + C \quad (\text{where } C \text{ is a constant})$$

and

$$x^2 + y^2 = 2C = C' \qquad (1)$$

Equation (1) represents the equation for the family of streamlines. For a given value of C' the equation gives a circle centered at the origin with C' the square of the radius.

For $x = 10$ and $y = 0$

$$10^2 + 0 = C' = 100$$

and the equation of the streamline passing through this point is

$$x^2 + y^2 = 100$$

which is a <u>circle of radius 10 centered at the origin.</u>

6.19

6.19 In a certain steady, two-dimensional flow field the fluid density varies linearly with respect to the coordinate x; that is, $\rho = Ax$ where A is a constant. If the x component of velocity u is given by the equation $u = y$, determine an expression for v.

For a variable density flow,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} \qquad \text{(Eq. 6.29)}$$

with

$$\rho u = (Ax)(y) = Axy$$

it follows that

$$\frac{\partial(\rho u)}{\partial x} = Ay$$

Thus,

$$\frac{\partial(\rho v)}{\partial y} = -Ay \qquad (1)$$

Integrate Eq. (1) with respect to y to obtain

$$\int d(\rho v) = -\int Ay\,dy + f_1(x)$$

or

$$\rho v = -\frac{Ay^2}{2} + f_1(x)$$

With $\rho = Ax$

$$v = -\left(\frac{1}{Ax}\right)\left(\frac{Ay^2}{2}\right) + \frac{f_1(x)}{Ax}$$

or

$$v = -\frac{y^2}{2x} + f(x)$$

where $f(x)$ is an arbitrary function of x.

6.20

6.20 In a two-dimensional, incompressible flow field, the x component of velocity is given by the equation $u = 2x$. **(a)** Determine the corresponding equation for the y component of velocity if $v = 0$ along the x axis. **(b)** For this flow field what is the magnitude of the average velocity of the fluid crossing the surface OA of Fig. P6.20? Assume that the velocities are in ft/s when x and y are in feet.

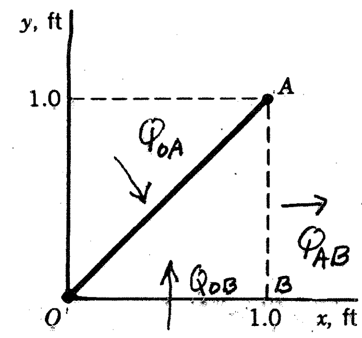


FIGURE P6.20

(consider a unit thickness = 1 ft)

(a) To satisfy the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since $\frac{\partial u}{\partial x} = 2$

it follows that

$$\frac{\partial v}{\partial y} = -2 \qquad (1)$$

Integration of Eq. (1) with respect to y yields

$$v = -2y + f(x)$$

If $v = 0$ along x-axis ($y=0$) then $f(x) = 0$ so that

$$\underline{\underline{v = -2y}}$$

(b) To satisfy conservation of mass

$$Q_{OA} = Q_{AB} - Q_{OB} \qquad \text{(see figure)}$$

Along AB $\quad u = 2(1) = 2\ \frac{ft}{s}$ $\quad$ so that

$$Q_{AB} = u\,A_{AB} = (2\ ft/s)(1\ ft)(1\ ft) = 2\ \frac{ft^3}{s}$$

Along OB $\quad v = 0$ $\quad$ so that $\quad Q_{OB} = 0$.

Thus,

$$Q_{OA} = Q_{AB} = 2\ \frac{ft^3}{s}$$

and

$$V_{AV} = \frac{Q_{OA}}{\text{area}_{OA}} = \frac{2\ \frac{ft^3}{s}}{\sqrt{2}\ ft^2} = \underline{\underline{1.41\ \frac{ft}{s}}}$$

6.21

6.21 The radial velocity component in an incompressible, two-dimensional flow field ($v_z = 0$) is

$$v_r = 2r + 3r^2 \sin\theta$$

Determine the corresponding tangential velocity component, v_θ, required to satisfy conservation of mass.

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(Eq. 6.35)}$$

Since $v_z = 0$,

$$\frac{\partial v_\theta}{\partial \theta} = -\frac{\partial (r v_r)}{\partial r} \qquad (1)$$

and with

$$r v_r = 2r^2 + 3r^3 \sin\theta$$

it follows that

$$\frac{\partial (r v_r)}{\partial r} = 4r + 9r^2 \sin\theta$$

Thus, Eq. (1) becomes

$$\frac{\partial v_\theta}{\partial \theta} = -(4r + 9r^2 \sin\theta) \qquad (2)$$

Equation (2) can be integrated with respect to θ to obtain

$$\int d v_\theta = -\int (4r + 9r^2 \sin\theta)\, d\theta + f(r)$$

or

$$v_\theta = \underline{\underline{-4r\theta - 9r^2\cos\theta + f(r)}}$$

where $f(r)$ is an undetermined function of r.

6.22

6.22 The stream function for an incompressible flow field is given by the equation

$$\psi = 3x^2y - y^3$$

where the stream function has the units of m^2/s with x and y in meters. **(a)** Sketch the streamline(s) passing through the origin. **(b)** Determine the rate of flow across the straight path AB shown in Fig. P6.22.

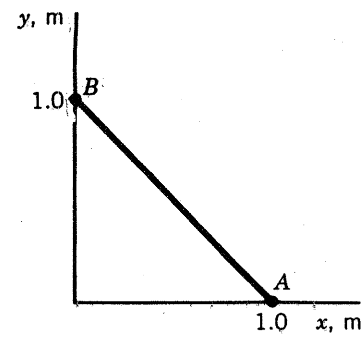


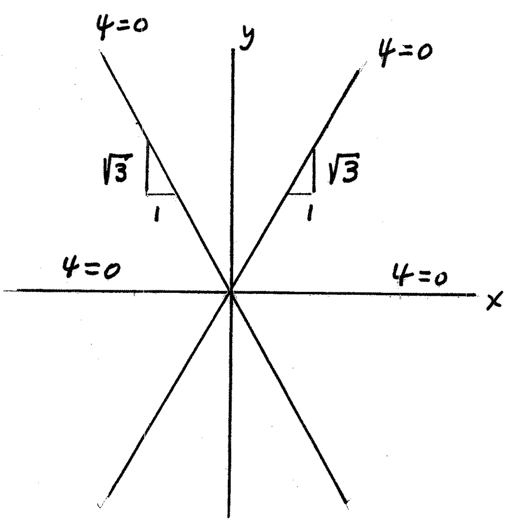
FIGURE P6.22

(a) Lines of constant ψ are streamlines. For $\psi = 3x^2y - y^3$ the streamline passing through the origin $(x=0, y=0)$ has a value $\psi = 0$. Thus, the equation for the streamlines through the origin is

$$0 = 3x^2y - y^3$$

or

$$y = \pm\sqrt{3}\,x$$



A sketch of these streamlines is shown in the figure.

(b) $q = \psi_B - \psi_A$

At B $\quad x = 0,\ y = 1\,m$ so that

$$\psi_B = 3(0)^2(1) - (1)^3 = -1\ m^3/s \text{ (per unit width)}$$

At A $\quad x = 1\,m,\ y = 0$ so that

$$\psi_A = 3(1)^2(0) - (0)^3 = 0$$

Thus,

$$q = \psi_B = \underline{\underline{-1\ m^3/s \text{ (per unit width)}}}$$

The negative sign indicates that the flow is from right to left as we look from A to B.

6.23

6.23 The streamlines in a certain incompressible, two-dimensional flow field are all concentric circles so that $v_r = 0$. Determine the stream function for **(a)** $v_\theta = Ar$ and for **(b)** $v_\theta = Ar^{-1}$, where A is a constant.

From the definition of the stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{(Eq. 6.42)}$$

so that with $v_r = 0$ it follows that $\frac{\partial \psi}{\partial \theta} = 0$
and therefore

$$\psi = f(r)$$

(a) For $v_\theta = Ar$

$$\frac{\partial \psi}{\partial r} = -Ar \qquad (1)$$

Integrate Eq. (1) with respect to r to obtain

$$\int d\psi = -\int Ar\,dr$$

or

$$\psi = -\frac{Ar^2}{2} + f_1(\theta)$$

However, since ψ is not a function of θ, it follows that

$$\underline{\underline{\psi = -\frac{Ar^2}{2} + C}}$$

where C is an arbitrary constant.

(b) Similarly, for $v_\theta = Ar^{-1}$

$$\int d\psi = -\int Ar^{-1}\,dr$$

or

$$\psi = \underline{\underline{-A \ln r + C}}$$

6.24*

6.24* The stream function for an incompressible, two-dimensional flow field is

$$\psi = 3x^2y + y$$

For this flow field plot several streamlines.

The equation for a streamline is found by setting ψ = constant in the equation for the stream function. Thus, for the given stream function

$$\psi = 3x^2y + y$$

it follows that the equation of a streamline is

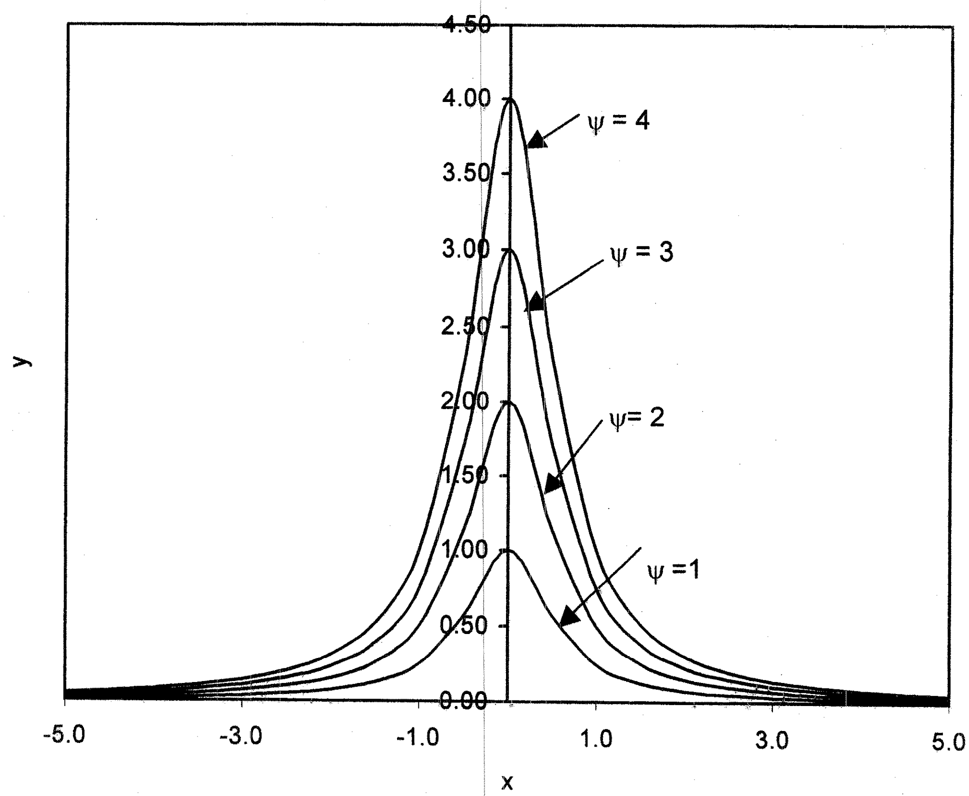
$$y = \frac{\psi}{1 + 3x^2}$$

where various constant values can be assigned to ψ to obtain a family of streamlines. Tabulated results for $\psi = 1, 2, 3, 4,$ and a plot showing the streamlines are given below.

	ψ =1	ψ =2	ψ =3	ψ=4
x	y	y	y	y
-5.0	0.0132	0.0263	0.0395	0.0526
-4.5	0.0162	0.0324	0.0486	0.0648
-4.0	0.0204	0.0408	0.0612	0.0816
-3.5	0.0265	0.0530	0.0795	0.1060
-3.0	0.0357	0.0714	0.1071	0.1429
-2.5	0.0506	0.1013	0.1519	0.2025
-2.0	0.0769	0.1538	0.2308	0.3077
-1.5	0.1290	0.2581	0.3871	0.5161
-1.0	0.2500	0.5000	0.7500	1.0000
-0.5	0.5714	1.1429	1.7143	2.2857
0.0	1.0000	2.0000	3.0000	4.0000
0.5	0.5714	1.1429	1.7143	2.2857
1.0	0.2500	0.5000	0.7500	1.0000
1.5	0.1290	0.2581	0.3871	0.5161
2.0	0.0769	0.1538	0.2308	0.3077
2.5	0.0506	0.1013	0.1519	0.2025
3.0	0.0357	0.0714	0.1071	0.1429
3.5	0.0265	0.0530	0.0795	0.1060
4.0	0.0204	0.0408	0.0612	0.0816
4.5	0.0162	0.0324	0.0486	0.0648
5.0	0.0132	0.0263	0.0395	0.0526

(con't)

6.24* (Con't)



4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
y
ψ = 4
ψ = 3
ψ= 2
ψ =1
-5.0
-3.0
-1.0
1.0
3.0
5.0
x

6.25*

6.25* The stream function for an incompressible, two-dimensional flow field is

$$\psi = 2r^3 \sin 3\theta$$

For this flow field plot several streamlines for $0 \leq \theta \leq \pi/3$.

The equation for a streamline is found by setting $\psi =$ constant in the equation for the stream function. Thus, for the given stream function

$$\psi = 2r^3 \sin 3\theta$$

it follows that the equation of a streamline is

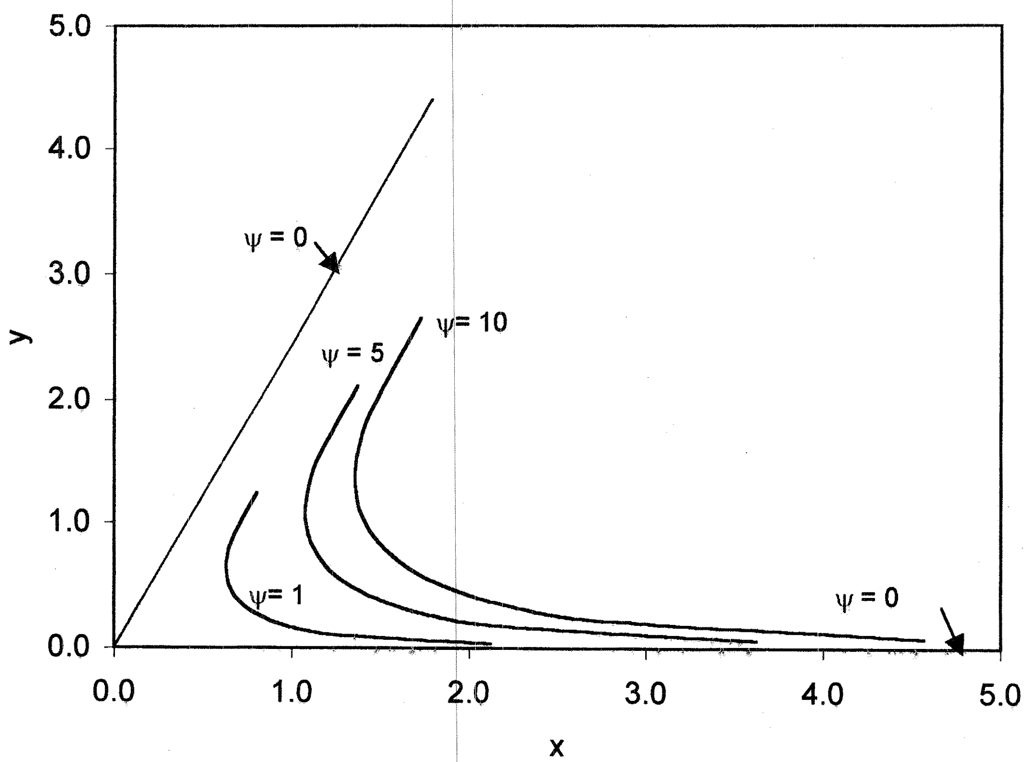
$$r = \left(\frac{\psi}{2 \sin 3\theta}\right)^{1/3}$$

where various constant values can be assigned to ψ to obtain a family of streamlines. Tabulated results and a plot of the data for $\psi = 1, 5,$ and 10 are given below where $x = r\cos\theta$ and $y = r\sin\theta$.

theta	sin 3*theta	r	$\psi = 1$		$\psi = 5$		$\psi = 10$	
			x	y	x	y	x	y
0.0175	0.0523	2.122	2.121	0.037	3.623	0.063	4.570	0.080
0.0873	0.2588	1.245	1.241	0.109	2.120	0.185	2.673	0.234
0.1571	0.4540	1.033	1.020	0.162	1.743	0.276	2.197	0.348
0.2269	0.6293	0.926	0.902	0.208	1.542	0.356	1.944	0.449
0.2967	0.7771	0.863	0.826	0.252	1.411	0.431	1.779	0.544
0.3665	0.8910	0.825	0.770	0.296	1.316	0.505	1.659	0.637
0.4363	0.9659	0.803	0.728	0.339	1.244	0.580	1.568	0.731
0.5061	0.9986	0.794	0.695	0.385	1.187	0.658	1.496	0.829
0.5760	0.9877	0.797	0.668	0.434	1.143	0.742	1.440	0.935
0.6458	0.9336	0.812	0.649	0.489	1.109	0.835	1.397	1.053
0.7156	0.8387	0.842	0.635	0.552	1.086	0.944	1.368	1.190
0.7854	0.7071	0.891	0.630	0.630	1.077	1.077	1.357	1.357
0.8552	0.5446	0.972	0.638	0.734	1.090	1.254	1.374	1.580
0.9250	0.3584	1.117	0.672	0.892	1.149	1.525	1.449	1.922
0.9948	0.1564	1.473	0.802	1.235	1.371	2.111	1.728	2.661

(con't)

6.25* (con't)



6.26

6.26 A two-dimensional flow field for a nonviscous, incompressible fluid is described by the velocity components

$$u = U_0 + 2y$$
$$v = 0$$

where U_0 is a constant. If the pressure at the origin (Fig. P6.26) is p_0, determine an expression for the pressure at **(a)** point A, and **(b)** point B. Explain clearly how you obtained your answer. Assume the units are consistent and body forces may be neglected.

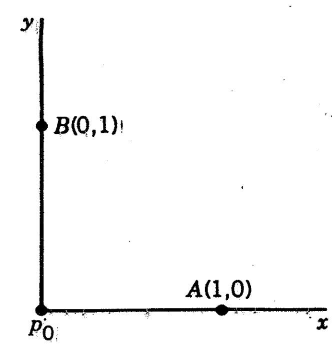


FIGURE P6.26

Check to see if flow is irrotational. Since

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.12)}$$

and for the given velocity distribution, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 2$, it follows that $\omega_z \neq 0$. Since flow is not irrotational cannot apply the Bernoulli equation between any two points in the flow field.

(a) Since $v = 0$, the origin and point A are on the same streamline. Thus,

$$\frac{p_0}{\gamma} + \frac{V_0^2}{2g} = \frac{p_A}{\gamma} + \frac{V_A^2}{2g} \qquad (1)$$

At the origin $V_0 = U_0$ and at A $V_A = U_0$ so that from Eq. (1)

$$\underline{\underline{p_A = p_0}}$$

(b) Point B is not on same streamline as origin so cannot apply Bernoulli equation between B and O. To find p_B use the y-component of Euler's equations:

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] \qquad \text{(Eq. 6.51b)}$$

Since $v = 0$ and $g_y = 0$,

$$\frac{\partial p}{\partial y} = 0$$

So that

$$\underline{\underline{p_B = p_0}}$$

6.27

6.27 In a certain two-dimensional flow field the velocity is constant with components $u = -4$ ft/s and $v = -2$ ft/s. Determine the corresponding stream function and velocity potential for this flow field. Sketch the equipotential line $\phi = 0$ which passes through the origin of the coordinate system.

From the definition of the stream function

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Eqs. 6.37)}$$

so that for the velocity components given

$$\frac{\partial \psi}{\partial y} = -4 \qquad (1)$$

$$\frac{\partial \psi}{\partial x} = 2 \qquad (2)$$

Integrate Eq. (1) with respect to y to obtain

$$\int d\psi = \int -4\, dy + f_1(x)$$

or

$$\psi = -4y + f_1(x) \qquad (3)$$

Similarly, integrate Eq. (2) with respect to x to obtain

$$\int d\psi = \int 2\, dx + f_2(y)$$

or

$$\psi = 2x + f_2(y) \qquad (4)$$

Thus, to satisfy both Eqs. (3) and (4)

$$\psi = \underline{\underline{2x - 4y + C}}$$

where C is an arbitrary constant.

From the definition of the velocity potential

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad \text{(Eqs. 6.64)}$$

so that for the velocity components given

$$\frac{\partial \phi}{\partial x} = -4 \qquad (5)$$

$$\frac{\partial \phi}{\partial y} = -2 \qquad (6)$$

(con't)

6.27 (Con't)

Integrate Eq. (5) with respect to x to obtain

$$\int d\phi = \int -4\,dx + f_3(y)$$

or

$$\phi = -4x + f_3(y) \qquad (7)$$

Integrate Eq. (6) with respect to y to obtain

$$\int d\phi = \int -2\,dy + f_4(x)$$

or

$$\phi = -2y + f_4(x) \qquad (8)$$

Thus, to satisfy both Eqs. (7) and (8)

$$\phi = \underline{\underline{-4x - 2y + C}} \qquad (9)$$

where C is an arbitrary constant.

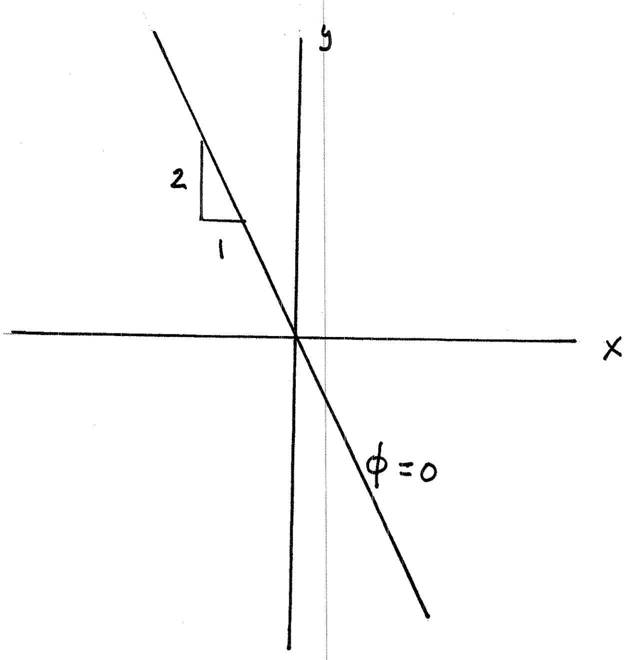
Since the equipotential line, $\phi = 0$, passes through the origin ($x = y = 0$), then $C = 0$ in Eq. (9) so that the equation of the $\phi = 0$ equipotential line is

$$2y = -4x$$

or

$$y = -2x$$

A sketch of this line is shown in the figure.



6.28

6.28 The stream function for a given two-dimensional flow field is

$$\psi = 5x^2y - (5/3)y^3$$

Determine the corresponding velocity potential.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 5x^2 - 5y^2 \qquad (1)$$

Integrate with respect to x to obtain

$$\int d\phi = \int (5x^2 - 5y^2)\,dx$$

or

$$\phi = \frac{5}{3}x^3 - 5xy^2 + f_1(y) \qquad (2)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -10xy \qquad (3)$$

and

$$\int d\phi = -\int 10xy\,dy$$

or

$$\phi = -5xy^2 + f_2(x) \qquad (4)$$

To satisfy both Eqs. (2) and (4)

$$\underline{\underline{\phi = \left(\frac{5}{3}\right)x^3 - 5xy^2 + C}}$$

where C is an arbitrary constant.

6.29

6.29 Determine the stream function corresponding to the velocity potential

$$\phi = x^3 - 3xy^2$$

Sketch the streamline $\psi = 0$, which passes through the origin.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 3x^2 - 3y^2$$

Integrate with respect to y to obtain

$$\int d\psi = \int (3x^2 - 3y^2)\, dy$$

or

$$\psi = 3\left(x^2 y - \frac{y^3}{3}\right) + f_1(x) \qquad (1)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -6xy$$

and integrating with respect to x yields

$$\int d\psi = \int 6xy\, dx$$

or

$$\psi = 3x^2 y + f_2(y) \qquad (2)$$

To satisfy both Eqs. (1) and (2)

$$\psi = 3x^2 y - y^3 + C$$

where C is an arbitrary constant. Since the streamline $\psi = 0$ passes through the origin $(x=0, y=0)$ it follows that $C = 0$ and

$$\underline{\underline{\psi = 3x^2 y - y^3}} \qquad (3)$$

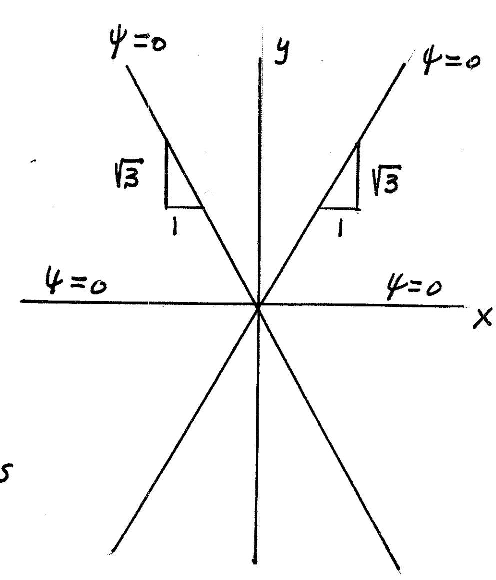
The equation of the streamline passing through the origin is found by setting $\psi = 0$ in Eq. (3) to yield

$$y(3x^2 - y^2) = 0$$

which is satisfied for $y = 0$

and

$$y = \pm\sqrt{3}\, x$$



A sketch of the $\psi = 0$ streamlines are shown in the figure.

6.30

6.30 A certain flow field is described by the stream function

$$\psi = A\,\theta + B\,r\sin\theta$$

where A and B are positive constants. Determine the corresponding velocity potential and locate any stagnation points in this flow field.

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} = \frac{A}{r} + B\cos\theta \qquad (1)$$

Integrate with respect to r to obtain

$$\int d\phi = \int\left(\frac{A}{r} + B\cos\theta\right)dr$$

or

$$\phi = A\ln r + Br\cos\theta + f_1(\theta) \qquad (2)$$

Similarly,

$$v_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -B\sin\theta \qquad (3)$$

and

$$\int d\phi = -\int B\,r\sin\theta\,d\theta$$

or

$$\phi = Br\cos\theta + f_2(r) \qquad (4)$$

To satisfy both Eqs. (2) and (4)

$$\underline{\underline{\phi = A\ln r + Br\cos\theta + C}}$$

where C is an arbitrary constant.

Stagnation points occur where $v_r = 0$ and $v_\theta = 0$. From Eq. (3) $v_\theta = 0$ at $\theta = 0$ and $\theta = \pi$. From Eq. (1) with $\theta = 0$

$$v_r = \frac{A}{r} + B$$

so that $v_r = 0$ for $r = -\frac{A}{B}$. However, since A and B are both positive constants this result indicates a negative value for r which is not defined.

At $\theta = \pi$

$$v_r = \frac{A}{r} + B\cos\pi = \frac{A}{r} - B$$

so that $v_r = 0$ for $r = \frac{A}{B}$. Thus, a stagnation point occurs at

$$\underline{\underline{\theta = \pi \quad \text{and} \quad r = \frac{A}{B}}}$$

6.31

6.31 It is known that the velocity distribution for two-dimensional flow of a viscous fluid between wide parallel plates (Fig. P6.31) is parabolic; that is

$$u = U_c\left[1 - \left(\frac{y}{h}\right)^2\right]$$

with $v = 0$. Determine, if possible, the corresponding stream function and velocity potential.

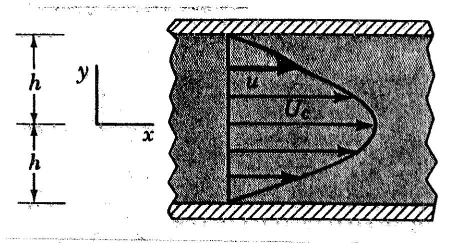


FIGURE P6.31

To determine the stream function let

$$u = \frac{\partial \psi}{\partial y} = U_c\left[1 - \left(\frac{y}{h}\right)^2\right]$$

and integrate with respect to y to obtain

$$\int d\psi = \int U_c\left[1 - \left(\frac{y}{h}\right)^2\right] dy$$

or

$$\psi = U_c\left[y - \frac{y^3}{3h^2}\right] + f_1(x)$$

Since $v = -\frac{\partial \psi}{\partial x} = 0$, ψ is not a function of x so that

$$\psi = U_c\, y\left[1 - \frac{1}{3}\left(\frac{y}{h}\right)^2\right] + C$$

where C is an arbitrary constant.

To determine the velocity potential let

$$u = \frac{\partial \phi}{\partial x} = U_c\left[1 - \left(\frac{y}{h}\right)^2\right]$$

and integrate with respect to x to obtain

$$\int d\phi = \int U_c\left[1 - \left(\frac{y}{h}\right)^2\right] dx$$

or

$$\phi = U_c\left[x - \left(\frac{y}{h}\right)^2 x\right] + f_2(y)$$

However,

$$v = \frac{\partial \phi}{\partial y} = 0 = -\frac{2 U_c\, x\, y}{h^2} + \frac{\partial f_2(y)}{\partial y}$$

and this relationship cannot be satisfied for all values of x and y. Thus, there is not a velocity potential that describes this flow (the flow is not irrotational).

6.32

6.32 The velocity potential for a certain inviscid flow field is

$$\phi = -(3x^2y - y^3)$$

where ϕ has the units of ft^2/s when x and y are in feet. Determine the pressure difference (in psi) between the points (1, 2) and (4, 4), where the coordinates are in feet, if the fluid is water and elevation changes are negligible.

Since the flow field is described by a velocity potential the flow is irrotational and the Bernoulli equation can be applied between any two points. Thus,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \qquad (1)$$

Also,

$$u = \frac{\partial \phi}{\partial x} = -6xy \qquad v = \frac{\partial \phi}{\partial y} = -3x^2 + 3y^2$$

At $x = 1\ ft$, $y = 2\ ft$

$$u_1 = -6(1)(2) = -12\ \frac{ft}{s}$$

$$v_1 = -3(1)^2 + 3(2)^2 = 9\ \frac{ft}{s}$$

So that $V_1^2 = u_1^2 + v_1^2 = \left(-12\ \frac{ft}{s}\right)^2 + \left(9\ \frac{ft}{s}\right)^2 = 225\left(\frac{ft}{s}\right)^2$

At $x = 4\ ft$, $y = 4\ ft$

$$u_2 = -6(4)(4) = -96\ \frac{ft}{s}$$

$$v_2 = -3(4)^2 + 3(4)^2 = 0$$

So that $V_2^2 = \left(-96\ \frac{ft}{s}\right)^2$

Thus, from Eq. (1)

$$p_1 - p_2 = \frac{1}{2}\frac{\gamma}{g}\left[V_2^2 - V_1^2\right]$$

$$= \frac{1}{2}\frac{\left(62.4\ \frac{lb}{ft^3}\right)}{\left(32.2\ \frac{ft}{s^2}\right)}\left[\left(-96\ \frac{ft}{s}\right)^2 - 225\left(\frac{ft}{s}\right)^2\right]$$

$$= 8710\ \frac{lb}{ft^2} = \left(8710\ \frac{lb}{ft^2}\right)\left(\frac{ft^2}{144\ in.^2}\right) = \underline{\underline{60.5\ psi}}$$

6.33

6.33 The velocity potential for a flow is given by

$$\phi = \frac{a}{2}(x^2 - y^2)$$

where a is a constant. Determine the corresponding stream function and sketch the flow pattern.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = ax$$

To determine ψ integrate with respect to y to obtain

$$\int d\psi = \int ax\,dy$$

or

$$\psi = axy + f_1(x) \qquad (1)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -ay$$

so that

$$\int d\psi = \int ay\,dx$$

or

$$\psi = axy + f_2(y) \qquad (2)$$

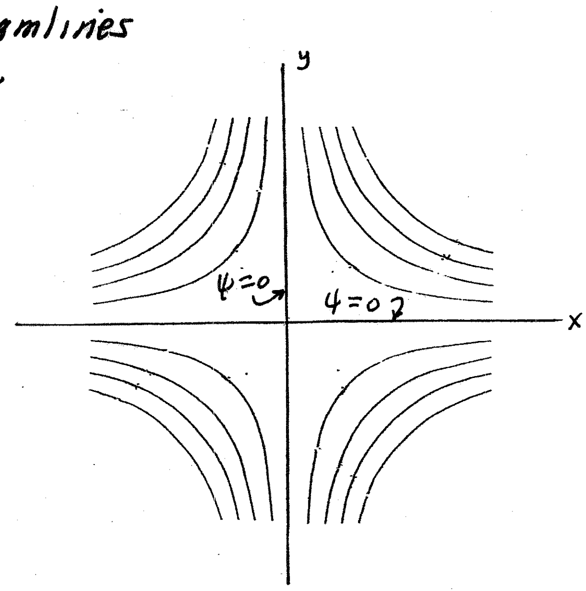
To satisfy both Eqs. (1) and (2)

$$\underline{\underline{\psi = axy + C}}$$

where C is an arbitrary constant. Let $C=0$ so that

$$\frac{\psi}{a} = xy \qquad (3)$$

For a given a the streamline pattern is obtained by setting ψ equal to various constants. For $\psi = 0$ the x and y axes are streamlines and for other values of ψ the streamlines are rectangular hyperbolas as shown in the sketch.



6.34

6.34 The stream function for a two-dimensional, nonviscous, incompressible flow field is given by the expression

$$\psi = -2(x - y)$$

where the stream function has the units of ft^2/s with x and y in feet. **(a)** Is the continuity equation satisfied? **(b)** Is the flow field irrotational? If so, determine the corresponding velocity potential. **(c)** Determine the pressure gradient in the horizontal x direction at the point $x = 2$ ft, $y = 2$ ft.

(a) To satisfy the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

For the stream function given,

$$u = \frac{\partial \psi}{\partial y} = 2\ \frac{ft}{s} \qquad v = -\frac{\partial \psi}{\partial x} = 2\ \frac{ft}{s}$$

so that

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 0$$

and the continuity equation is satisfied. Yes.

(Note: When a flow field is defined by a stream function the continuity equation is always identically satisfied.)

(b) Since

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.12)}$$

and

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

it follows that $\omega_z = 0$ and the flow field is irrotational. Yes.

Thus,

$$u = \frac{\partial \phi}{\partial x} = 2 \qquad v = \frac{\partial \phi}{\partial y} = 2$$

and integration yields

$$\phi = 2(x + y) + C$$

where C is an arbitrary constant.

(c) With the x-axis horizontal, $g_x = 0$, and

$$-\frac{\partial p}{\partial x} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.51a)}$$

and at $x = 2$ ft, $y = 2$ ft

$$\frac{\partial p}{\partial x} = -\rho\left[2\ \frac{ft}{s}(0) + 2\ \frac{ft}{s}(0)\right] = 0$$

6.35

6.35 The velocity potential for a flow is given by

$$\phi = U_\infty x + c\left(\cos\frac{2\pi x}{\ell}\right)e^{-\left(\frac{2\pi}{\ell}y\right)}$$

Determine the stream function for this flow.

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = U_\infty + c\left(\sin\frac{2\pi}{\ell}x\right)\left(-\frac{2\pi}{\ell}\right)e^{-\left(\frac{2\pi}{\ell}y\right)}$$

To determine ψ integrate with respect to y to obtain

$$\int d\psi = \int\left[U_\infty + c\left(\sin\frac{2\pi}{\ell}x\right)\left(-\frac{2\pi}{\ell}\right)e^{-\frac{2\pi}{\ell}y}\right]dy$$

or

$$\psi = U_\infty y + c\left(\sin\frac{2\pi}{\ell}x\right)e^{-\left(\frac{2\pi}{\ell}y\right)} + f_1(x) \qquad (1)$$

Similarly,

$$v = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} = c\left(\cos\frac{2\pi}{\ell}x\right)\left(-\frac{2\pi}{\ell}\right)e^{-\left(\frac{2\pi}{\ell}y\right)}$$

so that

$$\int d\psi = -\int c\left(\cos\frac{2\pi}{\ell}x\right)\left(-\frac{2\pi}{\ell}\right)e^{-\left(\frac{2\pi}{\ell}y\right)}dx$$

or

$$\psi = c\left(\sin\frac{2\pi}{\ell}x\right)e^{-\left(\frac{2\pi}{\ell}y\right)} + f_2(y) \qquad (2)$$

To satisfy both Eqs. (1) and (2)

$$\underline{\underline{\psi = U_\infty y + c\left(\sin\frac{2\pi}{\ell}x\right)e^{-\left(\frac{2\pi}{\ell}y\right)}}}$$

6.36

6.36 The velocity potential for a certain inviscid, incompressible flow field is given by the equation

$$\phi = 2x^2y - (\tfrac{2}{3})y^3$$

where ϕ has the units of m^2/s when x and y are in meters. Determine the pressure at the point $x = 2$ m, $y = 2$ m if the pressure at $x = 1$ m, $y = 1$ m is 200 kPa. Elevation changes can be neglected and the fluid is water.

Since the flow is irrotational,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \qquad (1)$$

with $V^2 = u^2 + v^2$. For the velocity potential given,

$$u = \frac{\partial \phi}{\partial x} = 4xy \qquad v = \frac{\partial \phi}{\partial y} = 2x^2 - 2y^2$$

At point 1 let $x = 1\,m$ and $y = 1\,m$ so that

$$u_1 = 4(1)(1) = 4\,\frac{m}{s} \qquad v_1 = 2(1)^2 - 2(1)^2 = 0$$

and

$$V_1^2 = \left(4\,\frac{m}{s}\right)^2 = 16\,\frac{m^2}{s^2}$$

At point 2 $x = 2\,m$ and $y = 2\,m$ so that

$$u_2 = 4(2)(2) = 16\,\frac{m}{s} \qquad v_2 = 2(2)^2 - 2(2)^2 = 0$$

and

$$V_2^2 = \left(16\,\frac{m}{s}\right)^2 = 256\,\frac{m^2}{s^2}$$

Thus, from Eq. (1)

$$p_2 = p_1 + \frac{\gamma}{2g}\left(V_1^2 - V_2^2\right)$$

$$= 200 \times 10^3\,\frac{N}{m^2} + \frac{\left(9.80 \times 10^3\,\frac{N}{m^3}\right)}{2\left(9.81\,\frac{m}{s^2}\right)}\left(16\,\frac{m^2}{s^2} - 256\,\frac{m^2}{s^2}\right)$$

$$= \underline{\underline{80.1\ kPa}}$$

6.37

6.37 A steady, uniform, incompressible, inviscid, two-dimensional flow makes an angle of 30° with the horizontal x axis. **(a)** Determine the velocity potential and the stream function for this flow. **(b)** Determine an expression for the pressure gradient in the vertical y direction. What is the physical interpretation of this result?

(a) From Eqs. 6.80 and 6.81

$$\phi = U(x\cos\alpha + y\sin\alpha) \qquad \text{(Eq. 6.80)}$$

and for $\alpha = 30°$

$$\phi = U(x\cos 30° + y\sin 30°) = \underline{\underline{U(0.866x + 0.500y)}}$$

Similarly,

$$\psi = U(y\cos\alpha - x\sin\alpha) \qquad \text{(Eq. 6.81)}$$

and for $\alpha = 30°$

$$\psi = U(y\cos 30° - x\sin 30°) = \underline{\underline{U(0.866y - 0.500x)}}$$

(b) Since

$$u = \frac{\partial\phi}{\partial x} \quad \text{and} \quad v = \frac{\partial\phi}{\partial y}$$

it follows that

$$u = 0.866U \quad \text{and} \quad v = 0.500U$$

From the Euler equation in the vertical y-direction

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \qquad \text{(Eq. 6.51b)}$$

and with $v =$ constant and $g_y = -g$

$$\frac{\partial p}{\partial y} = -\rho g$$

or

$$\underline{\underline{\frac{\partial p}{\partial y} = -\gamma}}$$

This result indicates that the pressure distribution is hydrostatic. This is not a surprising result since the Bernoulli equation indicates that if there is no change in velocity the change in pressure is simply due to the weight of the fluid, i.e., a hydrostatic variation.

6.38

6.38 The streamlines for an incompressible, inviscid, two-dimensional flow field are all concentric circles and the velocity varies directly with the distance from the common center of the streamlines; that is

$$v_\theta = Kr$$

where K is a constant. **(a)** For this *rotational* flow determine, if possible, the stream function. **(b)** Can the pressure difference between the origin and any other point be determined from the Bernoulli equation? Explain.

(a) $$v_\theta = -\frac{\partial \psi}{\partial r} = Kr \qquad (1)$$

Integrate Eq. (1) with respect to r to obtain

$$\int d\psi = -\int Kr\, dr$$

or

$$\psi = -\frac{Kr^2}{2} + f_1(\theta)$$

Since

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

it follows that ψ is not a function of θ and therefore

$$\underline{\underline{\psi = -\frac{Kr^2}{2} + C}}$$

where C is an arbitrary constant.

(b) The flow is rotational and therefore the Bernoulli equation cannot be applied between the origin and any point, since these points are not on the same streamline. <u>No.</u>
(Refer to discussion associated with derivation of Eq. 6.57.)

6.39

6.39 The velocity potential

$$\phi = -k(x^2 - y^2) \qquad (k = \text{constant})$$

may be used to represent the flow against an infinite plane boundary as illustrated in Fig. P6.39. For flow in the vicinity of a stagnation point it is frequently assumed that the pressure gradient along the surface is of the form

$$\frac{\partial p}{\partial x} = Ax$$

where A is a constant. Use the given velocity potential to show that this is true.

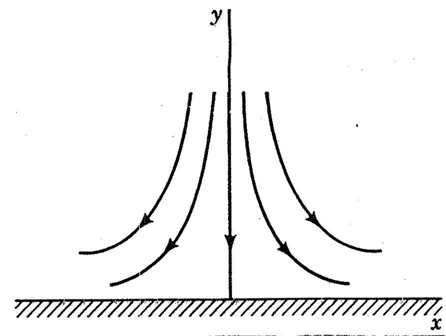


FIGURE P6.39

For the velocity potential given

$$u = \frac{\partial \phi}{\partial x} = -2kx \qquad (1)$$

$$v = \frac{\partial \phi}{\partial y} = -2ky \qquad (2)$$

and the stagnation point occurs at the origin.
For this steady, two-dimensional flow

$$-\frac{\partial p}{\partial x} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.51a)}$$

and along the surface ($y=0$) $v=0$ so that

$$\frac{\partial p}{\partial x} = \rho u \frac{\partial u}{\partial x} \qquad (3)$$

From Eq. (1) $u = -2kx$ and therefore

$$\frac{\partial u}{\partial x} = -2k$$

and Eq. (3) becomes

$$\frac{\partial p}{\partial x} = \rho(-2kx)(-2k) = 4k^2 x$$

or

$$\underline{\underline{\frac{\partial p}{\partial x} = Ax}}$$

where $A = 4k^2$.

6.40

6.40 Water is flowing between wedge shaped walls into a small opening as shown in Fig. P6.40. The velocity potential with units m^2/s for this flow is $\phi = -2 \ln r$ with r in meters. Determine the pressure differential between points A and B.

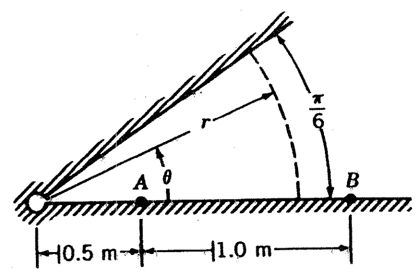


FIGURE P6.40

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} \qquad (1)$$

Along the horizontal surface , $v_\theta = 0$, and

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r}$$

so that

$$V = v_r = -\frac{2}{r}$$

Thus,

$$V_A = -\frac{2}{0.5} = -4\ \frac{m}{s} \qquad V_B = -\frac{2}{1.5} = -\frac{4}{3}\ \frac{m}{s}$$

and from Eq.(1)

$$p_A - p_B = \frac{\gamma}{2g}\left[V_B^2 - V_A^2\right]$$

$$= \frac{9.80 \times 10^3\ \frac{N}{m^3}}{2\left(9.81\ \frac{m}{s^2}\right)}\left[\left(-\frac{4}{3}\ \frac{m}{s}\right)^2 - \left(-4\ \frac{m}{s}\right)^2\right]$$

$$= \underline{\underline{-7.10\ kPa}}$$

6.41

6.41 An ideal fluid flows between the inclined walls of a two-dimensional channel into a sink located at the origin (Fig. P6.41). The velocity potential for this flow field is

$$\phi = \frac{m}{2\pi} \ln r$$

where m is a constant. **(a)** Determine the corresponding stream function. Note that the value of the stream function along the wall OA is zero. **(b)** Determine the equation of the streamline passing through the point B, located at $x = 1$, $y = 4$.

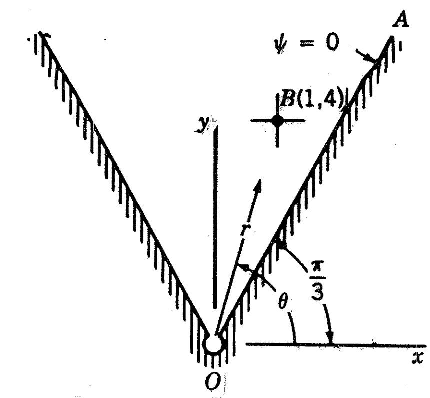


FIGURE P6.41

(a) $$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \qquad (1)$$

Integrate Eq. (1) with respect to θ to obtain

$$\int d\psi = \int \frac{m}{2\pi} d\theta$$

or

$$\psi = \frac{m\theta}{2\pi} + f_1(r)$$

Since

$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \qquad (2)$$

ψ is not a function of r so Eq. (2) becomes

$$\psi = \frac{m\theta}{2\pi} + C$$

Where C is a constant. Also, $\psi = 0$ for $\theta = \frac{\pi}{3}$ so that

$$C = -\frac{m}{6}$$

and

$$\underline{\underline{\psi = m\left(\frac{\theta}{2\pi} - \frac{1}{6}\right)}} \qquad (3)$$

(b) At B $\tan\theta = \frac{4}{1}$ so that $\theta = 1.33$ rad. From Eq. (3) the value of ψ passing through this point is

$$\psi = m\left(\frac{1.33}{2\pi} - \frac{1}{6}\right) = 0.0450\,m$$

and therefore the equation of the streamline passing through B is

$$0.0450\,m = m\left(\frac{\theta}{2\pi} - \frac{1}{6}\right)$$

or

$$\underline{\underline{\theta = 1.33 \text{ rad}}}$$

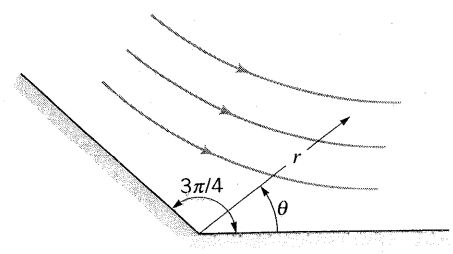
(Note: It can be seen from Eq. (3) that the streamlines are all straight lines passing through the origin.)

6.42

6.42 It is suggested that the velocity potential for the flow of an incompressible, nonviscous, two-dimensional flow along the wall shown in Fig. P6.42 is

$$\phi = r^{4/3} \cos \tfrac{4}{3}\theta$$

Is this a suitable velocity potential for flow along the wall? Explain.



■ FIGURE P6.42

If this is a suitable ϕ the corresponding ψ must have a constant value along the wall (since the wall must correspond to a streamline).

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = \frac{4}{3} r^{1/3} \cos \frac{4}{3}\theta \qquad (1)$$

Integrate Eq. (1) with respect to θ to obtain

$$\int d\psi = \int \frac{4}{3} r^{4/3} \cos \frac{4}{3}\theta$$

or

$$\psi = r^{4/3} \sin \frac{4}{3}\theta + f_1(r) \qquad (2)$$

Similarly,

$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{4}{3} r^{1/3} \sin \frac{4}{3}\theta$$

and

$$\int d\psi = \int \frac{4}{3} r^{1/3} \sin \frac{4}{3}\theta \, dr$$

or

$$\psi = r^{\frac{4}{3}} \sin \frac{4}{3}\theta + f_2(\theta) \qquad (3)$$

To satisfy both Eqs. (2) and (3)

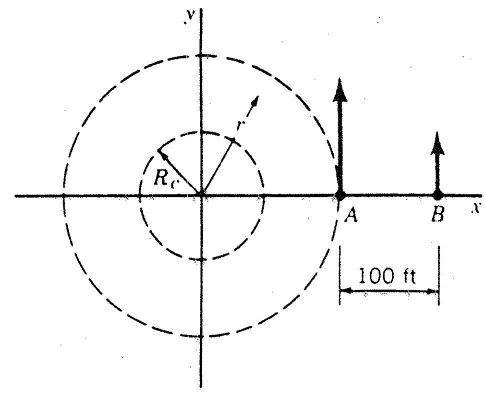
$$\psi = r^{4/3} \sin \frac{4}{3}\theta + C$$

where C is an arbitrary constant.

Along one section of the wall, $\theta = 0$, and $\psi = C$. Along the other section $\theta = \frac{3\pi}{4}$ and $\psi = C$. Thus, ψ has a constant value along the wall and the given velocity potential can be used to represent flow along the wall. Yes.

6.43

6.43 As illustrated in Fig. P6.43 a tornado can be approximated by a free vortex of strength Γ for $r > R_c$, where R_c is the radius of the core. Velocity measurements at points A and B indicate that $V_A = 125$ ft/s and $V_B = 60$ ft/s. Determine the distance from point A to the center of the tornado. Why can the free vortex model not be used to approximate the tornado throughout the flow field ($r \geq 0$)?



■ FIGURE P6.43

For a free vortex

$$v_\theta = \frac{K}{r} \qquad \text{(Eq. 6.86)}$$

Thus, at r_A, $v_\theta = 125\ \frac{ft}{s}$, so that $K = 125\, r_A$

and at r_B, $v_\theta = 60\ \frac{ft}{s}$, so that $K = 60\, r_B$.

Therefore,

$$125\, r_A = 60\, r_B$$

and since

$$r_B - r_A = 100\ ft$$

it follows that

$$125\, r_A = 60\,(100 + r_A)$$

or

$$\underline{\underline{r_A = 92.3\ ft}}$$

The free vortex cannot be used to approximate a tornado throughout the flow field since at $r=0$ the velocity becomes infinite.

6.44

6.44 The motion of a liquid in an open tank is that of a combined vortex consisting of a forced vortex for $0 \le r \le 2$ ft and a free vortex for $r > 2$ ft. The velocity profile and the corresponding shape of the free surface are shown in Fig. P6.44. The free surface at the center of the tank is a depth h below the free surface at $r = \infty$. Determine the value of h. Note that $h = h_{forced} + h_{free}$, where h_{forced} and h_{free} are the corresponding depths for the forced vortex and the free vortex, respectively. (See Section 2.12.2 for further discussion regarding the forced vortex.)

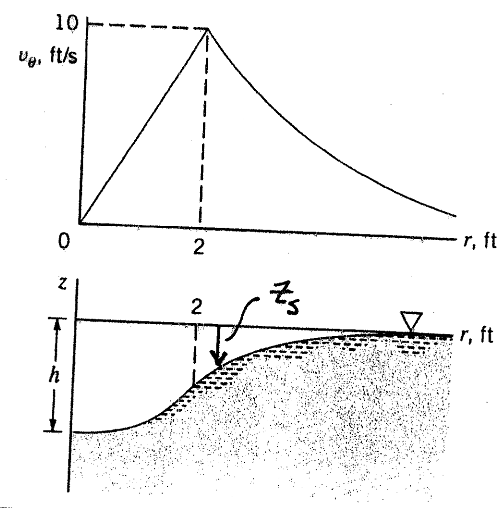


FIGURE P6.44

For forced vortex

$$z = \frac{\omega^2 r^2}{2g} + C \qquad \text{(Eq. 2.32)}$$

and with $z = 0$ at $r = 0$ it follows That $C = 0$.

Also, $v_\theta = r\omega$ and since $v_\theta = 10$ ft/s at $r = 2$ ft

$$\omega = \frac{10 \frac{ft}{s}}{2\ ft} = 5 \frac{rad}{s}$$

Thus, at $r = 2$ ft

$$z = \frac{\omega^2 r^2}{2g} = \frac{(5 \frac{rad}{s})^2 (2\ ft)^2}{2\ (32.2 \frac{ft}{s^2})} = 1.55\ ft$$

For free vortex (See Example 6.6)

$$z_s = \frac{\Gamma^2}{8\pi^2 r^2 g}$$

where $\Gamma = 2\pi r v_\theta$

so that

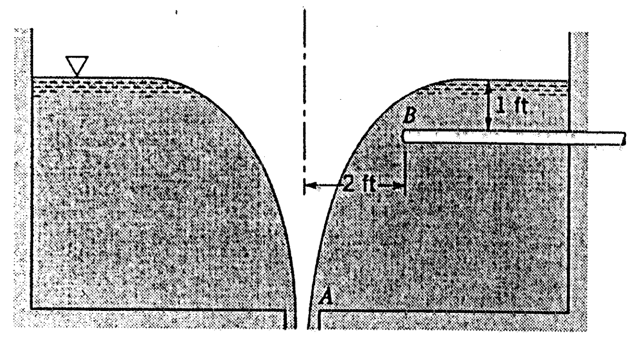
$$z_s = \frac{(2\pi r v_\theta)^2}{8\pi^2 r^2 g} = \frac{4\pi^2 (2 ft)^2 (10 \frac{ft}{s})^2}{8\pi^2 (2 ft)^2 (32.2 \frac{ft}{s^2})} = 1.55\ ft$$

Thus,

$$h = h_{forced} + h_{free} = 1.55\ ft + 1.55\ ft = \underline{\underline{3.10\ ft}}$$

6.45

6.45 When water discharges from a tank through an opening in its bottom, a vortex may form with a curved surface profile as shown in Fig. P6.45 and **Video V6.2**. Assume that the velocity distribution in the vortex is the same as that for a free vortex. At the same time the water is being discharged from the tank at point *A* it is desired to discharge a small quantity of water through the pipe *B*. As the discharge through *A* is increased, the strength of the vortex, as indicated by its circulation, is increased. Determine the maximum strength that the vortex can have in order that no air is sucked in at *B*. Express your answer in terms of the circulation. Assume that the fluid level in the tank at a large distance from the opening at *A* remains constant and viscous effects are negligible.



■ FIGURE P6.45

From Example 6.6,

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$

Air will be sucked into pipe when $z_s = -1\ ft$ for $r = 2\ ft$.

Thus,

$$\Gamma^2 = -8\pi^2 r^2 g\, z_s = -8\pi^2 (2\,ft)^2 \left(32.2\,\frac{ft}{s^2}\right)(-1\,ft)$$

or

$$|\Gamma| = \underline{\underline{101\ \frac{ft^2}{s}}}$$

6.46

6.46 The streamlines in a particular two-dimensional flow field are all concentric circles, as shown in Fig. P6.46. The velocity is given by the equation $v_\theta = \omega r$ where ω is the angular velocity of the rotating mass of fluid. Determine the circulation around the path $ABCD$.

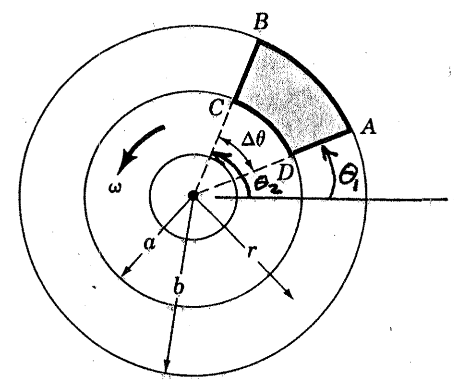


FIGURE P6.46

$$\Gamma = \oint_{ABCD} \vec{V} \cdot d\vec{s}$$

$$= \int_{AB} v_\theta \, b \, d\theta + \int_{BC} v_r \, dr + \int_{CD} v_\theta \, a \, d\theta + \int_{DA} v_r \, dr \qquad (1)$$

Since $v_r = 0$ and $v_\theta = \omega r$, Eq. (1) becomes

$$\Gamma = \int_{\theta_1}^{\theta_2} \omega b^2 \, d\theta + 0 + \int_{\theta_2}^{\theta_1} \omega a^2 \, d\theta + 0$$

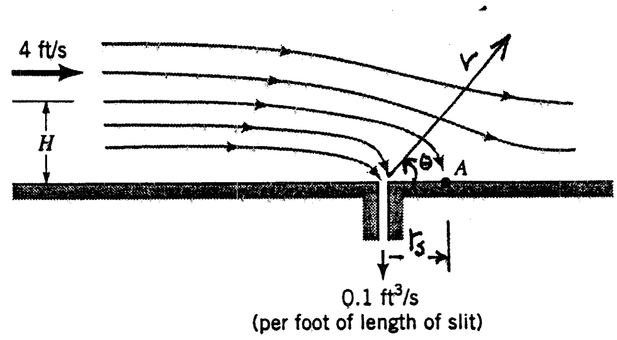
$$= \omega b^2 (\theta_2 - \theta_1) + \omega a^2 (\theta_1 - \theta_2)$$

or

$$\Gamma = \omega (\theta_2 - \theta_1)(b^2 - a^2) = \underline{\underline{\omega \Delta\theta (b^2 - a^2)}}$$

6.47

6.47 Water flows over a flat surface at 4 ft/s as shown in Fig. P6.47. A pump draws off water through a narrow slit at a volume rate of 0.1 ft^3/s per foot length of the slit. Assume that the fluid is incompressible and inviscid and can be represented by the combination of a uniform flow and a sink. Locate the stagnation point on the wall (point A) and determine the equation for the stagnation streamline. How far above the surface, H, must the fluid be so that it does not get sucked into the slit?



■ FIGURE P6.47

$$\psi = \psi_{\substack{uniform\\flow}} + \psi_{sink} = U r \sin\theta - \frac{m}{2\pi}\theta \qquad (1)$$

Thus,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta - \frac{m}{2\pi r} \qquad (2)$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U\sin\theta$$

Along the wall $v_\theta = 0$, and the stagnation point occurs where $v_r = 0$, so that from Eq. (2)

$$0 = U\cos(0°) - \frac{m}{2\pi r_s}$$

and therefore

$$r_s = \frac{m}{2\pi U}$$

For $U = 4\,\frac{ft}{s}$ and $m = 0.2\,\frac{ft^2}{s}$ (note that a source strength of $0.2\,\frac{ft^2}{s}$ must be used to obtain $0.1\,\frac{ft^2}{s}$ through slit which is only one half of a "full" sink). Thus,

$$r_s = \frac{0.2\,\frac{ft^2}{s}}{2\pi\left(4\,\frac{ft}{s}\right)} = 0.00796\ ft$$

and the stagnation point is on the wall <u>0.00796 ft to the right of slit.</u>

(con't)

6.47 (con't)

The value of ψ at the stagnation point ($r = 0.00796\ ft$, $\theta = 0°$) is zero (Eq. 1) so that the equation of the stagnation streamline is

$$0 = U r \sin\theta - \frac{m}{2\pi}\theta$$

or

$$r \sin\theta = \frac{m}{2\pi U}\theta$$

Since $y = r\sin\theta$ the equation of the stagnation streamline can be written as

$$\underline{\underline{y = \frac{m}{2\pi U}\theta}}$$

Fluid above the stagnation streamline will not be sucked into slit. The maximum distance, H, for the stagnation streamline occurs as $\theta \to \pi$ so that

$$H = \frac{m\pi}{2\pi U} = \frac{0.2\ \frac{ft^2}{s}}{2\left(4\ \frac{ft}{s}\right)} = \underline{\underline{0.0250\ ft}}$$

(Note: All the fluid below the stagnation streamline must pass through the slit. Thus, from conservation of mass

$$HU = \text{flow into slit}$$

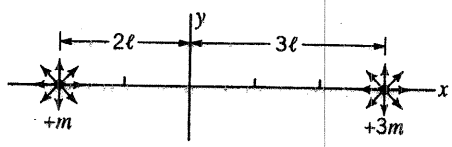
or

$$H = \frac{0.1\ \frac{ft^2}{s}}{4\ \frac{ft}{s}} = 0.0250\ ft$$

which checks with the answer above.)

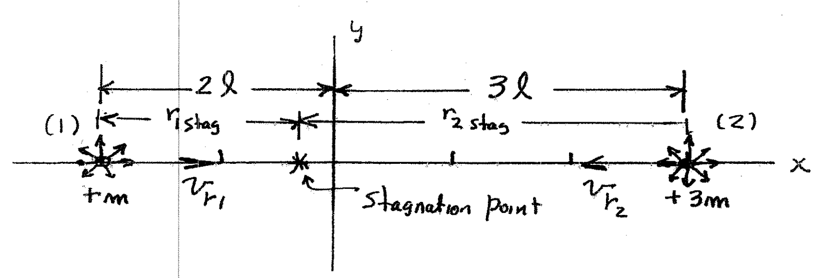
6.48

6.48 Two sources, one of strength m and the other with strength $3m$, are located on the x axis as shown in Fig. P6.48. Determine the location of the stagnation point in the flow produced by these sources.



■ **FIGURE P6.48**

Since the flow from each source is in the radial direction, it is only along the x-axis that the two radial components can cancel and create a stagnation point.



For source (1)

$$v_{r_1} = \frac{m}{2\pi r_1}$$

and for source (2)

$$v_{r_2} = \frac{3m}{2\pi r_2}$$

The stagnation point occurs where $v_{r_1} = v_{r_2}$ so that

$$\frac{m}{2\pi\, r_{1\,stag}} = \frac{3m}{2\pi\, r_{2\,stag}}$$

and

$$\frac{r_{2\,stag}}{r_{1\,stag}} = 3$$

Also,

$$r_{1\,stag} + r_{2\,stag} = 2\ell + 3\ell = 5\ell$$

so that

$$r_{1\,stag} + 3r_{1\,stag} = 5\ell$$

$$r_{1\,stag} = \frac{5}{4}\ell$$

Thus,

$$x_{stag} = -\left(2\ell - \frac{5}{4}\ell\right) = \underline{\underline{-0.75\ell}}$$

6.49

6.49 The velocity potential for a spiral vortex flow is given by $\phi = (\Gamma/2\pi)\,\theta - (m/2\pi)\ln r$, where Γ and m are constants. Show that the angle, α, between the velocity vector and the radial direction is constant throughout the flow field (see Fig. P6.49).

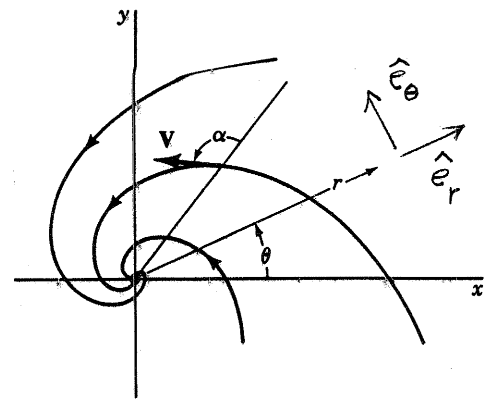


FIGURE P6.49

For the velocity potential given,

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{m}{2\pi r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

Since $\quad \vec{V}\cdot\hat{e}_r = |\vec{V}|\cos\alpha$

and $\quad \vec{V} = v_r\hat{e}_r + v_\theta\hat{e}_\theta$

then

$$\cos\alpha = \frac{\vec{V}\cdot\hat{e}_r}{|\vec{V}|} = \frac{v_r}{\sqrt{v_r^2+v_\theta^2}}$$

$$= \frac{1}{\sqrt{1+\left(\frac{v_\theta}{v_r}\right)^2}} = \frac{1}{\sqrt{1+\dfrac{\left(\frac{\Gamma}{2\pi r}\right)^2}{\left(-\frac{m}{2\pi r}\right)^2}}}$$

$$= \frac{1}{\sqrt{1+\left(\frac{\Gamma}{m}\right)^2}}$$

Thus, for a given Γ and m the angle α is a constant.

6.50

6.50 For a free vortex (see Video V6.2) determine an expression for the pressure gradient (a) along a streamline, and (b) normal to a streamline. Assume the streamline is in a horizontal plane, and express your answer in terms of the circulation.

For a free vortex

$$\phi = -\frac{\Gamma}{2\pi} \ln r \qquad \text{(Eq. 6.91)}$$

So that

$$v_r = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \qquad v_\theta = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Since the free vortex represents an irrotational flow field, the Bernoulli equation

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = \text{constant} \qquad (1)$$

is valid between any two points.

(a) Along a streamline ($r =$ constant), v_θ is constant and $v_r = 0$ so that from Eq. (1) with z constant the pressure is constant, i.e.,

$$\underline{\underline{\frac{\partial p}{\partial \theta} = 0}}$$

(b) Normal to the streamline with $v_r = 0$ and $z =$ constant

$$\frac{p}{\gamma} + \frac{v_\theta^2}{2g} + z = \text{constant}$$

So that

$$\frac{\partial p}{\partial r} = -\frac{\gamma}{2g}\frac{\partial (v_\theta^2)}{\partial r} = -\rho v_\theta \frac{\partial v_\theta}{\partial r}$$

$$= -\rho\left(\frac{\Gamma}{2\pi r}\right)\left(-\frac{\Gamma}{2\pi r^2}\right)$$

$$= \underline{\underline{\frac{\rho \Gamma^2}{4\pi^2 r^3}}}$$

6.51

6.51 Potential flow against a flat plate (Fig. P6.51*a*) can be described with the stream function

$$\psi = Axy$$

where A is a constant. This type of flow is commonly called a "stagnation point" flow since it can be used to describe the flow in the vicinity of the stagnation point at O. By adding a source of strength, m, at O, stagnation point flow against a flat plate with a "bump" is obtained as illustrated in Fig. P6.51*b*. Determine the relationship between the bump height, h, the constant, A; and the source strength, m.

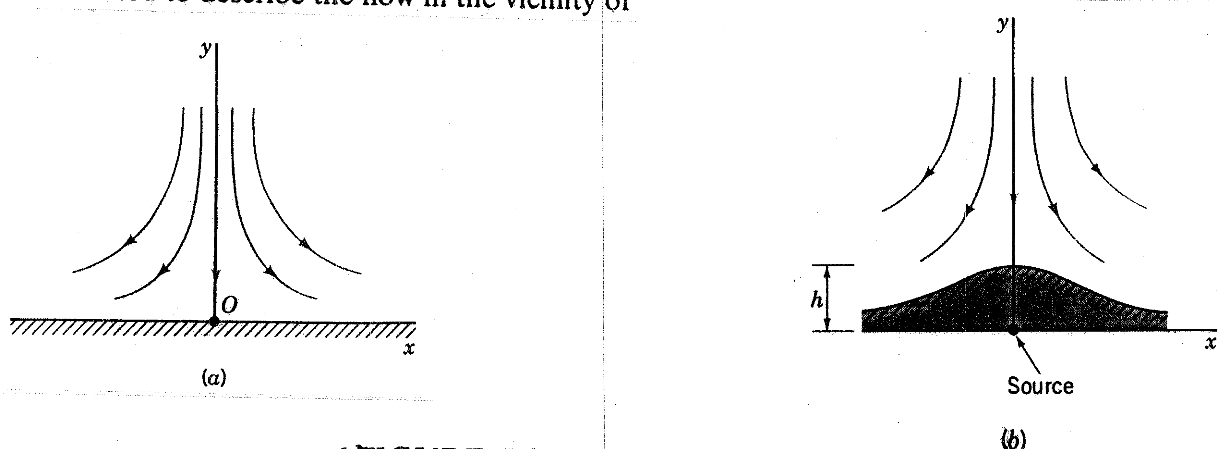


FIGURE P6.51

$$\psi = Axy + \frac{m}{2\pi}\theta = \frac{A}{2}r^2 \sin 2\theta + \frac{m}{2\pi}\theta$$

For the bump the stagnation point will occur at $x=0$, $y=h$ $(\theta = \frac{\pi}{2}, r=h)$. For the given stream function,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = Ar\cos 2\theta + \frac{m}{2\pi r} \qquad (1)$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = Ar \sin 2\theta$$

The point, $\theta = \frac{\pi}{2}$, $r=h$, will be a stagnation point if $v_r = 0$ since $v_\theta = 0$ at this point. Thus, from Eq. (1)

$$0 = Ah\cos \pi + \frac{m}{2\pi h}$$

or

$$Ah = \frac{m}{2\pi h}$$

and therefore

$$\underline{\underline{h^2 = \frac{m}{2\pi A}}}$$

6.52

6.52 The combination of a uniform flow and a source can be used to describe flow around a streamlined body called a half-body. (See **Video V6.3**.) Assume that a certain body has the shape of a half-body with a thickness of 0.5 m. If this body is placed in an air stream moving at 15 m/s, what source strength is required to simulate flow around the body?

The width of half-body $= 2\pi b$ (See Fig. 6.24)

so that

$$b = \frac{(0.5m)}{2\pi}$$

From Eq. 6.99

$$b = \frac{m}{2\pi U}$$

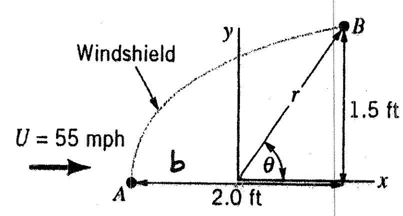
where m is the source strength, and therefore

$$m = 2\pi U b = 2\pi \left(15 \frac{m}{s}\right)\left(\frac{0.5m}{2\pi}\right)$$

$$= \underline{\underline{7.50 \frac{m^2}{s}}}$$

6.53

6.53 A vehicle windshield is to be shaped as a portion of a half-body with the dimensions shown in Fig. P6.53. **(a)** Make a scale drawing of the windshield shape. **(b)** For a free stream velocity of 55 mph, determine the velocity of the air at points A and B.



■ **FIGURE P6.53**

(a) From the figure

$$b + r\cos\theta = 2\text{ ft} \qquad (1)$$

$$r\sin\theta = 1.5\text{ft} \qquad (2)$$

and for a half-body

$$r = \frac{b(\pi - \theta)}{\sin\theta} \qquad (\text{Eq. } 6.100)$$

The above equations can be combined to give

$$\frac{1}{\pi-\theta} + \frac{1}{\tan\theta} = \frac{2}{1.5}$$

and a trial and error solution for θ gives

$$\theta = 0.839 \text{ rad} \quad (48.1°)$$

so that

$$b = \frac{r\sin\theta}{\pi - \theta} = \frac{1.5\text{ft}}{\pi - 0.839\text{ rad}} = 0.651\text{ ft}$$

Thus,

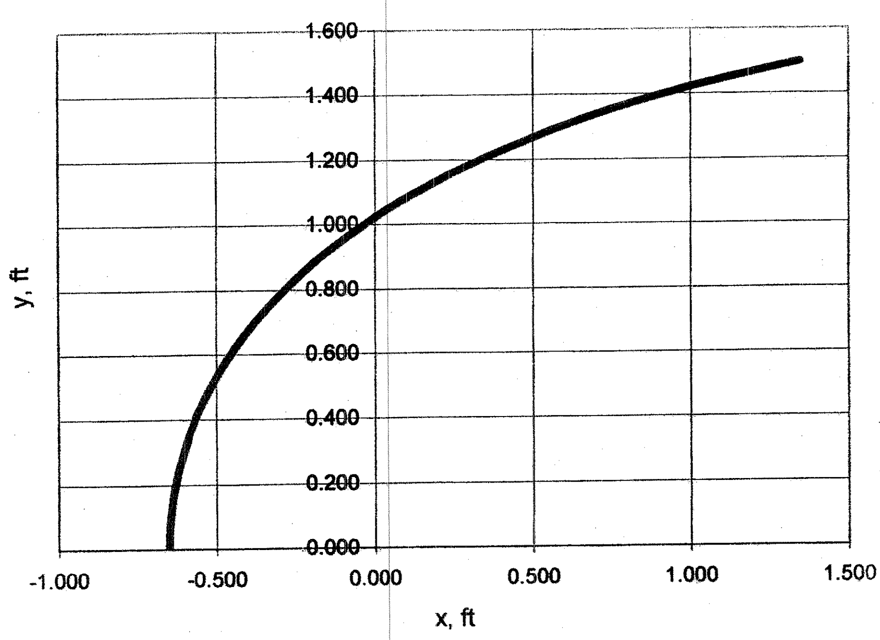
$$r = \frac{0.651\text{ ft}\,(\pi - \theta)}{\sin\theta} \qquad (3)$$

Equation (3) gives the profile of the windshield and with $x = r\cos\theta$ and $y = r\sin\theta$ the x and y coordinates can be obtained. Tabulated data and a plot of the data follows.

(con't)

6.53 (Con't)

Theta, rad	r, ft	x, ft	y, ft
3.142	0.651	-0.651	0.000
3.042	0.652	-0.649	0.065
2.942	0.655	-0.642	0.130
2.842	0.661	-0.631	0.195
2.742	0.669	-0.616	0.260
2.642	0.679	-0.596	0.326
2.542	0.692	-0.571	0.391
2.442	0.707	-0.541	0.456
2.342	0.726	-0.506	0.521
2.242	0.748	-0.465	0.586
2.142	0.774	-0.418	0.651
2.042	0.804	-0.364	0.716
1.942	0.838	-0.304	0.781
1.842	0.878	-0.235	0.846
1.742	0.925	-0.157	0.911
1.642	0.979	-0.069	0.977
1.542	1.042	0.030	1.042
1.442	1.116	0.144	1.107
1.342	1.203	0.273	1.172
1.242	1.307	0.423	1.237
1.142	1.432	0.596	1.302
1.042	1.584	0.800	1.367
0.942	1.771	1.042	1.432
0.839	2.015	1.346	1.499



(b) $V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$ (Eq. 6.101)

Point A is a stagnation point so that $\underline{\underline{V_A = 0}}$.

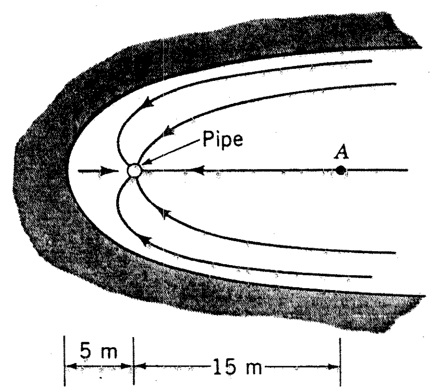
At the top of the windshield (point B) $\theta = 0.839$ rad and $r = 2.01$ ft so that

$$V_B^2 = (55\text{ mph})^2\left[1 + 2\left(\frac{0.651\text{ ft}}{2.01\text{ ft}}\right)\cos(0.839\text{ rad}) + \left(\frac{0.651\text{ ft}}{2.01\text{ ft}}\right)^2\right]$$

$V_B = \underline{\underline{68.2\text{ mph}}}$

6.54

6.54 One end of a pond has a shoreline that resembles a half-body as shown in Fig. P6.54. A vertical porous pipe is located near the end of the pond so that water can be pumped out. When water is pumped at the rate of $0.08\ m^3/s$ through a 3-m-long pipe, what will be the velocity at point A? *Hint*: Consider the flow *inside* a half-body. (See Video V6.3.)



■ FIGURE P6.54

For a half-body,

$$\psi = U r \sin\theta + \frac{m}{2\pi}\theta \qquad \text{(Eq. 6.97)}$$

so that

$$v_\theta = -\frac{\partial \psi}{\partial r} = U \sin\theta$$

and

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta + \frac{m}{2\pi r}$$

Thus, at point A, $\theta = 0$, $r = 15\,m$ and

$$v_\theta = 0$$

$$v_r = V_A = U + \frac{m}{2\pi(15)} \qquad (1)$$

For a flowrate of $0.06\frac{m^3}{s}$ in a 3-m long pipe, the source strength is $\frac{0.06}{3}\frac{m^2}{s}$. Since

$$b = \frac{m}{2\pi U} \qquad \text{(Eq. 6.99)}$$

then with $b = 5m$

$$U = \frac{m}{2\pi b} = \frac{\left(\frac{0.06}{3}\frac{m^2}{s}\right)}{2\pi(5m)} = 6.37 \times 10^{-4}\,\frac{m}{s}$$

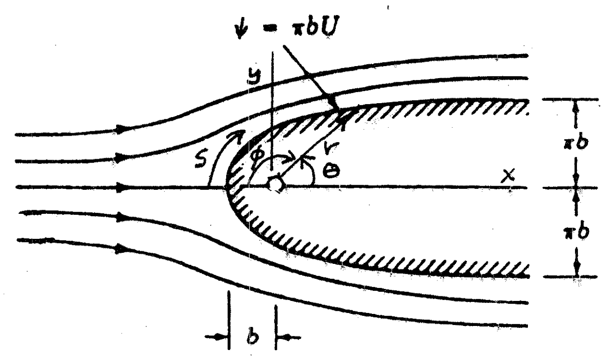
From Eq. (1)

$$V_A = 6.37 \times 10^{-4}\,\frac{m}{s} + \frac{\left(\frac{0.06}{3}\frac{m^2}{s}\right)}{2\pi\,(15m)}$$

$$= \underline{\underline{8.49 \times 10^{-4}\,\frac{m}{s}}}$$

6.55 *

6.55* For the half-body described in Section 6.6.1 show on a plot how the magnitude of the velocity on the surface, V_s, varies as a function of the distance, s (measured along the surface), from the stagnation point. Use the dimensionless variables V_s/U and s/b where U and b are defined in Fig. 6.24.



On the surface of the half-body

$$r = \frac{b(\pi-\theta)}{\sin\theta} \qquad \text{(Eq. 6.100)}$$

and

$$ds = \sqrt{(dx)^2+(dy)^2}$$

with $x = r\cos\theta$ and $y = r\sin\theta$. It follows that

$$dx = r(-\sin\theta)\,d\theta + \cos\theta\,dr$$
$$dy = r(\cos\theta)\,d\theta + \sin\theta\,dr$$

and therefore

$$ds = \sqrt{r^2(d\theta)^2+(dr)^2}$$

or

$$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Let $s^* = s/b$ and $r^* = r/b$ so that

$$ds^* = \sqrt{(r^*)^2+\left(\frac{dr^*}{d\theta}\right)^2}\,d\theta \qquad (1)$$

From Eq. 6.100

$$\frac{dr^*}{d\theta} = -\frac{\sin\theta+(\pi-\theta)\cos\theta}{\sin^2\theta} \qquad (2)$$

Thus, the arc length s^* is given by

$$s^* = \int_{\pi}^{\pi-\phi}\sqrt{(r^*)^2+\left(\frac{dr^*}{d\theta}\right)^2}\,d\theta \qquad (3)$$

for $0 \le \phi \le \pi$.

(con't)

6.55* (Con't)

The velocity, V_s, on the surface of the half-body can be obtained from Eq. 6.101 written in the form

$$V^* = \frac{V_s}{U} = \left[1 + \frac{2\cos\theta}{r^*} + \frac{1}{(r^*)^2}\right]^{1/2} \qquad (4)$$

Thus, for a given θ, r^* can be obtained from Eq. 6.100, s^* from Eq. (3), and V^* from Eq. (4). Equation (3) can be integrated using the trapezoidal rule, i.e.,

$I = \frac{1}{2}\sum_{i=1}^{m-1}(y_i + y_{i+1})(x_{i+1} - x_i)$ where $y \sim \sqrt{(r^*)^2 + \left(\frac{dr^*}{d\theta}\right)^2}$ and $x \sim \theta$. Tabulated data are given below and a plot of the data is given on the next page.

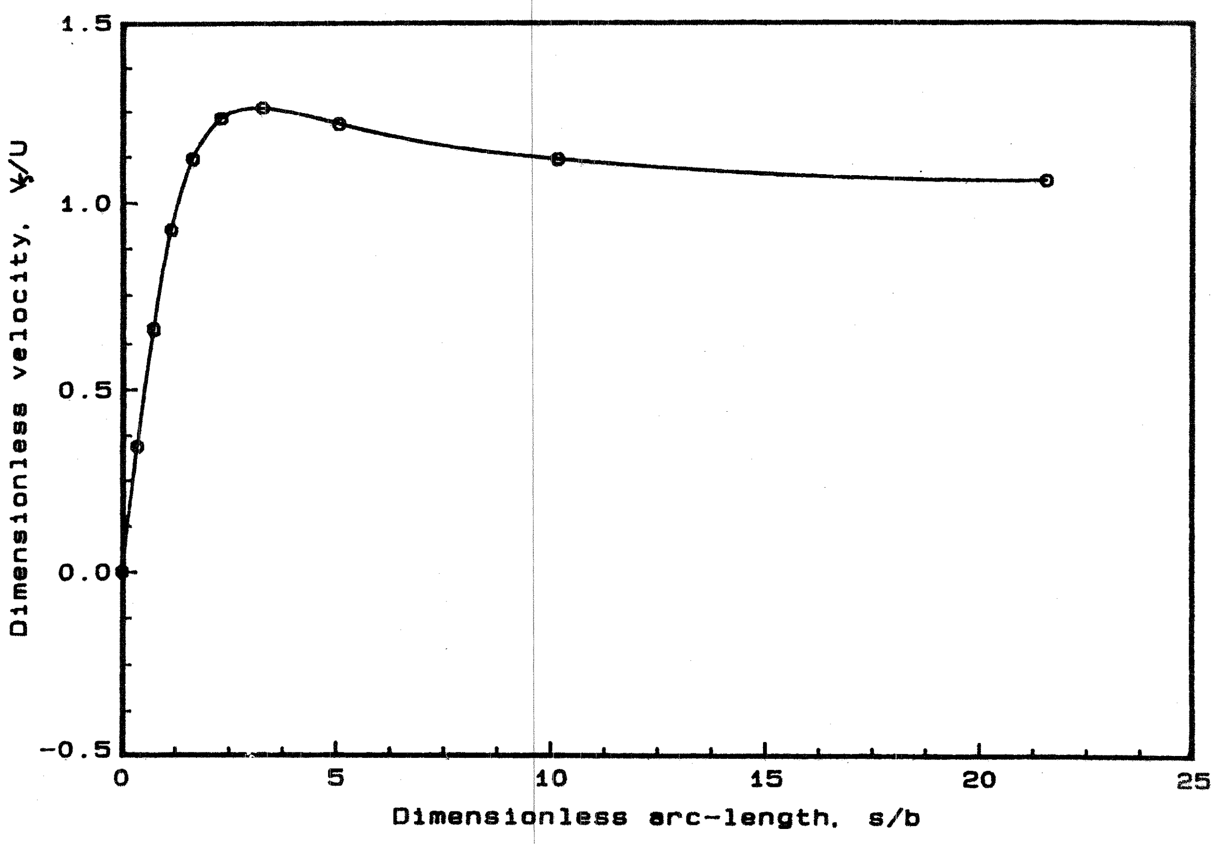
Theta, deg	Arc length, s/b	Velocity, Vs/U
180	0.000	0.000
170	0.175	0.174
160	0.353	0.344
150	0.535	0.508
140	0.725	0.661
130	0.927	0.801
120	1.144	0.926
110	1.381	1.032
100	1.646	1.119
90	1.949	1.185
80	2.305	1.231
70	2.737	1.255
60	3.281	1.259
50	4.008	1.244
40	5.054	1.213
30	6.749	1.169
20	10.142	1.116
10	21.549	1.058

Calculated from Eq.(3)

Calculated from Eq. (4)

(Con't)

6.55* (con't)



1.5
1.0
0.5
0.0
−0.5
Dimensionless velocity, V_s/U
0
5
10
15
20
25
Dimensionless arc-length, s/b

6.56

6.56 Two free vortices of equal strength, but opposite direction of rotation, are superimposed with a uniform flow as shown in Fig. P6.56. The stream functions for these two vorticies are $\psi = -[\pm\Gamma/(2\pi)] \ln r$. **(a)** Develop an equation for the x-component of velocity, u, at point $P(x,y)$ in terms of Cartesian coordinates x and y. **(b)** Compute the x-component of velocity at point A and show that it depends on the ratio Γ/H.

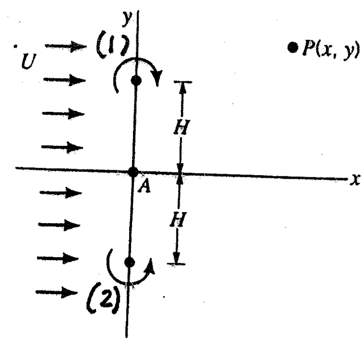


FIGURE P6.56

(a) For vortex (1), $\psi_1 = \frac{\Gamma}{2\pi} \ln r_1$

and $v_{\theta 1} = -\frac{\Gamma}{2\pi r_1}$ as shown.

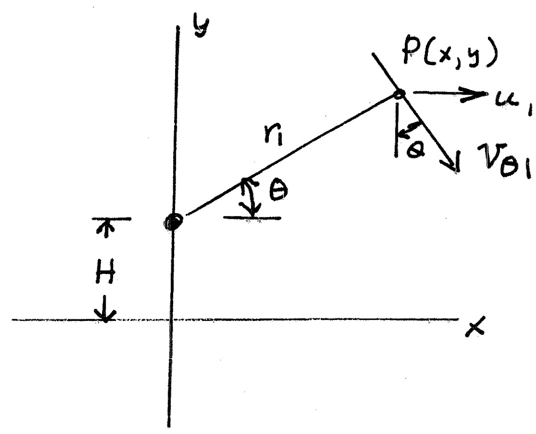
$u_1 = v_{\theta 1} \sin\theta$

where $\sin\theta = \frac{y-H}{\left[(y-H)^2 + x^2\right]^{1/2}}$

and $r_1 = \left[(y-H)^2 + x^2\right]^{1/2}$

so that

$$u_1 = \left(\frac{\Gamma}{2\pi}\right)\left(\frac{y-H}{(y-H)^2 + x^2}\right)$$



For vortex (2), $\psi_2 = -\frac{\Gamma}{2\pi} \ln r_2$

and $v_{\theta 2} = \frac{\Gamma}{2\pi r_2}$ as shown.

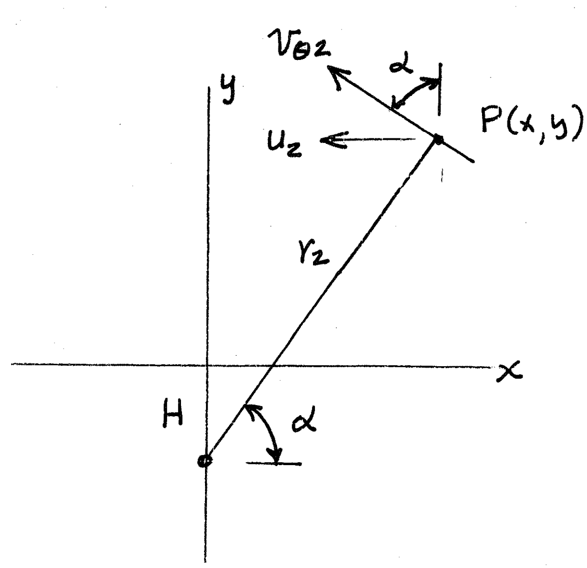
$u_2 = -v_{\theta 2} \sin\alpha$

where $\sin\alpha = \frac{y+H}{\left[(y+H)^2 + x^2\right]^{1/2}}$

and $r_2 = \left[(y+H)^2 + x^2\right]^{1/2}$

so that

$$u_2 = -\left(\frac{\Gamma}{2\pi}\right)\left(\frac{y+H}{(y+H)^2 + x^2}\right)$$



(con't)

6.56 (con't)

Thus, combining the two vortices with the uniform flow gives the x-component of velocity

$$u = u_1 + u_2 + U$$

$$= \frac{\Gamma}{2\pi}\left[\frac{y-H}{(y-H)^2+x^2} - \frac{y+H}{(y+H)^2+x^2}\right] + U \qquad (1)$$

(b) At point A where $x=y=0$, Eq. (1) gives

$$u_A = U - \frac{\Gamma}{\pi H}$$

6.57

6.57 A Rankine oval is formed by combining a source–sink pair, each having a strength of 36 ft^2/s, and separated by a distance of 12 ft along the x axis, with a uniform velocity of 10 ft/s (in the positive x direction). Determine the length and thickness of the oval.

$$\frac{\ell}{a} = \left[\frac{m}{\pi U a} + 1\right]^{1/2} \qquad \text{(Eq. 6.107)}$$

$$\frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right] \tan\left[2\left(\frac{\pi U a}{m}\right)\frac{h}{a}\right] \qquad \text{(Eq. 6.109)}$$

For $m = 36\ \frac{ft^2}{s}$, $a = 6\ ft$, and $U = 10\ \frac{ft}{s}$,

$$\frac{\pi U a}{m} = \frac{\pi\left(10\frac{ft}{s}\right)(6\ ft)}{36\ \frac{ft^2}{s}} = 5.24$$

Thus, length $= 2\ell$ and from Eq. 6.107

$$\underline{\underline{\text{length}}} = 2\,(6\ ft)\left[\frac{1}{5.24} + 1\right]^{1/2} = \underline{\underline{13.1\ ft}}$$

The thickness, $2h$, can be determined from Eq. 6.109 by trial and error. Assume value for h/a and compare with right hand side of Eq. 6.109. (See table below.)

$\frac{h}{a}$	$\frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right]\tan\left[2\,(5.24)\frac{h}{a}\right]$	
0.250	0.269	
0.251	0.262	
0.252	0.256	
0.253	0.250	← use

Thus, $\frac{h}{a} \approx 0.253$

and $\underline{\underline{\text{thickness}}} = 2h = 2\,(6\ ft)(0.253) = \underline{\underline{3.04\ ft}}$

6.58 *

6.58* Make use of Eqs. 6.107 and 6.109 to construct a table showing how ℓ/a, h/a, and ℓ/h for Rankine ovals depend on the parameter $\pi Ua/m$. Plot ℓ/h versus $\pi Ua/m$ and describe how this plot could be used to obtain the required values of m and a for a Rankine oval having a specific value of ℓ and h when placed in a uniform fluid stream of velocity, U.

For a Rankine oval

$$\frac{\ell}{a} = \left[\frac{m}{\pi U a} + 1\right]^{1/2} \qquad \text{(Eq. 6.107)}$$

and

$$\frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right]\tan\left[2\left(\frac{\pi U a}{m}\right)\frac{h}{a}\right] \qquad \text{(Eq. 6.109)}$$

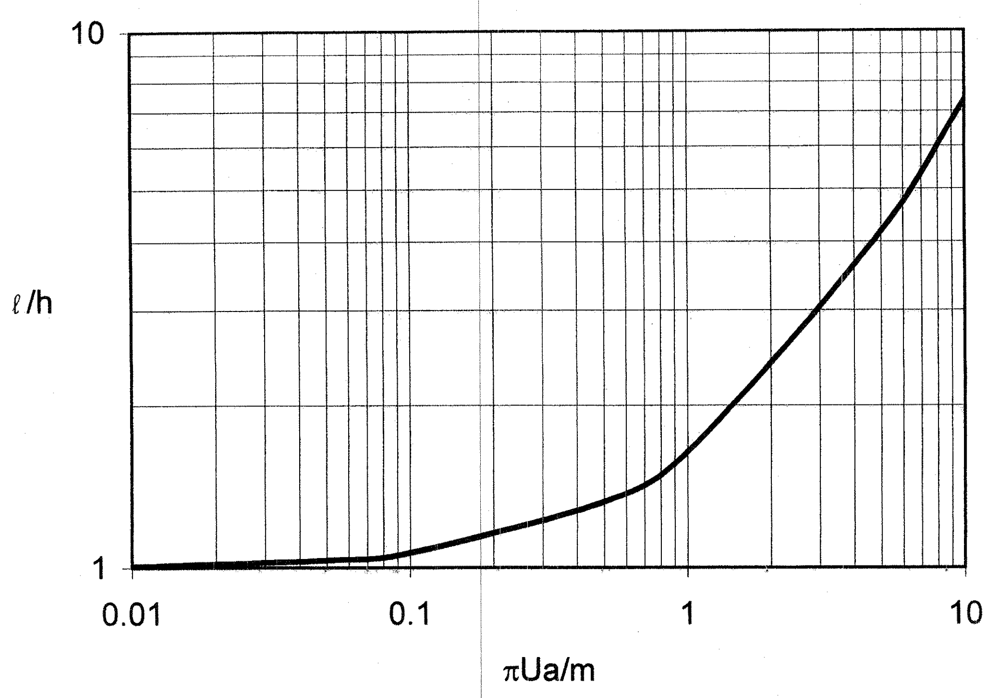
where the length of the body is 2ℓ and the width is $2h$. For a given value of $\pi Ua/m$, Eq. 6.107 can be solved for ℓ/a, and Eq. 6.109 can be solved (using an iteration procedure) for h/a. The ratio ℓ/h can then be determined. Tabulated data are given below.

$\pi Ua/m$	ℓ/a	h/a	ℓ/h
10	1.049	0.143	7.342
5	1.095	0.263	4.169
1	1.414	0.860	1.644
0.5	1.732	1.306	1.326
0.1	3.317	3.111	1.066
0.05	4.583	4.435	1.033
0.01	10.050	9.983	1.007

A plot of the data is shown on the next page.

(con't)

6.58* (con't)



For a Rankine oval with ℓ and h specified the following steps could be followed to determine m and a:

(1) For a given ℓ/h determine the required value of $\pi U a/m$ from the graph.

(2) Using this value of $\pi U a/m$ calculate ℓ/a from Eq. 6.107.

(3) With the value of ℓ/a determined, and ℓ specified, determine the value of a.

(4) With $\pi U a/m$ and a determined, the value of U/m is known, and for a given U the value of m is fixed.

6.59

6.59 An ideal fluid flows around a fixed cylinder as shown in Fig. P6.59. Note that the uniform velocity is in the negative x direction. Show that the pressure gradient, $\partial p/\partial s$, is proportional to s near the stagnation point. The coordinate s is measured along the cylinder surface as shown.

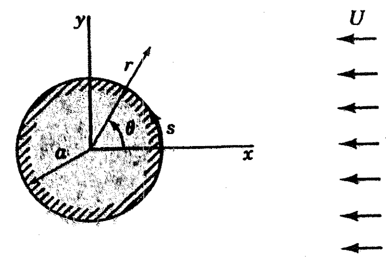


FIGURE P6.59

On the surface of the cylinder

$$p_s = p_0 + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta) \qquad \text{(Eq. 6.116)}$$

(Note: Because of the symmetry of the flow Eq. 6.116 is valid for uniform flow in either the positive or negative x-direction.)

Since $s = a\theta$,

$$\frac{\partial p_s}{\partial s} = \frac{\partial p_s}{\partial \theta}\frac{\partial \theta}{\partial s} = \frac{1}{a}\frac{\partial p_s}{\partial \theta}$$

Thus,

$$\frac{\partial p_s}{\partial \theta} = \frac{1}{2}\rho U^2(-8\sin\theta\cos\theta)$$

so that

$$\frac{\partial p_s}{\partial s} = -4\rho U^2 \frac{\sin\theta\cos\theta}{a} \qquad (1)$$

Near the stagnation point $(\theta = 0)$, $\sin\theta \approx \theta$ and $\cos\theta \approx 1.0$, and Eq. (1) can be expressed as

$$\frac{\partial p}{\partial s} \approx -4\rho U^2\frac{\theta}{a} = -\frac{4\rho U^2}{a^2}s$$

Thus,

$$\underline{\underline{\frac{\partial p}{\partial s} \propto s}}$$

6.60

6.60 An ideal fluid flows past an infinitely long semicircular "hump" located along a plane boundary as shown in Fig. P6.60. Far from the hump the velocity field is uniform, and the pressure is p_0. **(a)** Determine expressions for the maximum and minimum values of the pressure along the hump, and indicate where these points are located. Express your answer in terms of ρ, U, and p_0. **(b)** If the solid surface is the $\psi = 0$ streamline, determine the equation of the streamline passing through the point $\theta = \pi/2$, $r = 2a$.

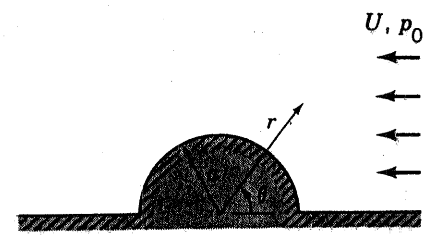


FIGURE P6.60

(a) On the surface of the hump,

$$p_s = p_0 + \frac{1}{2}\rho U^2 (1 - 4\sin^2\theta) \qquad \text{(Eq. 6.116)}$$

The maximum pressure occurs where $\sin\theta = 0$, or at $\theta = 0, \pi$, and at these points

$$\underline{\underline{p_s(\max) = p_0 + \frac{1}{2}\rho U^2}} \qquad \underline{\underline{(\text{at } \theta = 0 \text{ and } \pi)}}$$

The minimum pressure occurs where $\sin\theta = 1$, or at $\theta = \frac{\pi}{2}$, and at this point

$$\underline{\underline{p_s(\min) = p_0 - \frac{3}{2}\rho U^2}} \qquad \underline{\underline{(\text{at } \theta = \frac{\pi}{2})}}$$

(b) For uniform flow in the <u>negative</u> x-direction,

$$\psi = -Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

(refer to discussion associated with the derivation of Eq. 6.112).

At $\theta = \frac{\pi}{2}$, $r = 2a$

$$\psi = -2aU\left(1 - \frac{a^2}{(2a)^2}\right)\sin\frac{\pi}{2} = -\frac{3}{2}aU$$

and thus the equation of the streamline passing through this point is

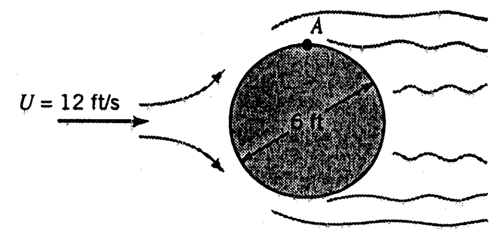
$$-\frac{3}{2}aU = -Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

or

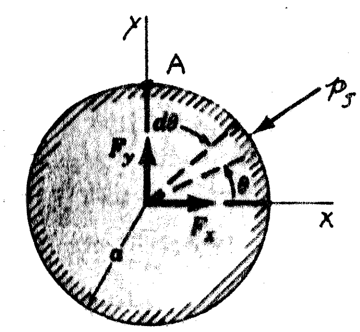
$$\underline{\underline{\frac{2}{3}\frac{r}{a}\left(1 - \frac{a^2}{r^2}\right)\sin\theta = 1}}$$

6.61

6.61 Water flows around a 6-ft diameter bridge pier with a velocity of 12 ft/s. Estimate the force (per unit length) that the water exerts on the pier. Assume that the flow can be approximated as an ideal fluid flow around the front half of the cylinder, but due to flow separation (see Video V6.4), the average pressure on the rear half is constant and approximately equal to $\frac{1}{2}$ the pressure at point A (see Fig. P6.61).



■ FIGURE P6.61



From Fig. 6.28 it follows that the drag on a section (between $\theta = 0$ and $\theta = \alpha$) of a circular cylinder is given by the equation

$$Drag = F_x = -\int_0^{\alpha} p_s \cos\theta \, a\, d\theta$$

For the force on the front half of the cylinder (per unit length)

$$F_{x_1} = -2\int_{\pi/2}^{\pi} p_s \cos\theta \, a\, d\theta \qquad (1)$$

and due to symmetry $F_y = 0$. From Eq. 6.116

$$p_s = p_0 + \frac{1}{2}\rho U^2 (1 - 4\sin^2\theta) \qquad (Eq. 6.116)$$

and since we are only interested in the force due to the flowing fluid we will let $p_0 = 0$. Thus, from Eq. (1)

$$F_{x_1} = -2\int_{\pi/2}^{\pi} \frac{1}{2}\rho U^2 (1 - 4\sin^2\theta)\cos\theta \; a\, d\theta \qquad (2)$$

Since

$$\int_{\pi/2}^{\pi} \cos\theta \, d\theta = \sin\theta \Big]_{\pi/2}^{\pi} = -1$$

and

$$\int_{\pi/2}^{\pi} \sin^2\theta \cos\theta \, d\theta = \frac{\sin^3\theta}{3}\Big]_{\pi/2}^{\pi} = -\frac{1}{3}$$

(con't)

6.61 (con't)

It follows from Eq. (2) that

$$F_{x_1} = -\frac{\rho U^2 a}{3}$$

Note that the negative sign indicates that the water is actually "pulling" on the cylinder (front half) in the upstream direction. However, when the effect of the rear half of the cylinder is taken into account (in a real fluid) there will be a net drag in the direction of flow.

The pressure at the top of the cylinder (point A) is given by

$$p_s = p_0 + \frac{1}{2}\rho U^2 (1 - 4\sin^2\theta) \qquad \text{(Eq. 6.116)}$$

and with $\theta = \pi/2$

$$p_A = p_0 - \frac{3}{2}\rho U^2$$

Since $p_0 = 0$

$$p_A = -\frac{3}{2}\rho U^2$$

Note that the negative pressure will give a positive F_x and

$$F_{x_2} = -\frac{p_A}{2} \times \text{projected area} = -\frac{p_A}{2} \times 2a(1)$$

So that

$$F_{x_2} = \frac{3}{4}\rho U^2 (2a)(1) = \frac{3}{2}\rho U^2 a$$

Thus,

$$F_x = F_{x_1} + F_{x_2}$$

$$= -\frac{\rho U^2 a}{3} + \frac{3\rho U^2 a}{2}$$

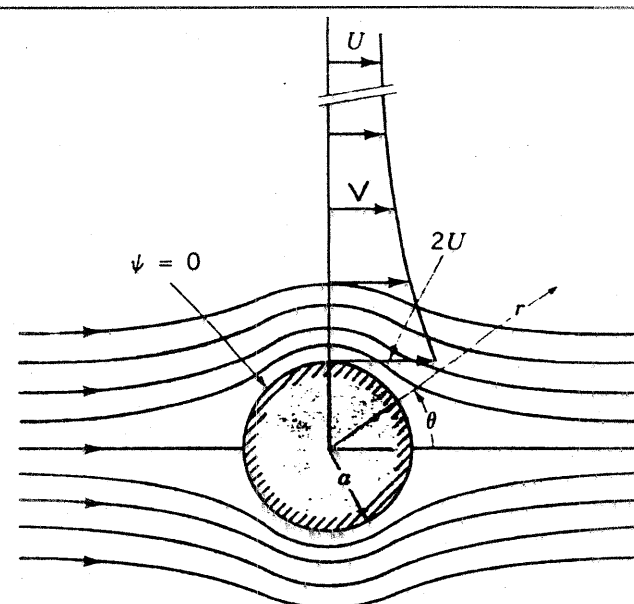
$$= \frac{7}{6}\rho U^2 a$$

With the data given,

$$F_x = \frac{7}{6}\left(1.94\,\frac{\text{slugs}}{\text{ft}^3}\right)\left(12\,\frac{\text{ft}}{\text{s}}\right)^2 (3\text{ ft}) = \underline{\underline{978\,\frac{\text{lb}}{\text{ft}}}}$$

6.62*

*6.62 Consider the steady potential flow around the circular cylinder shown in Fig. 6.26. Show on a plot the variation of the magnitude of the dimensionless fluid velocity, V/U, along the positive y axis. At what distance, y/a (along the y axis), is the velocity within 1% of the free-stream velocity?



■ FIGURE 6.26

Along the y-axis $v_r = 0$ so that the magnitude of the velocity, V, is equal to $|v_\theta|$. Since

$$v_\theta = -U\left(1+\frac{a^2}{r^2}\right)\sin\theta \qquad \text{(Eq. 6.115)}$$

it follows that along the positive y-axis $(\theta = \frac{\pi}{2},\ r = y)$

$$V = |v_\theta| = U\left(1+\frac{a^2}{y^2}\right)$$

or

$$\frac{V}{U} = 1 + \frac{a^2}{y^2} = 1 + \frac{1}{\left(\frac{y}{a}\right)^2} \qquad (1)$$

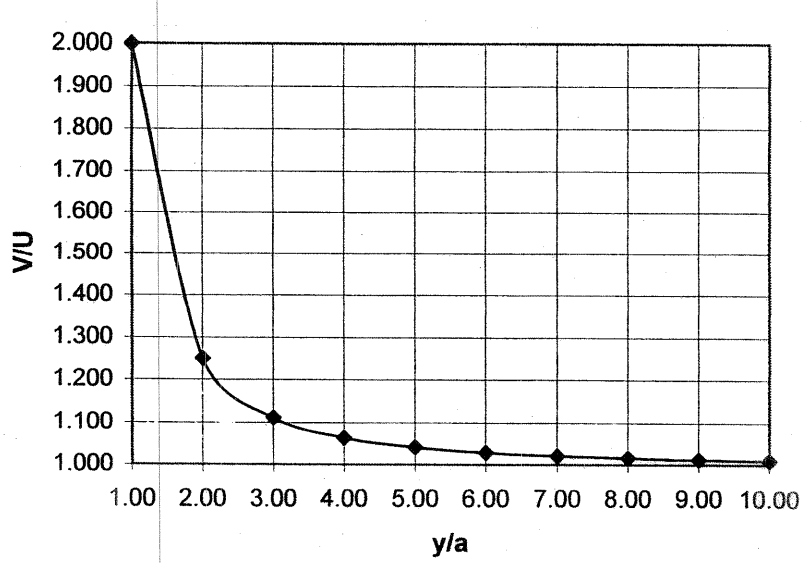
Tabulated data and a plot of the data are given below. It can be seen from these results that for

$$\underline{\underline{\frac{y}{a} \geq 10}}$$

the velocity V is within 1% of the free-stream velocity U.

y/a	V/U
1.00	2.000
2.00	1.250
3.00	1.111
4.00	1.063
5.00	1.040
6.00	1.028
7.00	1.020
8.00	1.016
9.00	1.012
10.00	1.010

Calculated from Eq. (1)



6.63

6.63 The velocity potential for a cylinder (Fig. P6.63) rotating in a uniform stream of fluid is

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

where Γ is the circulation. For what value of the circulation will the stagnation point be located at: **(a)** point A, **(b)** point B?

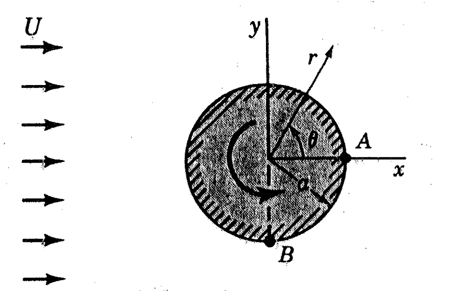


FIGURE P6.63

(a) $\sin\theta_{stag} = \dfrac{\Gamma}{4\pi U a}$ (Eq. 6.122)

At point A, $\theta_{stag} = 0$ and it follows that $\underline{\underline{\Gamma = 0}}$.

(b) At point B, $\theta_{stag} = \dfrac{3\pi}{2}$, and from Eq. 6.122

$$\Gamma = 4\pi U a \sin\frac{3\pi}{2} = \underline{\underline{-4\pi U a}}$$

6.64

6.64 A fixed circular cylinder of infinite length is placed in a steady, uniform stream of an incompressible, nonviscous fluid. Assume that the flow is irrotational. Prove that the drag on the cylinder is zero. Neglect body forces.

$$\text{Drag} = F_x = -\int_0^{2\pi} p_s \cos\theta \; a \, d\theta \qquad (\text{Eq. } 6.117)$$

$$p_s = p_0 + \tfrac{1}{2}\rho U^2 \left(1 - 4\sin^2\theta\right) \qquad (\text{Eq. } 6.116)$$

Thus,

$$\text{Drag} = -\left\{ a p_0 \int_0^{2\pi} \cos\theta \, d\theta + \frac{a}{2}\rho U^2 \int_0^{2\pi} \cos\theta \, d\theta - 2a\rho U^2 \int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta \right\}$$

Since,

$$\int_0^{2\pi} \cos\theta \, d\theta = \sin\theta \Big]_0^{2\pi} = 0$$

and

$$\int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta = \frac{\sin^3\theta}{3}\Big]_0^{2\pi} = 0$$

it follows that

$$\underline{\underline{\text{Drag} = 0}}$$

6.65

6.65 Repeat Problem 6.64 for a rotating cylinder for which the stream function and velocity potential are given by Eqs. 6.119 and 6.120, respectively. Verify that the lift is not zero and can be expressed by Eq. 6.124.

$$Drag = F_x = -\int_0^{2\pi} p_s \cos\theta \, a \, d\theta \qquad \text{(Eq. 6.117)}$$

$$p_s = p_0 + \frac{1}{2}\rho U^2 \left(1 - 4\sin^2\theta + \frac{2\Gamma \sin\theta}{\pi a U} - \frac{\Gamma^2}{4\pi^2 a^2 U^2}\right) \qquad \text{(Eq. 6.123)}$$

Thus,

$$Drag = -\left\{ a p_0 \int_0^{2\pi} \cos\theta \, d\theta + \frac{a}{2}\rho U^2 \left[\int_0^{2\pi} \cos\theta \, d\theta - 4\int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta + \frac{2\Gamma}{\pi a U}\int_0^{2\pi} \cos\theta \sin\theta \, d\theta - \frac{\Gamma^2}{4\pi^2 a^2 U^2}\int_0^{2\pi} \cos\theta \, d\theta \right]\right\}$$

Since,

$$\int_0^{2\pi} \cos\theta \, d\theta = \sin\theta \Big]_0^{2\pi} = 0$$

and

$$\int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta = \frac{\sin^3\theta}{3}\Big]_0^{2\pi} = 0$$

and

$$\int_0^{2\pi} \cos\theta \sin\theta \, d\theta = \frac{\sin^2\theta}{2}\Big]_0^{2\pi} = 0$$

it follows that

$$\underline{\underline{Drag = 0}}$$

(con't)

6.65 (Con't)

$$Lift = F_y = -\int_0^{2\pi} p_s \sin\theta \; a\,d\theta \qquad \text{(Eq. 6.118)}$$

with p_s given by Eq. 6.123 it follows that

$$Lift = -\left\{ a\,p_0 \int_0^{2\pi} \sin\theta\, d\theta + \frac{a}{2}\rho U^2 \left[\int_0^{2\pi} \sin\theta\, d\theta - 4\int_0^{2\pi} \sin^3\theta\, d\theta + \frac{2\Gamma}{\pi a U}\int_0^{2\pi} \sin^2\theta\, d\theta - \frac{\Gamma^2}{4\pi^2 a^2 U^2}\int_0^{2\pi} \sin\theta\, d\theta \right]\right\}$$

Since,

$$\int_0^{2\pi} \sin\theta\, d\theta = -\cos\theta \Big]_0^{2\pi} = 0$$

and

$$\int_0^{2\pi} \sin^3\theta\, d\theta = -\frac{\cos\theta}{3}\left(\sin^2\theta + 2\right)\Big]_0^{2\pi} = 0$$

and

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\Big]_0^{2\pi} = \pi$$

it follows that

$$Lift = -\frac{a}{2}\rho U^2 \left(\frac{2\Gamma}{\pi a U}\right)(\pi)$$

Thus,

$$\underline{\underline{Lift = -\rho U \Gamma}}$$

(which is Eq. 6.124).

6.66

6.66 A source of strength m is located a distance ℓ from a vertical solid wall as shown in Fig. P6.66. The velocity potential for this incompressible, irrotational flow is given by

$$\phi = \frac{m}{4\pi}\{\ln[(x-\ell)^2 + y^2] + \ln[(x+\ell)^2 + y^2]\}$$

(a) Show that there is no flow through the wall. **(b)** Determine the velocity distribution along the wall. **(c)** Determine the pressure distribution along the wall, assuming $p = p_0$ far from the source. Neglect the effect of the fluid weight on the pressure.

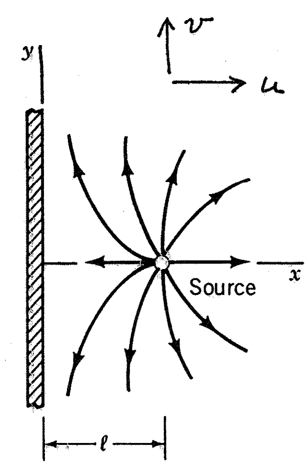


FIGURE P6.66

(a) $$u = \frac{\partial \phi}{\partial x}$$

Since, $$\frac{\partial}{\partial x}\ln\left[(x-\ell)^2 + y^2\right] = \frac{2(x-\ell)}{(x-\ell)^2+y^2}$$

and $$\frac{\partial}{\partial x}\ln\left[(x+\ell)^2 + y^2\right] = \frac{2(x+\ell)}{(x+\ell)^2+y^2}$$

it follows that

$$u = \frac{m}{4\pi}\left[\frac{2(x-\ell)}{(x-\ell)^2+y^2} + \frac{2(x+\ell)}{(x+\ell)^2+y^2}\right]$$

Along the wall, $x=0$, so that

$$u = \frac{m}{4\pi}\left[\frac{-2\ell}{\ell^2+y^2} + \frac{2\ell}{\ell^2+y^2}\right] = 0$$

Thus, there is no flow through the wall.

(b) The velocity along wall, $V_w = v$ since $u=0$. Also

$$v = \frac{\partial \phi}{\partial y}$$

and with the given velocity potential

$$v = \frac{m}{4\pi}\left[\frac{2y}{(x-\ell)^2+y^2} + \frac{2y}{(x+\ell)^2+y^2}\right] \qquad (1)$$

(con't)

6.66 (con't)

Along the wall, $x=0$, and from Eq.(1)

$$V_w = v = \frac{m}{4\pi}\left[\frac{2y}{\ell^2+y^2} + \frac{2y}{\ell^2+y^2}\right]$$

or

$$\underline{\underline{V_w = \frac{m}{\pi}\left(\frac{y}{\ell^2+y^2}\right)}} \qquad (2)$$

(c) Far from the source, $p=p_0$ and $V \approx 0$. Thus,

$$\frac{p_0}{\gamma} = \frac{p_w}{\gamma} + \frac{V_w^2}{2g}$$

where p_w is the pressure at the wall, so that

$$p_w = p_0 - \frac{1}{2}\rho V_w^2$$

With V_w given by Eq.(2),

$$\underline{\underline{p_w = p_0 - \frac{\rho m^2}{2\pi^2}\left(\frac{y}{\ell^2+y^2}\right)^2}}$$

6.67

6.67 A long porous pipe runs parallel to a horizontal plane surface as shown in Fig. P6.67. The longitudinal axis of the pipe is perpendicular to the plane of the paper. Water flows radially from the pipe at a rate of $0.5\,\pi$ ft³/s per foot of pipe. Determine the difference in pressure (in lb/ft²) between point B and point A. The flow from the pipe may be approximated by a two-dimensional source. *Hint:* To develop the stream function or velocity potential for this type of flow, place (symmetrically) another equal source on the other side of the wall. With this combination there is no flow across the x-axis, and this axis can be replaced with a solid boundary. This technique is called the *method of images.*

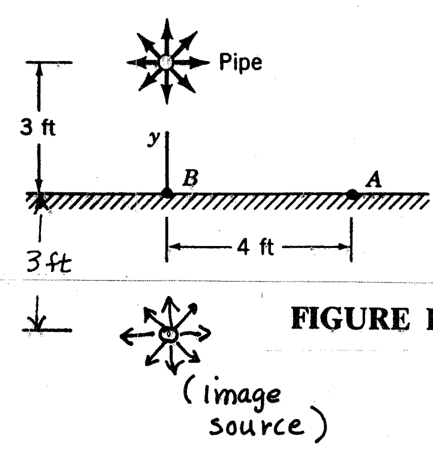


FIGURE P6.67

For a source,

$$\phi = \frac{m}{2\pi}\ln r = \frac{m}{4\pi}\ln r^2$$

where r is measured from the source. With the coordinate system shown in figure

$$r^2 = x^2 + (y-3)^2 \qquad \text{(for upper source)}$$

and

$$r^2 = x^2 + (y+3)^2 \qquad \text{(for lower source)}$$

so that for the combined sources

$$\phi = \frac{m}{4\pi}\left\{\ln\left[x^2+(y-3)^2\right] + \ln\left[x^2+(y+3)^2\right]\right\}$$

Since,

$$u = \frac{\partial \phi}{\partial x}$$

and

$$\frac{\partial}{\partial x}\ln\left[x^2+(y-3)^2\right] = \frac{2x}{x^2+(y-3)^2}$$

$$\frac{\partial}{\partial x}\ln\left[x^2+(y+3)^2\right] = \frac{2x}{x^2+(y+3)^2}$$

it follows that

$$u = \frac{m}{4\pi}\left[\frac{2x}{x^2+(y-3)^2} + \frac{2x}{x^2+(y+3)^2}\right]$$

Along the wall, $y=0$, $v=0$ and therefore

$$V_w = u = \frac{m}{4\pi}\left(\frac{4x}{x^2+9}\right)$$

(con't)

6.67 (con't)

At point A, $x = 4\,ft$, and with $m = 0.5\pi\ \frac{ft^2}{s}$,

$$V_{wA} = \frac{0.5\pi\ \frac{ft^2}{s}}{4\pi}\left[\frac{4\ (4\,ft)}{(4\,ft)^2 + 9\,ft^2}\right] = \frac{2}{25}\ \frac{ft}{s}$$

At point B, $x = 0$, and

$$V_{wB} = 0$$

Thus, from the Bernoulli equation

$$\frac{p_B}{\gamma} + \frac{V_{wB}^2}{2g} = \frac{p_A}{\gamma} + \frac{V_{wA}^2}{2g}$$

or

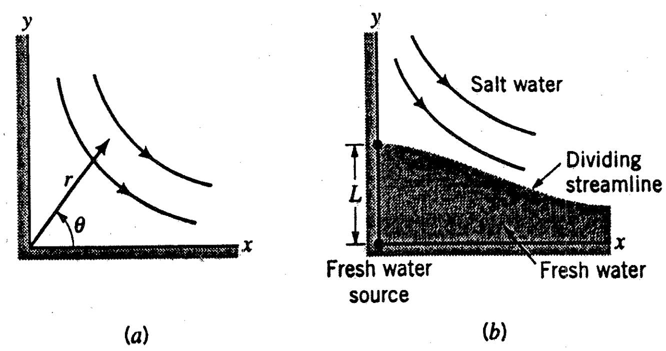
$$p_B - p_A = \frac{1}{2}\frac{\gamma}{g}\ V_{wA}^2$$

$$= \frac{\left(62.4\ \frac{lb}{ft^3}\right)}{2\left(32.2\ \frac{ft}{s^2}\right)}\left(\frac{2}{25}\ \frac{ft}{s}\right)^2 = \underline{\underline{0.00620\ psf}}$$

Master Typing Sheet
10% Reduction
8 1/2 x 11 trim size

6.68

6.68 At a certain point at the beach, the coast line makes a right angle bend as shown in Fig. 6.68*a*. The flow of salt water in this bend can be approximated by the potential flow of an incompressible fluid in a right angle corner. **(a)** Show that the stream function for this flow is $\psi = A r^2 \sin 2\theta$, where A is a positive constant. **(b)** A fresh water reservoir is located in the corner. The salt water is to be kept away from the reservoir to avoid any possible seepage of salt water into the fresh water (Fig. 6.68*b*). The fresh water source can be approximated as a line source having a strength m, where m is the volume rate of flow (per unit length) emanating from the source. Determine m if the salt water is not to get closer than a distance L to the corner. *Hint:* Find the value of m (in terms of A and L) so that a stagnation point occurs at $y = L$. **(c)** The streamline passing through the stagnation point would represent the line dividing the fresh water from the salt water. Plot this streamline.



■ FIGURE P6.68

(a) For the given stream function,

$$\psi = A r^2 \sin 2\theta$$

along $\theta = 0$ $\psi = 0$ and $\theta = \pi/2$ $\psi = 0$.

Thus, the rays $\theta = 0$ and $\theta = \pi/2$ can be replaced with a solid boundary along which the stream function must be constant. This boundary forms a right angle and therefore this stream function can be used to represent flow in a right angle corner.

(b) Since

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 2 A r \cos 2\theta$$

at $\theta = \pi/2$

$$v_r = 2 A r \cos \pi = -2Ar$$

For a source located at the origin

$$\psi = \frac{m}{2\pi}\theta$$

and $v_{r_s} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r}$

To create a stagnation point at $r = L$ and $\theta = \frac{\pi}{2}$

let $v_r = v_{r_s}$.

(con't)

6.68 (con't)

Thus,

$$2AL = \frac{m}{2\pi L}$$

and

$$\underline{m = 4\pi A L^2}$$

gives a stagnation point at $r = L$, $\theta = \pi/2$.

(c) The combined stream function is

$$\psi = A r^2 \sin 2\theta + \frac{m}{2\pi}\theta$$

and with $m = 4\pi A L^2$

$$\psi = A r^2 \sin 2\theta + 2AL^2\theta$$

The value of ψ at the stagnation point ($r = L$, $\theta = \pi/2$) is

$$\psi_{stag} = A L^2 \sin \pi + 2AL^2\left(\frac{\pi}{2}\right)$$

$$= AL^2\pi$$

Thus, the equation for the streamline passing through the stagnation point is

$$AL^2\pi = A r^2 \sin 2\theta + 2AL^2\theta$$

or

$$r = \sqrt{\frac{\pi L^2 - 2L^2\theta}{\sin 2\theta}}$$

and

$$r' = \frac{r}{L} = \sqrt{\frac{\pi - 2\theta}{\sin 2\theta}} \qquad (1)$$

For plotting let

$$x' = r' \cos\theta \quad \text{and} \quad y' = r' \sin\theta$$

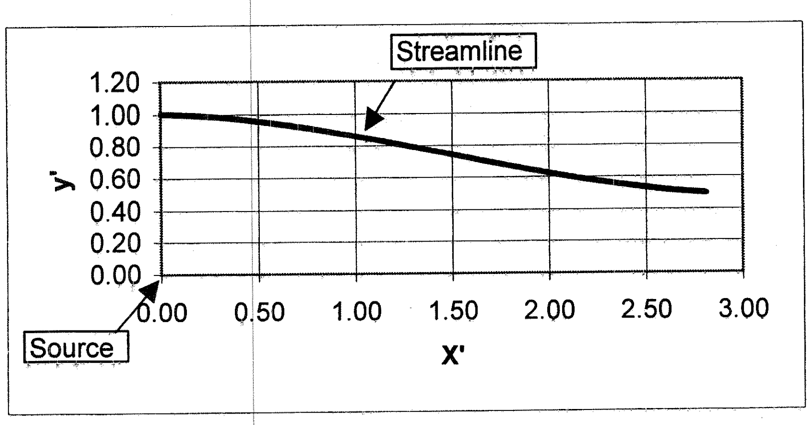
and a plot of the dividing streamline from Eq. (1) is shown on the following page.

(con't)

Master Typing Sheet
10% Reduction
8 1/2 x 11 trim size

6.68 (con't)

Theta(deg)	Theta(rad)	r/L	x'	y'
10	0.175	2.857	2.814	0.496
20	0.349	1.950	1.832	0.667
30	0.524	1.555	1.347	0.778
40	0.698	1.331	1.020	0.856
50	0.873	1.191	0.765	0.912
60	1.047	1.100	0.550	0.952
70	1.222	1.042	0.356	0.979
80	1.396	1.010	0.175	0.995
90	1.571	1.000	0.000	1.000



6.69

6.69 The two-dimensional velocity field for an incompressible, Newtonian fluid is described by the relationship

$$\mathbf{V} = (12xy^2 - 6x^3)\hat{\mathbf{i}} + (18x^2y - 4y^3)\hat{\mathbf{j}}$$

where the velocity has units of m/s when x and y are in meters. Determine the stresses σ_{xx}, σ_{yy}, and τ_{xy} at the point $x = 0.5$ m, $y = 1.0$ m if pressure at this point is 6 kPa and the fluid is glycerin at 20 °C. Show these stresses on a sketch.

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \qquad \text{(Eq. 6.125a)}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \qquad \text{(Eq. 6.125b)}$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \text{(Eq. 6.125d)}$$

For the given velocity distribution, with $x = 0.5\,m$ and $y = 1.0\,m$:

$$\frac{\partial u}{\partial x} = 12y^2 - 18x^2 = 12(1.0)^2 - 18(0.5)^2 = 7.50\,\frac{1}{s}$$

$$\frac{\partial u}{\partial y} = 24xy = 24(0.5)(1.0) = 12.0\,\frac{1}{s}$$

$$\frac{\partial v}{\partial x} = 36xy = 36(0.5)(1.0) = 18.0\,\frac{1}{s}$$

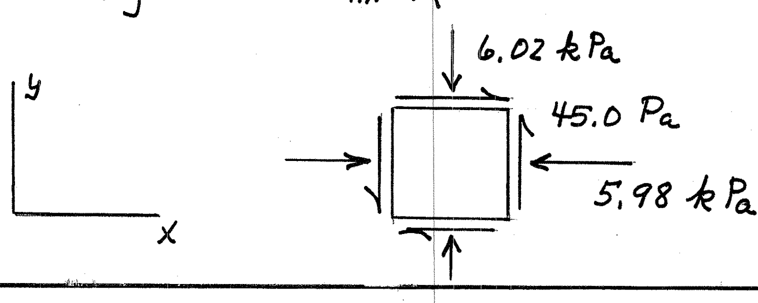
$$\frac{\partial v}{\partial y} = 18x^2 - 12y^2 = 18(0.5)^2 - 12(1.0)^2 = -7.50\,\frac{1}{s}$$

Thus, for $p = 6 \times 10^3 \frac{N}{m^2}$ and $\mu = 1.50 \frac{N \cdot s}{m^2}$,

$$\sigma_{xx} = -6\times 10^3 \frac{N}{m^2} + 2\left(1.50\,\frac{N\cdot s}{m^2}\right)\left(7.50\,\frac{1}{s}\right) = \underline{\underline{-5.98\ kPa}}$$

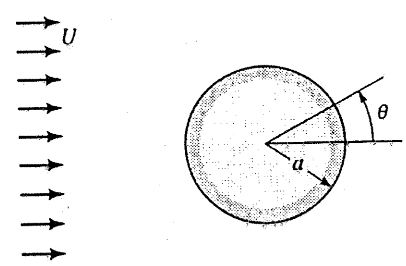
$$\sigma_{yy} = -6\times 10^3 \frac{N}{m^2} + 2\left(1.50\,\frac{N\cdot s}{m^2}\right)\left(-7.50\,\frac{1}{s}\right) = \underline{\underline{-6.02\ kPa}}$$

$$\tau_{xy} = \left(1.50\,\frac{N\cdot s}{m^2}\right)\left(12.0\,\frac{1}{s} + 18.0\,\frac{1}{s}\right) = \underline{\underline{45.0\ Pa}}$$



6.70

6.70 Typical inviscid flow solutions for flow around bodies indicate that the fluid flows smoothly around the body, even for blunt bodies as shown in **Video V6.4**. However, experience reveals that due to the presence of viscosity, the main flow may actually separate from the body creating a wake behind the body. As discussed in a later section (Section 9.2.6), whether or not separation takes place depends on the pressure gradient along the surface of the body, as calculated by inviscid flow theory. If the pressure decreases in the direction of flow (a *favorable* pressure gradient), no separation will occur. However, if the pressure increases in the direction of flow (an *adverse* pressure gradient), separation may occur. For the circular cylinder of Fig. P6.70 placed in a uniform stream with velocity, U, determine an expression for the pressure gradient in the direction flow on the surface of the cylinder. For what range of values for the angle θ will an adverse pressure gradient occur?



■ FIGURE P6.70

From Eq. 6.116

$$p_s = p_0 + \frac{1}{2}\rho U^2 \left(1 - 4\sin^2\theta\right)$$

Thus,

$$\frac{\partial p_s}{\partial \theta} = \underline{\underline{4\rho U^2 \sin\theta \cos\theta}} \qquad (1)$$

Since an adverse pressure gradient occurs for a positive $\partial p_s/\partial\theta$, it follows from Eq. (1) that θ falls in the range of $\pm 90°$ for an adverse pressure gradient. This range corresponds to the rear half of the cylinder.

6.71

6.71 For a two-dimensional incompressible flow in the x–y plane show that the z component of the vorticity, ζ_z, varies in accordance with the equation

$$\frac{D\zeta_z}{Dt} = \nu \nabla^2 \zeta_z$$

What is the physical interpretation of this equation for a nonviscous fluid? *Hint:* This *vorticity transport equation* can be derived from the Navier–Stokes equations by differentiating and eliminating the pressure between Eqs. 6.127a and 6.127b.

For two-dimensional flow with $w=0$, Eq. 6.127a reduces to

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (1)$$

and Eq. 6.127b reduces to

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

Differentiate Eq. (1) with respect to y and Eq. (2) with respect to x, and subtract Eq. (1) from Eq. (2) to obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) =$$

$$\frac{\mu}{\rho}\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] \quad (3)$$

By definition (see Eq. 6.17)

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Re-write Eq. (3) to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) =$$

$$\frac{\mu}{\rho}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \quad (4)$$

(con't)

6.71 (con't)

Since each term in parenthesis in Eq. (4) is ζ_z it follows that

$$\frac{\partial \zeta_z}{\partial t} + u\frac{\partial \zeta_z}{\partial x} + v\frac{\partial \zeta_z}{\partial y} = \frac{\mu}{\rho}\left(\frac{\partial^2 \zeta_z}{\partial x^2} + \frac{\partial^2 \zeta_z}{\partial y^2}\right) \qquad (5)$$

The left side of Eq. (5) can be expressed as (see Eq. 4.5) $\frac{D\zeta_z}{Dt}$ where the operator $\frac{D(\ )}{Dt}$ is the material derivative. The right hand side of Eq. (5) can be expressed as

$$\nu \nabla^2 \zeta_z$$

where $\nu = \mu/\rho$ so that Eq. (5) can be written as

$$\underline{\underline{\frac{D\zeta_z}{Dt} = \nu \nabla^2 \zeta_z}}$$

For a nonviscous fluid, $\nu = 0$, and in this case

$$\frac{D\zeta_z}{Dt} = 0$$

Thus, for a two-dimensional flow of an incompressible, nonviscous fluid, the change in the vorticity of a fluid particle as it moves through the flow field is zero.

6.72

6.72 The velocity of a fluid particle moving along a horizontal streamline that coincides with the x axis in a plane, two-dimensional incompressible flow field was experimentally found to be described by the equation $u = x^2$. Along this streamline determine an expression for: **(a)** the rate of change of the v-component of velocity with respect to y; **(b)** the acceleration of the particle; and **(c)** the pressure gradient in the x direction. The fluid is Newtonian.

(a) From the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

so that with $u = x^2$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \underline{\underline{-2x}} \qquad (1)$$

Also, Eq. (1) can be integrated with respect to y to obtain

$$\int dv = \int -2x\, dy$$

or

$$v = -2xy + f(x)$$

Since the x-axis is a streamline, $v = 0$ along this axis and therefore $f(x) = 0$ so that

$$v = -2xy$$

(b)

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (x^2)(2x) + (-2xy)(0) = 2x^3$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (x^2)(-2y) + (-2xy)(-2x) = 2x^2y$$

Along x-axis, $y=0$, and therefore $a_y = 0$. Thus,

$$\vec{a} = \underline{\underline{2x^3\,\hat{i}}}$$

(c) From Eq. 6.127a (with $g_x = 0$),

$$a_x = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

so that

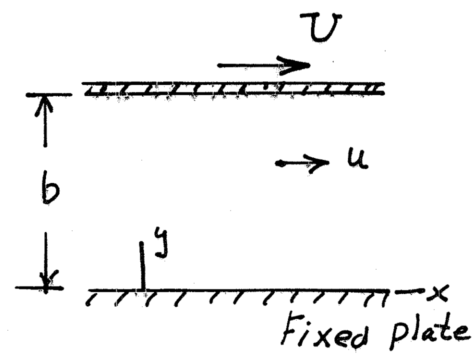
$$2x^3 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}(2+0)$$

and

$$\frac{\partial p}{\partial x} = \underline{\underline{2\mu - 2\rho x^3}}$$

6.73

6.73 Two horizontal, infinite, parallel plates are spaced a distance b apart. A viscous liquid is contained between the plates. The bottom plate is fixed and the upper plate moves parallel to the bottom plate with a velocity U. Because of the no-slip boundary condition (see **Video V6.5**), the liquid motion is caused by the liquid being dragged along by the moving boundary. There is no pressure gradient in the direction of flow. Note that this is a so-called simple *Couette flow* discussed in Section 6.9.2. **(a)** Start with the Navier–Stokes equations and determine the velocity distribution between the plates. **(b)** Determine an expression for the flowrate passing between the plates (for a unit width). Express your answer in terms of b and U.



(a) For steady flow with $v = w = 0$ it follows that the Navier-Stokes equations reduce to (in direction of flow)

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad \text{(Eq. 6.129)}$$

Thus, for zero pressure gradient

$$\frac{\partial^2 u}{\partial y^2} = 0$$

so that

$$u = C_1 y + C_2$$

At $y = 0$ $u = 0$ and it follows that $C_2 = 0$. Similarly, at $y = b$ $u = U$ and $C_1 = \frac{U}{b}$

Therefore,

$$\underline{\underline{u = \frac{U}{b} y}}$$

(b)

$$q = \int_0^b u\,(1)\,dy = \frac{U}{b}\int_0^b y\,dy = \frac{U}{b}\left.\frac{y^2}{2}\right|_0^b = \underline{\underline{\frac{Ub}{2}}}$$

where q is the flowrate per unit width.

6.74

6.74 Oil (SAE 30) at 15.6 °C flows steadily between fixed, horizontal, parallel plates. The pressure drop per unit length along the channel is 20 kPa/m, and the distance between the plates is 4mm. The flow is laminar. Determine: (a) the volume rate of flow (per meter of width), (b) the magnitude and direction of the shearing stress acting on the bottom plate, and (c) the velocity along the centerline of the channel.

(a) $$q = \frac{2h^3 \Delta p}{3\mu \ \ell} \qquad \text{(Eq. 6.136)}$$

For $h = \frac{4\,mm}{2} = 2\times10^{-3}\,m$, $\mu = 0.38\,\frac{N\cdot s}{m^2}$, and $\frac{\Delta p}{\ell} = 20\times10^{3}\,\frac{N}{m^3}$,

$$q = \frac{2\,(2\times10^{-3}\,m)^3(20\times10^{3}\,\frac{N}{m^3})}{3\,(0.38\,\frac{N\cdot s}{m^2})} = \underline{\underline{2.81\times10^{-4}\,\frac{m^2}{s}}}$$

(b) $$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \text{(Eq. 6.125d)}$$

Since
$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2) \qquad \text{(Eq. 6.134)}$$

and
$$v = 0$$

it follows that
$$\frac{\partial u}{\partial y} = \frac{1}{2\mu}\frac{\partial p}{\partial x}(2y) \qquad \frac{\partial v}{\partial x} = 0$$

and therefore
$$\tau_{yx} = \frac{\partial p}{\partial x}(y)$$

At the bottom plate, $y = -h$, and since $\frac{\partial p}{\partial x} = -\frac{\Delta p}{\ell}$,

$$\tau_{yx} = \frac{\Delta p}{\ell}(h) = (20\times10^{3}\,\frac{N}{m^3})(2\times10^{-3}\,m)$$

$= \underline{\underline{40\,\frac{N}{m^2}\text{ acting in the direction of flow}}}$

(c)
$$u_{max} = \frac{3}{2}V \qquad \text{(Eq. 6.138)}$$

$$= \frac{3}{2}\left(\frac{q}{2h}\right) = \frac{3}{2}\,\frac{(2.81\times10^{-4}\,\frac{m^2}{s})}{(2)(2\times10^{-3}\,m)} = \underline{\underline{0.105\,\frac{m}{s}}}$$

6.75

6.75 Two fixed, horizontal, parallel plates are spaced 0.4 in. apart. A viscous liquid ($\mu = 8 \times 10^{-3}\ \mathrm{lb \cdot s/ft^2}$, $SG = 0.9$) flows between the plates with a mean velocity of 0.5 ft/s. The flow is laminar. Determine the pressure drop per unit length in the direction of flow. What is the maximum velocity in the channel?

$$V = \frac{h^2}{3\mu}\frac{\Delta p}{\ell} \qquad \text{(Eq. 6.137)}$$

Thus,

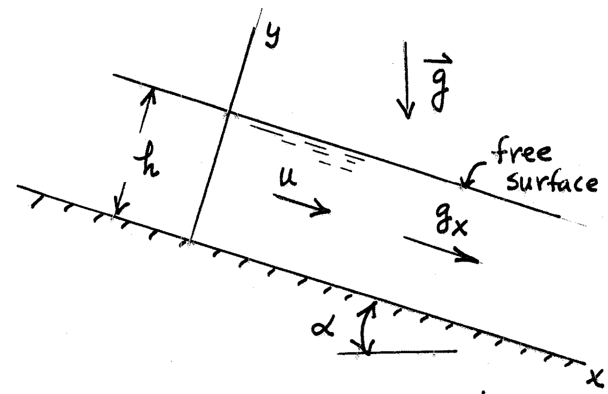
$$\frac{\Delta p}{\ell} = \frac{3\mu V}{h^2} = \frac{3\left(8\times10^{-3}\ \frac{lb \cdot s}{ft^2}\right)\left(0.5\ \frac{ft}{s}\right)}{\left(\frac{0.2\ in.}{12\ \frac{in.}{ft}}\right)^2} = \underline{\underline{43.2\ \frac{lb}{ft^2}\ \text{per ft}}}$$

$$u_{max} = \frac{3}{2}V \qquad \text{(Eq. 6.138)}$$

$$= \frac{3}{2}\left(0.5\ \frac{ft}{s}\right) = \underline{\underline{0.75\ \frac{ft}{s}}}$$

6.76

6.76 A layer of viscous liquid of constant thickness (no velocity perpendicular to plate) flows steadily down an infinite, inclined plane. Determine, by means of the Navier–Stokes equations, the relationship between the thickness of the layer and the discharge per unit width. The flow is laminar, and assume air resistance is negligible so that the shearing stress at the free surface is zero.



$g_x = g \sin\alpha$

With the coordinate system shown in the figure $v=0$, $w=0$, and from the continuity equation $\frac{\partial u}{\partial x}=0$. Thus, from the x-component of the Navier-Stokes equations (Eq. 6.127a),

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin\alpha + \mu \frac{d^2u}{dy^2} \qquad (1)$$

Also, since there is a free surface, there cannot be a pressure gradient in the x-direction so that $\frac{\partial p}{\partial x}=0$ and Eq. (1) can be written as

$$\frac{d^2u}{dy^2} = -\frac{\rho g}{\mu}\sin\alpha$$

Integration yields

$$\frac{du}{dy} = -\left(\frac{\rho g}{\mu}\sin\alpha\right)y + C_1 \qquad (2)$$

Since the shearing stress

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

equals zero at the free surface ($y=h$) it follows that

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y=h$$

so that the constant in Eq. (2) is

$$C_1 = \frac{\rho g}{\mu}\sin\alpha$$

Integration of Eq. (2) yields

$$u = -\left(\frac{\rho g}{\mu}\sin\alpha\right)\frac{y^2}{2} + \left(\frac{\rho g}{\mu}\sin\alpha\right)y + C_2$$

Since $u=0$ at $y=0$, it follows that $C_2=0$, and therefore

$$u = \frac{\rho g}{\mu}\sin\alpha\left(hy - \frac{y^2}{2}\right)$$

The flowrate per unit width can be expressed as $q = \int_0^h u\,dy$ so that

$$q = \int_0^h \frac{\rho g}{\mu}\sin\alpha\left(hy - \frac{y^2}{2}\right)dy = \underline{\underline{\frac{\rho g h^3 \sin\alpha}{3\mu}}}$$

6.77

6.77 A viscous, incompressible fluid flows between the two infinite, vertical, parallel plates of Fig. P6.77. Determine, by use of the Navier–Stokes equations, an expression for the pressure gradient in the direction of flow. Express your answer in terms of the mean velocity. Assume that the flow is laminar, steady, and uniform.

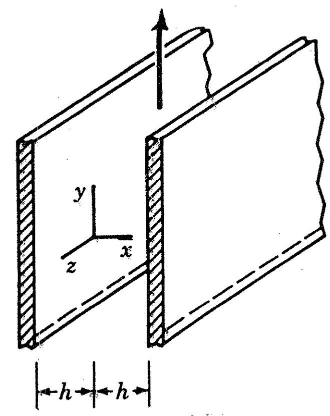


FIGURE P6.77

With the coordinate system shown $u=0$, $w=0$ and from the continuity equation $\frac{\partial v}{\partial y}=0$. Thus, from the y-component of the Navier-Stokes equations (Eq. 6.127b), with $g_y=-g$,

$$0=-\frac{\partial p}{\partial y}-\rho g+\mu\frac{d^2 v}{dx^2} \qquad (1)$$

Since the pressure is not a function of x, Eq. (1) can be written as

$$\frac{d^2 v}{dx^2}=\frac{P}{\mu}$$

(where $P=\frac{\partial p}{\partial y}+\rho g$) and integrated to obtain

$$\frac{dv}{dx}=\frac{P}{\mu}x+C_1 \qquad (2)$$

From symmetry $\frac{dv}{dx}=0$ at $x=0$ so that $C_1=0$. Integration of Eq. (2) yields

$$v=\frac{P}{\mu}\frac{x^2}{2}+C_2$$

Since at $x=\pm h$, $v=0$ it follows that $C_2=-\frac{P}{2\mu}(h^2)$ and therefore

$$v=\frac{P}{2\mu}(x^2-h^2)$$

The flowrate per unit width in the z-direction can be expressed as

$$q=\int_{-h}^{h} v\,dx=\int_{-h}^{h}\frac{P}{2\mu}(x^2-h^2)\,dx=-\frac{2}{3}\frac{Ph^3}{\mu}$$

Thus, with V (mean velocity) given by the equation

$$V=\frac{q}{2h}=-\frac{1}{3}\frac{Ph^2}{\mu}$$

it follows that

$$\underline{\underline{\frac{\partial p}{\partial y}=-\frac{3\mu V}{h^2}-\rho g}}$$

6.78

6.78 A fluid of density ρ flows steadily *downward* between the two vertical infinite, parallel plates shown in the figure for Problem 6.77. The flow is fully developed and laminar. Make use of the Navier–Stokes equation to determine the relationship between the discharge and the other parameters involved, for the case in which the change in pressure along the channel is zero.

See solution for Problem 6.83 to obtain

$$q = -\frac{2}{3}\frac{\mathcal{P}h^3}{\mu}$$

where q is the discharge per unit width and $\mathcal{P} = \frac{\partial p}{\partial y} + \rho g$. Thus,

$$\frac{\partial p}{\partial y} + \rho g = -\frac{3}{2}\frac{\mu q}{h^3}$$

or

$$\frac{\partial p}{\partial y} = -\frac{3}{2}\frac{\mu q}{h^3} - \rho g$$

For $\frac{\partial p}{\partial y} = 0$

$$\underline{\underline{q = -\frac{2}{3}\frac{\rho g h^3}{\mu}}}$$

(Note: The negative sign indicates that the direction of flow must be <u>downward</u> to create a zero pressure gradient.)

6.79

6.79 Due to the no-slip condition, as a solid is pulled out of a viscous liquid some of the liquid is also pulled along as described in Example 6.9 and shown in Video V6.5. Based on the results given in Example 6.9, show on a dimensionless plot the velocity distribution in the fluid film (v/V_0 vs. x/h) when the average film velocity, V, is 10% of the belt velocity, V_0.

From Example 6.9, the average velocity is given by the equation

$$V = V_0 - \frac{\gamma h^2}{3\mu} \qquad (1)$$

with the velocity distribution

$$v = \frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + V_0 \qquad (2)$$

If $V = 0.1 V_0$, then from Eq. (1)

$$0.1 V_0 = V_0 - \frac{\gamma h^2}{3\mu}$$

or

$$V_0 = \frac{\gamma h^2}{2.7\mu} \qquad (3)$$

In dimensionless form Eq. (2) becomes

$$\frac{v}{V_0} = \frac{\gamma h^2}{2\mu V_0}\left(\frac{x}{h}\right)^2 - \frac{\gamma h^2}{\mu V_0}\left(\frac{x}{h}\right) + 1 \qquad (4)$$

From Eq. (3)

$$\frac{\gamma h^2}{\mu V_0} = 2.7$$

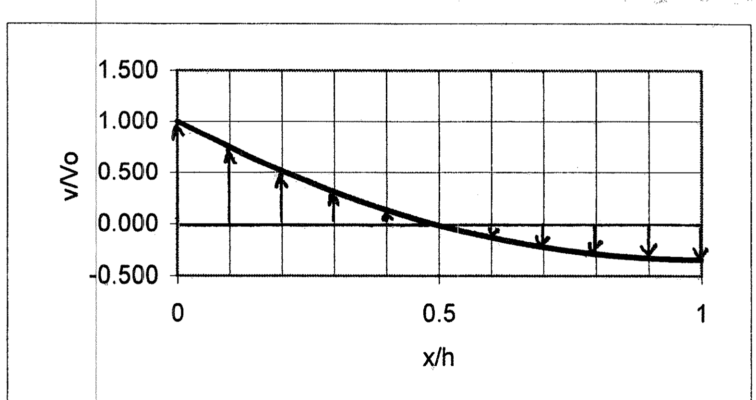
and Eq. (4) can be written as

$$\frac{v}{V_0} = 1.35\left(\frac{x}{h}\right)^2 - 2.7\left(\frac{x}{h}\right) + 1 \qquad (5)$$

A plot of the velocity distribution is shown below.

x/h	v/Vo
0	1.000
0.1	0.744
0.2	0.514
0.3	0.312
0.4	0.136
0.5	-0.013
0.6	-0.134
0.7	-0.229
0.8	-0.296
0.9	-0.337
1	-0.350

Calculated from Eq. (5)



6.80

6.80 An incompressible, viscous fluid is placed between horizontal, infinite, parallel plates as is shown in Fig. P6.80. The two plates move in opposite directions with constant velocities, U_1 and U_2, as shown. The pressure gradient in the x direction is zero and the only body force is due to the fluid weight. Use the Navier–Stokes equations to derive an expression for the velocity distribution between the plates. Assume laminar flow.

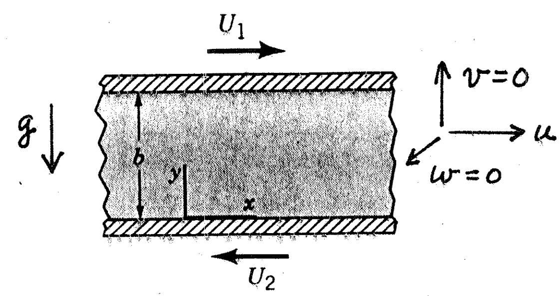


FIGURE P6.80

For the specified conditions, $v=0$, $w=0$, $\frac{\partial p}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) reduces to

$$\frac{d^2u}{dy^2}=0 \qquad (1)$$

Integration of Eq. (1) yields

$$u = C_1 y + C_2 \qquad (2)$$

For $y=0$, $u=-U_2$ and therefore from Eq. (2)

$$C_2 = -U_2$$

For $y=b$, $u=U_1$ so that

$$U_1 = C_1 b - U_2$$

or

$$C_1 = \frac{U_1+U_2}{b}$$

Thus,

$$\underline{\underline{u = \left(\frac{U_1+U_2}{b}\right) y - U_2}}$$

6.81

6.81 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P6.81). The bottom plate is fixed and the upper plate moves with a constant velocity U. Determine the velocity at the interface. Express your answer in terms of U, μ_1, and μ_2. The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress is continuous across the interface between the two fluids. Assume laminar flow.

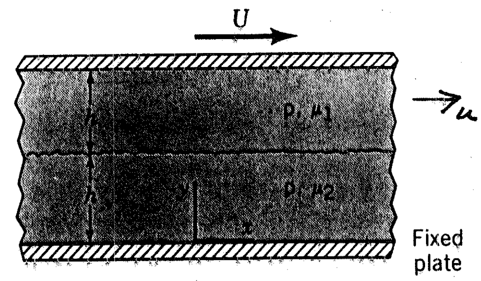


FIGURE P6.81

For the specified conditions, $v=0$, $w=0$, $\frac{\partial p}{\partial x}=0$, and $g_x=0$, so that the x-component of the Navier-Stokes equations (Eq. 6.127a) for either the upper or lower layer reduces to

$$\frac{d^2u}{dy^2}=0 \qquad (1)$$

Integration of Eq. (1) yields

$$u = Ay + B$$

which gives the velocity distribution in either layer.
In the upper layer at $y=2h$, $u=U$ so that

$$B_1 = U - A_1(2h)$$

where the subscript 1 refers to the upper layer.
For the lower layer at $y=0$, $u=0$ so that

$$B_2 = 0$$

where the subscript 2 refers to the lower layer. Thus,

$$u_1 = A_1(y-2h)+U$$

and

$$u_2 = A_2 y$$

At $y=h$, $u_1=u_2$ so that

$$A_1(h-2h)+U = A_2 h$$

or

$$A_2 = -A_1 + \frac{U}{h} \qquad (2)$$

(con't)

6.81 (con't)

Since the velocity distribution is linear in each layer the shearing stress

$$\tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{du}{dy}$$

is constant throughout each layer. For the upper layer

$$\tau_1 = \mu_1 A_1$$

and for the lower layer

$$\tau_2 = \mu_2 A_2$$

At the interface $\tau_1 = \tau_2$ so that

$$\mu_1 A_1 = \mu_2 A_2$$

or

$$\frac{A_1}{A_2} = \frac{\mu_2}{\mu_1} \qquad (3)$$

Substitution of Eq. (3) into Eq. (2) yields

$$A_2 = -\frac{\mu_2}{\mu_1} A_2 + \frac{U}{h}$$

or

$$A_2 = \frac{U/h}{1 + \mu_2/\mu_1}$$

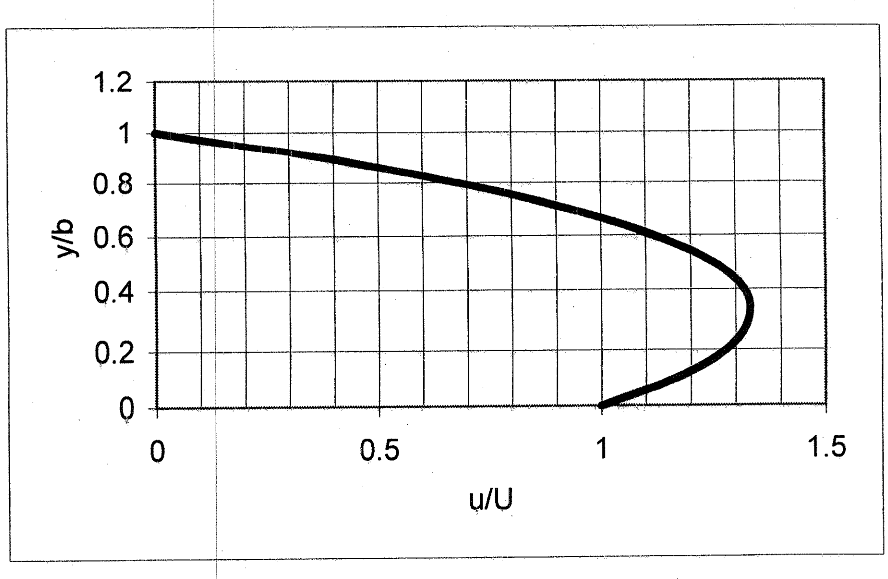
Thus, velocity at the interface is

$$u_2 (y = h) = A_2 h = \underline{\underline{\frac{U}{1 + \frac{\mu_2}{\mu_1}}}}$$

6.82 (con't)

u/U	y/b
1	0
1.17	0.1
1.28	0.2
1.33	0.3
1.32	0.4
1.25	0.5
1.12	0.6
0.93	0.7
0.68	0.8
0.37	0.9
0	1

Calculated from Eq. (2) with P =3.



To determine where the maximum velocity occurs differentiate Eq. (2) and set equal to zero. Thus,

$$\frac{d(u/U)}{dy} = -P\left[2\left(\frac{y}{b^2}\right) - \frac{1}{b}\right] - \frac{1}{b} = 0$$

and with $P = 3$

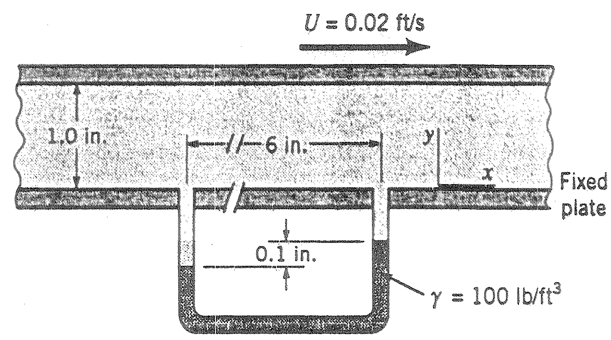
$$\frac{d(u/U)}{dy} = -3\left[\frac{1}{b}\left(2\frac{y}{b} - 1\right)\right] - \frac{1}{b} = 0$$

so that

$$\underline{\underline{\frac{y}{b} = \frac{1}{3}}}$$

6.83

6.83 A viscous fluid (specific weight = 80 lb/ft^3; viscosity = 0.03 lb · s/ft^2) is contained between two infinite, horizontal parallel plates as shown in Fig. P6.83. The fluid moves between the plates under the action of a pressure gradient, and the upper plate moves with a velocity U while the bottom plate is fixed. A U-tube manometer connected between two points along the bottom indicates a differential reading of 0.1 in. If the upper plate moves with a velocity of 0.02 ft/s, at what distance from the bottom plate does the maximum velocity in the gap between the two plates occur? Assume laminar flow.



■ FIGURE P6.83

$$u = U\frac{y}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by) \qquad \text{(Eq. 6.140)}$$

Maximum velocity will occur at distance y_m where $\frac{du}{dy} = 0$.

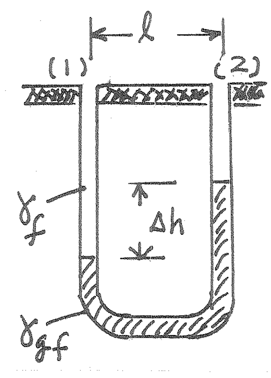
Thus,

$$\frac{du}{dy} = \frac{U}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(2y - b)$$

and for $\frac{du}{dy} = 0$

$$y_m = -\frac{\mu U}{b\left(\frac{\partial p}{\partial x}\right)} + \frac{b}{2} \qquad (1)$$

For manometer (see figure to right),



$$p_1 + \gamma_f\,\Delta h - \gamma_{gf}\,\Delta h = p_2$$

or

$$p_1 - p_2 = (\gamma_{gf} - \gamma_f)\,\Delta h$$

$$= \left(100\,\frac{\text{lb}}{\text{ft}^3} - 80\,\frac{\text{lb}}{\text{ft}^3}\right)\left(\frac{0.1\text{ in.}}{12\,\frac{\text{in.}}{\text{ft}}}\right) = 0.167\,\frac{\text{lb}}{\text{ft}^2}$$

Also,

$$-\frac{\partial p}{\partial x} = \frac{p_1 - p_2}{\ell} = \frac{0.167\,\frac{\text{lb}}{\text{ft}^2}}{\left(\frac{6\text{ in.}}{12\,\frac{\text{in.}}{\text{ft}}}\right)} = 0.334\,\frac{\text{lb}}{\text{ft}^3}$$

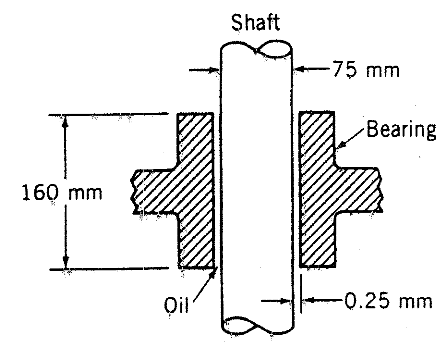
Thus, from Eq. (1)

$$y_m = -\frac{\left(0.03\,\frac{\text{lb}\cdot\text{s}}{\text{ft}^2}\right)\left(0.02\,\frac{\text{ft}}{\text{s}}\right)}{\left(\frac{1.0\text{ in.}}{12\,\frac{\text{in.}}{\text{ft}}}\right)\left(-0.334\,\frac{\text{lb}}{\text{ft}^3}\right)} + \frac{\frac{1.0\text{ in.}}{12\,\frac{\text{in.}}{\text{ft}}}}{2}$$

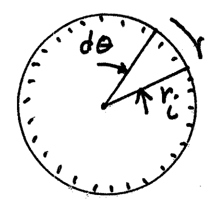
$$= 0.0632\text{ ft}\left(\frac{12\text{ in.}}{\text{ft}}\right) = \underline{\underline{0.759\text{ in.}}}$$

6.84

6.84 A vertical shaft passes through a bearing and is lubricated with an oil having a viscosity of 0.2 N·s/m^2 as shown in Fig. P6.84. Assume that the flow characteristics in the gap between the shaft and bearing are the same as those for laminar flow between infinite parallel plates with zero pressure gradient in the direction of flow. Estimate the torque required to overcome viscous resistance when the shaft is turning at 80 rev/min.



■ FIGURE P6.84



$dF = \tau\, dA$

$= \tau r_i \ell\, d\theta$

$\ell \sim$ shaft length

The torque due to force dF acting on a differential area, $dA = r_i \ell\, d\theta$, is (see figure at right)

$$d\mathcal{T} = r_i\, dF = r_i^2 \tau \ell\, d\theta$$

where τ is the shearing stress. Thus,

$$\mathcal{T} = r_i^2 \tau \ell \int_0^{2\pi} d\theta = 2\pi r_i^2 \tau \ell \qquad (1)$$

In the gap,

$$u = U\frac{y}{b} \qquad \text{(Eq. 6.142)}$$

where $U = r_i \omega$ and b is the gap width. Also,

$$\tau = \mu \frac{du}{dy} = \frac{\mu U}{b}$$

Thus, from Eq. (1)

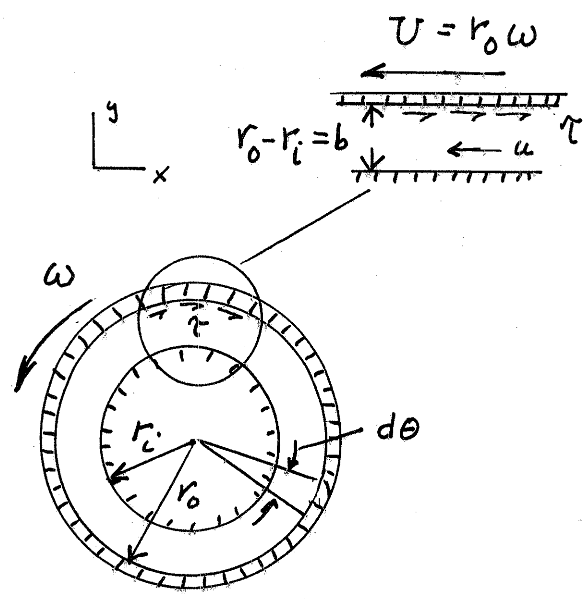
$$\mathcal{T} = 2\pi r_i^2 \left(\mu \frac{U}{b}\right)\ell = 2\pi r_i^3 \mu \omega \frac{\ell}{b}$$

$$= 2\pi \left(\frac{0.075}{2}\,m\right)^3 \left(0.2\,\frac{N\cdot s}{m^2}\right)\left[\left(80\,\frac{rev}{min}\right)\left(\frac{2\pi\ rad}{rev}\right)\left(\frac{min}{60\,s}\right)\right]\frac{(0.160\,m)}{(0.25\times 10^{-3}\,m)}$$

$$= \underline{\underline{0.355\ N\cdot m}}$$

6.85

6.85 A viscous fluid is contained between two long concentric cylinders. The geometry of the system is such that the flow between the cylinders is approximately the same as the laminar flow between two infinite parallel plates. **(a)** Determine an expression for the torque required to rotate the outer cylinder with an angular velocity ω. The inner cylinder is fixed. Express your answer in terms of the geometry of the system, the viscosity of the fluid, and the angular velocity. **(b)** For a small rectangular element located at the fixed wall determine an expression for the rate of angular deformation of this element. (See Video V6.1 and Fig. P6.9.)



$\ell \sim$ cylinder length
$\tau \sim$ shearing stress

(a) The torque which must be applied to outer cylinder to overcome the force due to the shearing stress is (see figure)

$$d\mathcal{T} = r_o\, dF = r_o\,(\tau\, r_o\, \ell\, d\theta) = r_o^2\, \tau\, \ell\, d\theta$$

so that

$$\mathcal{T} = r_o^2\, \tau\, \ell \int_0^{2\pi} d\theta = 2\pi r_o^2\, \tau\, \ell \qquad (1)$$

In the gap

$$u = U\frac{y}{b} \qquad (\text{Eq. } 6.142)$$

Since,

$$\tau = \mu\frac{du}{dy} = \frac{\mu U}{b}$$

and $b = r_o - r_i$, $U = r_o\omega$ (see figure), it follows from Eq. (1) that

$$\mathcal{T} = 2\pi r_o^2\left(\frac{\mu\, r_o\,\omega}{r_o - r_i}\right)\ell = \underline{\underline{\frac{2\pi r_o^3\,\mu\,\omega\,\ell}{r_o - r_i}}}$$

(Con't)

6.85 (con't)

(b) From Eq. 6.18

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

For the linear distribution

$$u = -\frac{r_o \omega}{r_o - r_i}\, y = -\frac{U y}{b}$$

so that

$$\frac{\partial u}{\partial y} = -\frac{U}{b}$$

and s $v = 0$

$$\dot{\gamma} = -\frac{U}{b}$$

The negative sign indicates that the original right angle shown in Fig. P6.9b is increasing.

6.86*

*6.86 Oil (SAE 30) flows between parallel plates spaced 5 mm apart. The bottom plate is fixed but the upper plate moves with a velocity of 0.2 m/s in the positive x direction. The pressure gradient is 60 kPa/m, and is negative. Compute the velocity at various points across the channel and show the results on a plot. Assume laminar flow.

The velocity distribution is given by the equation

$$u = U\frac{y}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - by\right) \qquad \text{(Eq. 6.140)}$$

and for the given data,

$$u = \frac{\left(0.2\,\frac{m}{s}\right)}{(0.005\,m)}\,y + \frac{1}{2\left(0.38\,\frac{N\cdot s}{m^2}\right)}\left(-60\times10^3\,\frac{N}{m^3}\right)\left[y^2 - (0.005\,m)\,y\right]$$

so that

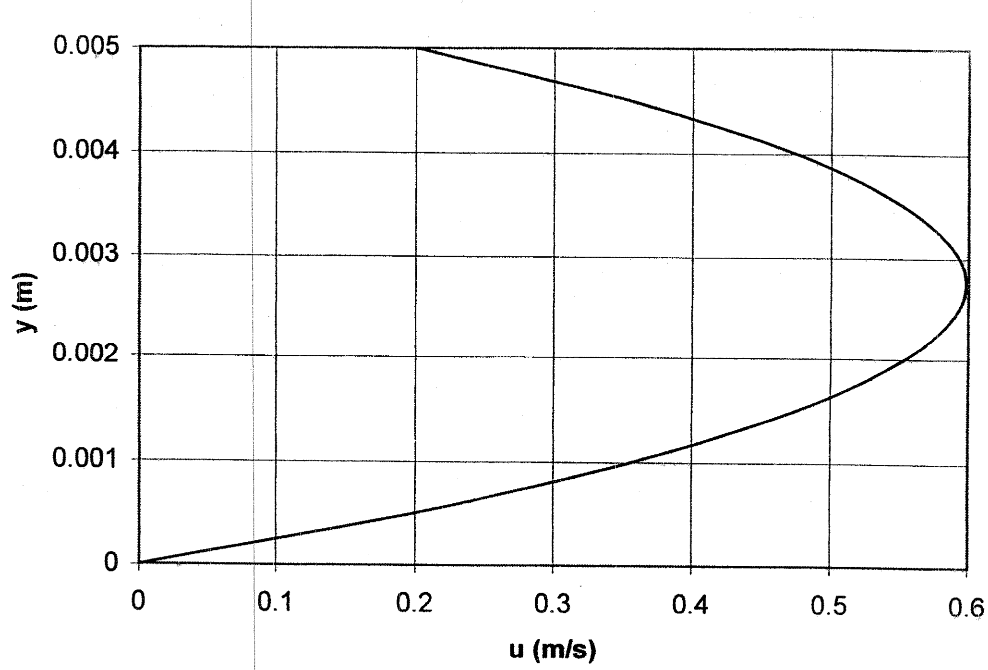
$$u = 40\,y + 7.89\times10^4\left(0.005\,y - y^2\right) \qquad (1)$$

with u in m/s when y is in m.

Tabulated data and a plot of the data are given below.

y, m	u, m/s
0	0
0.0005	0.1975
0.0010	0.3556
0.0015	0.4742
0.0020	0.5534
0.0025	0.5931
0.0030	0.5934
0.0035	0.5542
0.0040	0.4756
0.0045	0.3575
0.0050	0.2000

Calculated from Eq. (1)



6.87

6.87 Consider a steady, laminar flow through a straight horizontal tube having the constant elliptical cross section given by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The streamlines are all straight and parallel. Investigate the possibility of using an equation for the z component of velocity of the form

$$w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

as an exact solution to this problem. With this velocity distribution what is the relationship between the pressure gradient along the tube and the volume flowrate through the tube?

From the description of the problem, $u=0$, $v=0$, $g_z=0$, $w \neq f(t)$, and the continuity equation indicates that $\frac{\partial w}{\partial z}=0$. With these conditions the z-component of the Navier-Stokes equations (Eq. 6.127c) reduces to

$$\frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad (1)$$

Due to the no-slip boundary condition, $w=0$ on the elliptical boundary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Thus, the proposed velocity distribution satisfies this condition since on the boundary

$$w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = A\left[1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right] = A\left[1 - (1)\right] = 0$$

This result indicates that the proposed velocity distribution can be used as a solution. Substitution of the velocity distribution into Eq. (1) gives the relationship between the pressure gradient, $\frac{\partial p}{\partial z}$, and the velocity. Since,

$$\frac{\partial^2 w}{\partial x^2} = -\frac{2A}{a^2} \qquad \frac{\partial^2 w}{\partial y^2} = -\frac{2A}{b^2}$$

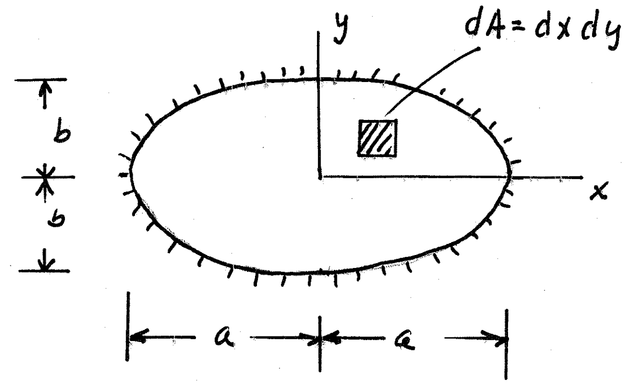
it follows that

$$\frac{\partial p}{\partial z} = -2A\mu\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \qquad (2)$$

(con't)

6.87 (con't)

The volume flowrate, Q, through the tube is given by the equation



$$Q = \int_{area} w\, dA$$

$$= 4\int_0^b \int_0^{a\sqrt{1-\frac{y^2}{b^2}}} w\, dx\, dy$$

Thus,

$$Q = 4A \int_0^b \int_0^{a\sqrt{1-\frac{y^2}{b^2}}} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\, dy$$

$$= 4A \int_0^b \left[x - \frac{x^3}{3a^2} - \frac{y^2}{b^2}x\right]_0^{a\sqrt{1-\frac{y^2}{b^2}}} dy$$

$$= 4A \int_0^b \left[a\sqrt{1-\frac{y^2}{b^2}}\left(1-\frac{y^2}{b^2}\right) - \frac{1}{3}a\sqrt{1-\frac{y^2}{b^2}}\left(1-\frac{y^2}{b^2}\right)\right] dy$$

$$= \frac{8Aa}{3}\int_0^b \left(1 - \frac{y^2}{b^2}\right)^{3/2} dy = \frac{8Aa}{3}\left(\frac{3b\pi}{16}\right) = \frac{A\pi ab}{2}$$

and therefore

$$A = \frac{2Q}{\pi ab}$$

From Eq. (2)

$$\frac{\partial p}{\partial z} = -\frac{4\mu Q}{\pi ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

6.88

6.88 A fluid is initially at rest between two horizontal, infinite, parallel plates. A constant pressure gradient in a direction parallel to the plates is suddenly applied and the fluid starts to move. Determine the appropriate differential equation(s), initial condition, and boundary conditions that govern this type of flow. You need not solve the equation(s).

Differential equations are the same as Eqs. 6.129, 6.130, and 6.131 except that $\frac{\partial u}{\partial t} \neq 0$ (since the flow is unsteady). Thus, Eq. 6.129 must include the local acceleration term, $\frac{\partial u}{\partial t}$, and the governing differential equations are:

(x-direction) $$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$ (with $\frac{\partial p}{\partial x}$ = constant)

(y-direction) $$0 = -\frac{\partial p}{\partial y} - \rho g$$

(z-direction) $$0 = -\frac{\partial p}{\partial z}$$

Initial condition: $u = 0$ for $t = 0$ for all y.

Boundary conditions: $u = 0$ for $y = \pm h$ for $t \geq 0$.

6.89

6.89 It is known that the velocity distribution for steady, laminar flow in circular tubes (either horizontal or vertical) is parabolic. (See Video V6.6.) Consider a 10-mm diameter horizontal tube through which ethyl alcohol is flowing with a steady mean velocity 0.15 m/s. (a) Would you expect the velocity distribution to be parabolic in this case? Explain. (b) What is the pressure drop per unit length along the tube?

(a) Check Reynolds number to determine if flow is laminar:

$$Re = \frac{\rho V (2R)}{\mu} = \frac{\left(789 \frac{kg}{m^3}\right)\left(0.15 \frac{m}{s}\right)(0.010\,m)}{1.19 \times 10^{-3} \frac{N \cdot s}{m^2}} = 995 < 2100$$

Thus, the flow is laminar and velocity distribution would be parabolic. $\underline{\underline{\text{Yes.}}}$

(b) Since the flow is laminar

$$V = \frac{R^2}{8\mu} \frac{\Delta p}{\ell} \qquad \text{(Eq. 6.152)}$$

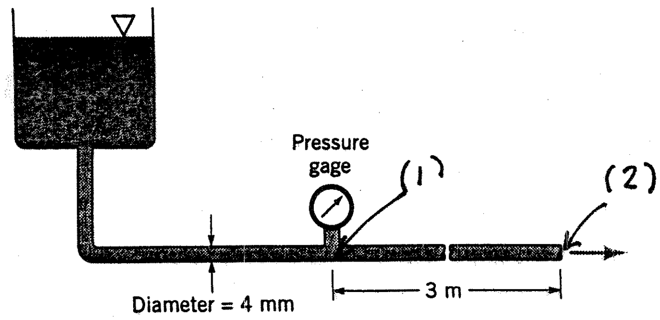
so that

$$\frac{\Delta p}{\ell} = \frac{8 \mu V}{R^2} = \frac{8 \left(1.19 \times 10^{-3} \frac{N \cdot s}{m^2}\right)\left(0.15 \frac{m}{s}\right)}{\left(\frac{0.010\,m}{2}\right)^2}$$

$$= \underline{\underline{57.1 \frac{N}{m^2} \text{ per m}}}$$

6.90

6.90 A simple flow system to be used for steady flow tests consists of a constant head tank connected to a length of 4-mm-diameter tubing as shown in Fig. P6.90. The liquid has a viscosity of 0.015 N · s/m², a density of 1200 kg/m³, and discharges into the atmosphere with a mean velocity of 2 m/s. **(a)** Verify that the flow will be laminar. **(b)** The flow is fully developed in the last 3 m of the tube. What is the pressure at the pressure gage? **(c)** What is the magnitude of the wall shearing stress, τ_{rz}, in the fully developed region?



■ FIGURE P6.90

(a) Check Reynolds number to determine if flow is laminar:

$$Re = \frac{\rho V(2R)}{\mu} = \frac{(1200\ \frac{kg}{m^3})(2\ \frac{m}{s})(0.004\,m)}{0.015\ \frac{N \cdot s}{m^2}} = 640$$

Since the Reynolds number is well below 2100 the flow is laminar.

(b) For laminar flow,

$$V = \frac{R^2}{8\mu}\frac{\Delta p}{\ell} \qquad (\text{Eq. } 6.152)$$

Since $\Delta p = p_1 - p_2 = p_1 - 0$ (see figure)

$$p_1 = \frac{8\mu V \ell}{R^2} = \frac{8(0.015\ \frac{N \cdot s}{m^2})(2\ \frac{m}{s})(3\,m)}{(\frac{0.004}{2}\,m)^2} = \underline{\underline{180\ kPa}}$$

(c)

$$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad (\text{Eq. } 6.126f)$$

For fully developed pipe flow, $v_r = 0$, so that

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

Also,

$$v_z = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (\text{Eq. } 6.154)$$

and with $v_{max} = 2V$, where V is the mean velocity

$$\tau_{rz} = 2V\mu\left(-\frac{2r}{R^2}\right)$$

Thus, at the wall, $r = R$,

$$\left|(\tau_{rz})_{wall}\right| = \left|-\frac{4V\mu}{R}\right| = \left|-\frac{4(2\ \frac{m}{s})(0.015\ \frac{N \cdot s}{m^2})}{(\frac{0.004}{2}\,m)}\right| = \underline{\underline{60.0\ \frac{N}{m^2}}}$$

6.91

6.91 A highly viscous Newtonian liquid (ρ = 1,300 kg/m^3; μ = 6.0 N · s/m^2) is contained in a long, vertical, 150-mm diameter tube. Initially the liquid is at rest but when a valve at the bottom of the tube is opened flow commences. Although the flow is slowly changing with time, at any instant the velocity distribution is parabolic, that is, the flow is quasi-steady. (See Video V6.6.) Some measurements show that the average velocity, V, is changing in accordance with the equation $V = 0.1\,t$, with V in m/s when t is in seconds. (a) Show on a plot the velocity distribution (v_z vs. r) at t = 2 s, where v_z is the velocity and r is the radius from the center of the tube. (b) Verify that the flow is laminar at this instant.

(a) For parabolic velocity distribution

$$\frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2 \qquad \text{(Eq. 6.154)}$$

Since $v_{max} = 2V$

$$v_z = 2V\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (1)$$

With $V = 0.1t$, at $t = 2\,s$ $V = 0.2\,\frac{m}{s}$ and $R = \frac{150\,mm}{2} = 75\,mm$. Thus, Eq. (1) becomes

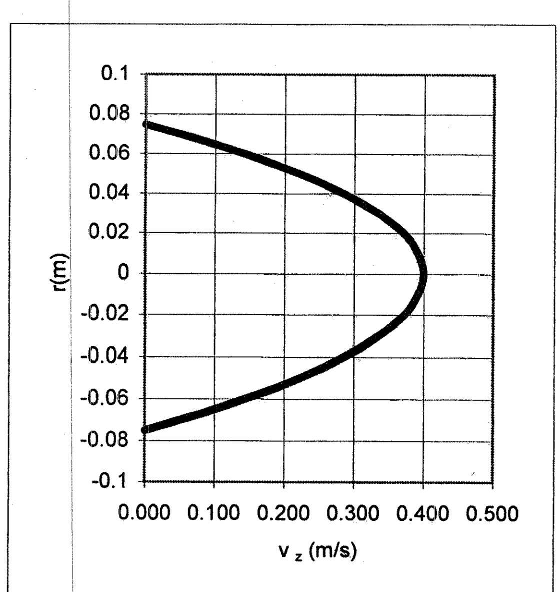
$$v_z = 2\left(0.2\,\frac{m}{s}\right)\left[1 - \frac{r^2}{(0.075\,m)^2}\right]$$

and

$$v_z = 0.4\left(1 - 178\,r^2\right)$$

A plot of this velocity distribution is shown below.

v_z (m/s)	r(m)
0.000	0.075
0.100	0.065
0.185	0.055
0.256	0.045
0.313	0.035
0.356	0.025
0.384	0.015
0.400	0
0.384	-0.015
0.356	-0.025
0.313	-0.035
0.256	-0.045
0.256	-0.045
0.185	-0.055
0.100	-0.065
0.000	-0.075



(b)
$$Re = \frac{\rho V D}{\mu} = \frac{\left(1300\,\frac{kg}{m^3}\right)\left(0.2\,\frac{m}{s}\right)\left(0.150\,m\right)}{6.0\,\frac{N\,s}{m^2}}$$

$$= 6.5 \ll 2100 \qquad \text{(Flow is laminar)}$$

6.92

6.92 **(a)** Show that for Poiseuille flow in a tube of radius R the magnitude of the wall shearing stress, τ_{rz}, can be obtained from the relationship

$$|(\tau_{rz})_{\text{wall}}| = \frac{4\mu Q}{\pi R^3}$$

for a Newtonian fluid of viscosity μ. The volume rate of flow is Q. **(b)** Determine the magnitude of the wall shearing stress for a fluid having a viscosity of 0.004 N·s/m² flowing with an average velocity of 130 mm/s in a 2-mm-diameter tube.

(a) $$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad \text{(Eq. 6.126f)}$$

For Poiseuille flow in a tube, $v_r = 0$, and therefore

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

Since,

$$v_z = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad \text{(Eq. 6.154)}$$

and $v_{max} = 2V$, where V is the mean velocity, it follows that

$$\frac{\partial v_z}{\partial r} = -\frac{4Vr}{R^2}$$

Thus, at the wall ($r=R$),

$$(\tau_{rz})_{wall} = -\frac{4\mu V}{R}$$

and with $Q = \pi R^2 V$

$$|(\tau_{rz})_{wall}| = \underline{\underline{\frac{4\mu Q}{\pi R^3}}}$$

(b) $$|(\tau_{rz})_{wall}| = \frac{4\mu V}{R} = \frac{4\left(0.004 \frac{N\cdot s}{m^2}\right)\left(0.130 \frac{m}{s}\right)}{\left(\frac{0.002}{2} m\right)}$$

$$= \underline{\underline{2.08 \text{ Pa}}}$$

6.93

6.93 An incompressible, Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Fig. P6.93. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric, and fully developed.

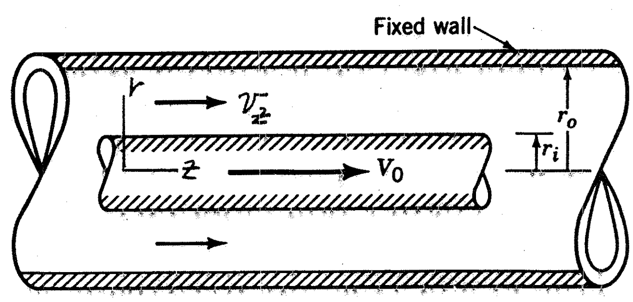


FIGURE P6.93

Equation 6.147, which was developed for flow in circular tubes, applies in the annular region. Thus,

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r^2 + c_1 \ln r + c_2 \qquad (1)$$

With boundary conditions, $r = r_o$, $v_z = 0$, and $r = r_i$, $v_z = V_0$, it follows that:

$$0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r_o^2 + c_1 \ln r_o + c_2 \qquad (2)$$

$$V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r_i^2 + c_1 \ln r_i + c_2 \qquad (3)$$

Subtract Eq. (2) from Eq. (3) to obtain

$$V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_i^2 - r_o^2\right) + c_1 \ln \frac{r_i}{r_o}$$

so that

$$c_1 = \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_i^2 - r_o^2\right)}{\ln \frac{r_i}{r_o}}$$

The drag on the inner cylinder will be zero if

$$\left(\tau_{rz}\right)_{r=r_i} = 0$$

Since,

$$\tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad \text{(Eq. 6.126f)}$$

and with $v_r = 0$, it follows that

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r}$$

(con't)

6.93 (Con't)

Differentiate Eq.(1) with respect to r to obtain

$$\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right) r + \frac{c_1}{r}$$

so that at $r = r_i$

$$\left(\tau_{rz}\right)_{r=r_i} = \mu\left[\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right) r_i + \frac{V_o - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_i^2 - r_o^2\right)}{r_i \ln \frac{r_i}{r_o}}\right]$$

Thus, in order for the drag to be zero,

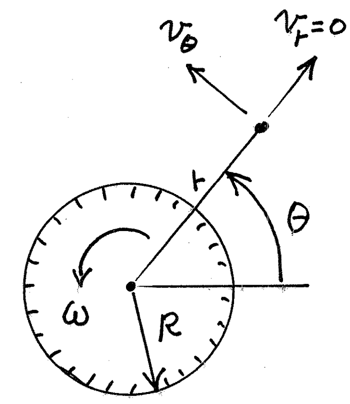
$$\frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right) r_i + \frac{V_o - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_i^2 - r_o^2\right)}{r_i \ln \frac{r_i}{r_o}} = 0$$

or

$$V_o = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[2 r_i^2 \ln \frac{r_i}{r_o} - \left(r_i^2 - r_o^2\right)\right]$$

6.94

6.94 An infinitely long, solid, vertical cylinder of radius R is located in an infinite mass of an incompressible fluid. Start with the Navier–Stokes equation in the θ direction and derive an expression for the velocity distribution for the steady flow case in which the cylinder is rotating about a fixed axis with a constant angular velocity ω. You need not consider body forces. Assume that the flow is axisymmetric and the fluid is at rest at infinity.



For this flow field, $v_r = 0$, $v_z = 0$, and from the continuity equation,

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(Eq. 6.35)}$$

it follows that

$$\frac{\partial v_\theta}{\partial \theta} = 0 \qquad \text{(See figure for notation.)}$$

Thus, the Navier-Stokes equation in the θ-direction (Eq. 6.128b) for steady flow reduces to

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2}\right]$$

Due to the symmetry of the flow,

$$\frac{\partial p}{\partial \theta} = 0$$

so that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} = 0$$

or

$$\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} = 0 \qquad (1)$$

Since v_θ is a function of only r, Eq. (1) can be expressed as an ordinary differential equation, and re-written as

$$\frac{d^2 v_\theta}{dr^2} + \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0 \qquad (2)$$

Equation (2) can be integrated to yield

$$\frac{d v_\theta}{dr} + \frac{v_\theta}{r} = C_1$$

or

$$r\frac{d v_\theta}{dr} + v_\theta = C_1 r \qquad (3)$$

(con't)

6.94 (con't)

Equation (3) can be expressed as

$$\frac{d\,(r v_\theta)}{dr} = c_1 r$$

and a second integration yields

$$r v_\theta = \frac{c_1 r^2}{2} + c_2$$

or

$$v_\theta = \frac{c_1 r}{2} + \frac{c_2}{r}$$

As $r \to \infty$, $v_\theta \to 0$, (since fluid is at rest at infinity) so that $c_1 = 0$. Thus,

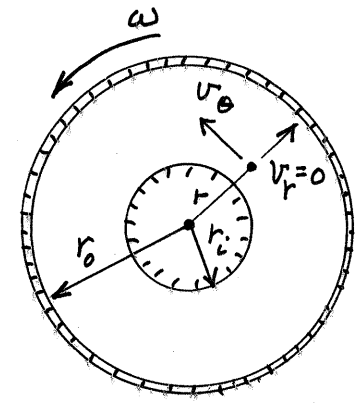
$$v_\theta = \frac{c_2}{r}$$

and since at $r = R$, $v_\theta = R\omega$, it follows that $c_2 = R^2\omega$ and

$$\underline{\underline{v_\theta = \frac{R^2\omega}{r}}}$$

6.95

6.95 A viscous fluid is contained between two infinitely long vertical concentric cylinders. The outer cylinder has a radius r_o and rotates with an angular velocity ω. The inner cylinder is fixed and has a radius r_i. Make use of the Navier–Stokes equations to obtain an exact solution for the velocity distribution in the gap. Assume that the flow in the gap is axisymmetric (neither velocity nor pressure are functions of angular position θ within gap) and that there are no velocity components other than the tangential component. The only body force is the weight.



The velocity distribution in the annular space is given by the equation

$$v_\theta = \frac{C_1 r}{2} + \frac{C_2}{r} \qquad (1)$$

(See solution to Problem 6.94 for derivation.)

With the boundary conditions $r = r_i$, $v_\theta = 0$, and $r = r_o$, $v_\theta = r_o\omega$ (see figure for notation), it follows from Eq. (1) that:

$$0 = \frac{C_1 r_i}{2} + \frac{C_2}{r_i}$$

$$r_o\omega = \frac{C_1 r_o}{2} + \frac{C_2}{r_o}$$

Therefore,

$$C_1 = \frac{2\omega}{1 - \frac{r_i^2}{r_o^2}}$$

and

$$C_2 = \frac{-r_i^2\omega}{1 - \frac{r_i^2}{r_o^2}}$$

so that

$$v_\theta = \frac{r\omega}{1 - \frac{r_i^2}{r_o^2}} - \frac{r_i^2\omega}{r\left(1 - \frac{r_i^2}{r_o^2}\right)}$$

or

$$v_\theta = \frac{r\omega}{\left(1 - \frac{r_i^2}{r_o^2}\right)}\left[1 - \frac{r_i^2}{r^2}\right]$$

Master Typing Sheet
10% Reduction
8 1/2 x 11 trim size

6.96

6.96 For flow between concentric cylinders, with the outer cylinder rotating at an angular velocity ω and the inner cylinder fixed, it is commonly assumed that the tangential velocity (v_θ) distribution in the gap between the cylinders is linear. Based on the exact solution to this problem (see Problem 6.95) the velocity distribution in the gap is not linear. For an outer cylinder with radius $r_o = 2.00$ in. and an inner cylinder with radius $r_i = 1.80$ in., show, with the aid of a plot, how the dimensionless velocity distribution, $v_\theta/r_o\omega$, varies with the dimensionless radial position, r/r_o, for the exact and approximate solutions.

For a linear velocity distribution (approximate solution)

$$v_\theta = (r_o\omega)\left(\frac{r-r_i}{r_o-r_i}\right)$$

and in nondimensional form

$$\frac{v_\theta}{r_o\omega} = \frac{\frac{r}{r_o} - \frac{r_i}{r_o}}{1 - \frac{r_i}{r_o}} \qquad (1)$$

For the exact solution (see Problem 6.95)

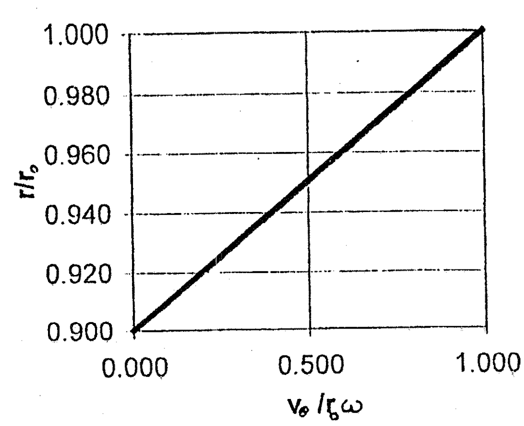
$$v_\theta = \frac{r\omega}{\left(1 - \frac{r_i^2}{r_o^2}\right)}\left[1 - \frac{r_i^2}{r^2}\right]$$

and in nondimensional form

$$\frac{v_\theta}{r_o\omega} = \frac{\frac{r}{r_o}}{\left(1 - \frac{r_i^2}{r_o^2}\right)}\left[1 - \frac{r_i^2}{r_o^2}\left(\frac{r}{r_o}\right)^{-2}\right] \qquad (2)$$

For $r_i = 1.80$ in. and $r_o = 2.00$ in., some tabulated values and a graph are shown below. Note that there is little difference between the exact and approximate solutions for this small gap width. For all practical purposes both solutions fall on the single curve shown.

Linear	Exact	
$v_\theta/r_o\omega$	$v_\theta/r_o\omega$	r/r_o
0.000	0.000	0.900
0.125	0.131	0.913
0.250	0.260	0.925
0.375	0.387	0.938
0.500	0.512	0.950
0.625	0.637	0.963
0.750	0.759	0.975
0.875	0.880	0.988
1.000	1.000	1.000



6.97

6.97 A viscous liquid ($\mu = 0.012 \text{ lb}\cdot\text{s/ft}^2$, $\rho = 1.79 \text{ slugs/ft}^3$) flows through the annular space between two horizontal, fixed, concentric cylinders. If the radius of the inner cylinder is 1.5 in. and the radius of the outer cylinder is 2.5 in., what is the pressure drop along the axis of the annulus per foot when the volume flowrate is $0.14 \text{ ft}^3/\text{s}$?

Check Reynolds number to determine if flow is laminar:

$$Re = \frac{\rho V D_h}{\mu}$$

where $D_h = 2(r_o - r_i)$ and $V = \dfrac{Q}{\pi(r_o^2 - r_i^2)}$.

Thus,

$$Re = \frac{2\rho Q}{\pi\mu(r_o + r_i)} = \frac{2\left(1.79 \frac{\text{slugs}}{\text{ft}^3}\right)\left(0.14 \frac{\text{ft}^3}{\text{s}}\right)}{\pi\left(0.012 \frac{\text{lb}\cdot\text{s}}{\text{ft}^2}\right)\left(\frac{2.5\text{in.} + 1.5\text{in.}}{12 \frac{\text{in.}}{\text{ft}}}\right)}$$

$$= 39.9 < 2100$$

Since the Reynolds number is well below 2100 the flow is laminar and

$$Q = \frac{\pi}{8\mu}\frac{\Delta p}{\ell}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln\frac{r_o}{r_i}}\right] \qquad \text{(Eq. 6.156)}$$

so that

$$\frac{\Delta p}{\ell} = \frac{\dfrac{8\mu Q}{\pi}}{r_o^4 - r_i^4 - \dfrac{(r_o^2 - r_i^2)^2}{\ln\frac{r_o}{r_i}}}$$

$$= \frac{8\left(0.012 \frac{\text{lb}\cdot\text{s}}{\text{ft}^2}\right)\left(0.14 \frac{\text{ft}^3}{\text{s}}\right)/\pi}{\left(\frac{2.5\text{ in.}}{12 \frac{\text{in.}}{\text{ft}}}\right)^4 - \left(\frac{1.5\text{ in.}}{12 \frac{\text{in.}}{\text{ft}}}\right)^4 - \dfrac{\left[\left(\frac{2.5\text{ in}}{12 \frac{\text{in.}}{\text{ft}}}\right)^2 - \left(\frac{1.5\text{ in}}{12 \frac{\text{in.}}{\text{ft}}}\right)^2\right]^2}{\ln\frac{2.5\text{ in.}}{1.5\text{ in.}}}}$$

$$= \underline{\underline{33.1 \frac{\text{lb}}{\text{ft}^2} \text{ per ft}}}$$

6.98 *

6.98 * Plot the velocity profile for the fluid flowing in the annular space described in Problem P6.97. Determine from the plot the radius at which the maximum velocity occurs and compare with the value predicted from Eq. 6.157.

The velocity distribution in the annulus is given by the equation

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln \frac{r_o}{r_i}} \ln \frac{r}{r_o}\right] \quad \text{(Eq. 6.155)}$$

From Problem 6.97

$$\frac{\partial p}{\partial z} = -\frac{\Delta p}{\ell} = -28.4 \, \frac{lb}{ft^3}$$

Thus, with $\mu = 0.016 \; lb\cdot s/ft^2$, $r_i = 1.5$ in., and $r_o = 2.5$ in. it follows that

$$v_z = -\frac{\left(28.4 \frac{lb}{ft^3}\right)}{4\left(0.016 \frac{lb \cdot s}{ft^2}\right)}\left[r^2 - \left(\frac{2.5}{12} ft\right)^2 + \frac{\left(\frac{1.5}{12} ft\right)^2 - \left(\frac{2.5}{12} ft\right)^2}{\ln \frac{2.5}{1.5}} \ln \frac{r}{\frac{2.5}{12} ft}\right]$$

or

$$v_z = -444\left(r^2 - 0.0434 - 0.0544 \ln \frac{r}{0.208}\right)$$

where v_z in ft/s with r in ft.

Tabulated data and a plot of the data are given below. From these data it is seen that the maximum velocity occurs at

$$r_m \approx \underline{\underline{0.165 \; ft}}$$

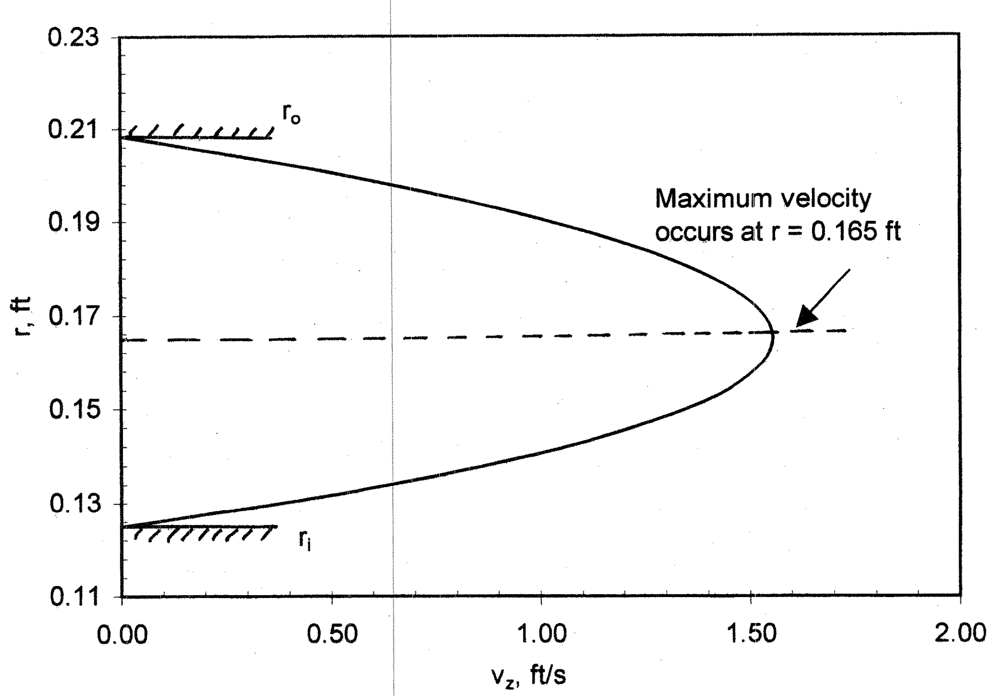
This value corresponds to the value calculated from Eq. 6.157:

$$r_m = \left[\frac{r_o^2 - r_i^2}{2 \ln \frac{r_o}{r_i}}\right]^{1/2} = \left[\frac{\left(\frac{2.5}{12} ft\right)^2 - \left(\frac{1.5}{12} ft\right)^2}{2 \ln \frac{2.5}{1.5}}\right]^{1/2} = 0.165 \; ft$$

(con't)

6.98* (con't)

v_z (ft/s)	r (ft)
0.00	0.125
0.422	0.131
0.770	0.136
1.051	0.142
1.267	0.147
1.421	0.153
1.516	0.158
1.554	0.164
1.537	0.169
1.466	0.175
1.344	0.181
1.171	0.186
0.949	0.192
0.680	0.197
0.364	0.203
0.00	0.208



6.99 *

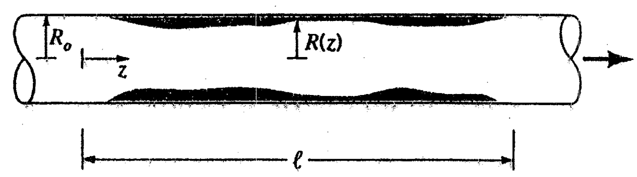
*6.99 As is shown by Eq. 6.150 the pressure gradient for laminar flow through a tube of constant radius is given by the expression:

$$\frac{\partial p}{\partial z} = -\frac{8\mu Q}{\pi R^4}$$

For a tube whose radius is changing very gradually, such as the one illustrated in Fig. P6.99, it is expected that this equation can be used to approximate the pressure change along the tube if the actual radius, $R(z)$, is used at each cross section. The following measurements were obtained along a particular tube.

z/ℓ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$R(z)/R_o$	1.00	0.73	0.67	0.65	0.67	0.80	0.80	0.71	0.73	0.77	1.00

Compare the pressure drop over the length ℓ for this nonuniform tube with one having the constant radius R_o. *Hint:* To solve this problem you will need to numerically integrate the equation for the pressure gradient given above.



■ FIGURE P6.99

From the equation given for the pressure gradient,

$$\int_{p_1}^{p_2} dp = -\int_0^{\ell} \frac{8\mu Q}{\pi [R(z)]^4} dz$$

Since $p_1 - p_2 = \Delta p$ (the pressure drop) it follows that

$$\Delta p = \frac{8\mu Q}{\pi} \int_0^{\ell} [R(z)]^{-4} dz$$

or, with $z^* = z/\ell$ and $R^* = R/R_o$,

$$\Delta p = \frac{8\mu Q \ell}{\pi R_o^4} \int_0^1 (R^*)^{-4} dz^*$$

For a constant radius tube (see Eq. 6.151),

$$\Delta p_{R=R_o} = \frac{8\mu Q \ell}{\pi R_o^4}$$

So that

$$\frac{\Delta p \text{ (nonuniform tube)}}{\Delta p \text{ (uniform tube)}} = \int_0^1 (R^*)^{-4} dz^*$$

This integral can be evaluated numerically using the trapezoidal rule, i.e.,

$$I = \frac{1}{2}\sum_{i=1}^{n-1} (y_i + y_{i+1})(x_{i+1} - x_i)$$ where

$y \sim (R^*)^{-4}$ and $x \sim z^*$.

(con't)

6.99 *

(con't)

z/ℓ	R/Ro	(R/Ro)^-4
0.0	1.00	1.00
0.1	0.73	3.52
0.2	0.67	4.96
0.3	0.65	5.60
0.4	0.67	4.96
0.5	0.80	2.44
0.6	0.80	2.44
0.7	0.71	3.94
0.8	0.73	3.52
0.9	0.77	2.84
1.0	1.00	1.00

Using the tabulated data above, the approximate value of the integral is 3.52. Thus,

$$\frac{\Delta p\ (\text{nonuniform tube})}{\Delta p\ (\text{uniform tube})} = \underline{\underline{3.52}}$$

6.100

6.100 Show how Eq. 6.155 is obtained.

From Eq. 6.147

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1 \ln r + c_2 \qquad \text{(Eq. 6.147)}$$

For flow in an annulus, $v_z = 0$ at $r = r_o$ and $v_z = 0$ at $r = r_i$. Thus, from Eq. 6.147

$$0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r_o^2 + c_1 \ln r_o + c_2$$

$$0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r_i^2 + c_1 \ln r_i + c_2$$

and solving for c_1 and c_2 we have

$$c_1 = \frac{-\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_o^2 - r_i^2\right)}{\ln\left(\frac{r_o}{r_i}\right)} \qquad (1)$$

$$c_2 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r_o^2 + \frac{r_o^2 - r_i^2}{\ln\left(\frac{r_o}{r_i}\right)}\ln r_o\right) \qquad (2)$$

Substitution of Eqs. (1) and (2) into Eq. 6.147 gives

$$\underline{\underline{v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln\left(\frac{r_o}{r_i}\right)}\ln\frac{r}{r_o}\right]}}$$

which is the desired equation (Eq. 6.155).

6.101

6.101 A wire of diameter d is stretched along the centerline of a pipe of diameter D. For a given pressure drop per unit length of pipe, by how much does the presence of the wire reduce the flowrate if **(a)** $d/D = 0.1$; **(b)** $d/D = 0.01$?

The volume flowrate is given by Eq. 6.156

$$Q = \frac{\pi \Delta p}{8\mu \ell}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(\frac{r_o}{r_i})}\right] \qquad \text{(Eq. 6.156)}$$

which can be written as

$$Q = \frac{\pi r_o^4 \Delta p}{8\mu \ell}\left\{1 - \left(\frac{r_i}{r_o}\right)^4 + \frac{\left[1-\left(\frac{r_i}{r_o}\right)^2\right]^2}{\ln(\frac{r_i}{r_o})}\right\} \qquad (1)$$

Since $\frac{r_i}{r_o} = \frac{d}{D}$, Eq. (1) can also be written as

$$Q = \frac{\pi r_o^4 \Delta p}{8\mu \ell}\left\{1 - \left(\frac{d}{D}\right)^4 + \frac{\left[1-\left(\frac{d}{D}\right)^2\right]^2}{\ln(\frac{d}{D})}\right\} \qquad (2)$$

Note that for $\frac{d}{D} = 0$ (no wire)

$$Q = \frac{\pi r_o^4 \Delta p}{8\mu \ell}$$

which corresponds to Poiseuille's Law (Eq. 6.151).

(a) For $\frac{d}{D} = 0.1$, Eq. (2) gives

$$Q = \frac{\pi r_o^4 \Delta p}{8\mu \ell}\left\{1 - (0.1)^4 + \frac{\left[1-(0.1)^2\right]^2}{\ln(0.1)}\right\} = 0.574$$

Thus, for the same Δp the flowrate is reduced by

$$\% \text{ reduction in } Q = (1 - 0.574) \times 100 = \underline{\underline{42.6\,\%}}$$

(b) Similarly, for $\frac{d}{D} = 0.01$ Eq. (2) gives

$$Q = \frac{\pi r_o^4 \Delta p}{8\mu \ell}\left\{1 - (0.01)^4 + \frac{\left[1-(0.01)^2\right]^2}{\ln(0.01)}\right\} = 0.783$$

and

$$\% \text{ reduction in } Q = (1 - 0.783) \times 100 = \underline{\underline{21.7\,\%}}$$

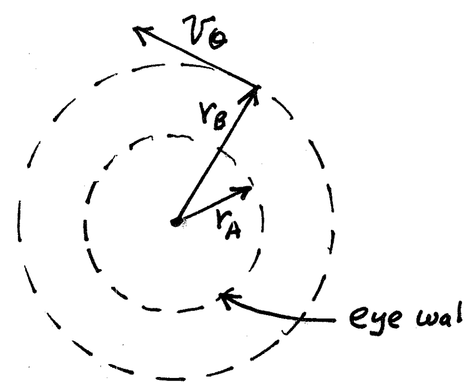
Note that the presence of even a very small wire along the tube centerline has a significant effect on the flowrate.

6.102

6.102 (See "**Some hurricane facts**," Section 6.5.3.) Consider a category five hurricane that has a maximum wind speed of 160 mph at the eye wall, 10 miles from the center of the hurricane. If the flow in the hurricane outside of the hurricane's eye is approximated as a free vortex, determine the wind speeds at locations 20 mi, 30 mi, and 40 mi from the center of the storm.

For free vortex

$$v_\theta = \frac{K}{r} \qquad \text{(Eq. 6.86)}$$



Thus, at eye wall

$$160\ mph = \frac{K}{10\ mi}$$

so that

$$K = (160\ mph)(10\ mi)$$

and

$$v_\theta = \frac{(160\ mph)(10\ mi)}{r_B}$$

For,

$r_B = 20\ mi \qquad v_\theta = \frac{(160\ mph)(10\ mi)}{20\ mi} = \underline{\underline{80.0\ mph}}$

$r_B = 30\ mi \qquad v_\theta = \frac{(160\ mph)(10\ mi)}{30\ mi} = \underline{\underline{53.3\ mph}}$

$r_B = 40\ mi \qquad v_\theta = \frac{(160\ mph)(10\ mi)}{40\ mi} = \underline{\underline{40.0\ mph}}$

6.103

6.103 (See "A **sailing ship without sails,**" Section 6.6.3.) Determine the magnitude of the total force developed by the two rotating cylinders on the Flettner "rotor-ship" due to the Magnus effect. Assume a wind speed relative to the ship of **(a)** 10 mph and **(b)** 30 mph. Each cylinder has a diameter of 9 ft, a length of 50 ft, and rotates at 750 rev/min. Use Eq. 6.124 and calculate the circulation by assuming the air sticks to the rotating cylinders. *Note*: This calculated force is at right angles to the direction of the wind and it is the component of this force in the direction of motion of the ship that gives the propulsive thrust. Also, due to viscous effects, the actual propulsive thrust will be smaller than that calculated from Eq. 6.124 which is based on inviscid flow theory.

$F_y = -\rho U \Gamma$ (force per unit length) (Eq. 6.124)

$\Gamma = \oint \vec{V} \cdot \vec{ds}$ (Eq. 6.89)

On the cylinder surface

$$\vec{V} = r\omega\, \hat{e}_\theta \quad \text{and} \quad \vec{ds} = r\,d\theta\, \hat{e}_\theta$$

so that

$$\Gamma = \int_0^{2\pi} (r\omega)(r\,d\theta)\, \hat{e}_\theta \cdot \hat{e}_\theta = 2\pi r^2 \omega$$

$$= (2\pi)(4.5\,\text{ft})^2 \left(750\,\frac{\text{rev}}{\text{min}}\right)\left(2\pi\,\frac{\text{rad}}{\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right)$$

$$= 9990\,\frac{\text{ft}^2}{\text{s}}$$

and

$$F_y = -\left(0.00238\,\frac{\text{slugs}}{\text{ft}^3}\right)\left(9990\,\frac{\text{ft}^2}{\text{s}}\right) U = -23.8\,U$$

(a) For a cylinder with length = 50 ft and number of cylinders = 2 and wind speed = 10 mph,

$$|F_y| = \left(23.8\,\frac{\text{lb}\cdot\text{s}}{\text{ft}^2}\right)\left(10\,\frac{\text{mi}}{\text{hr}}\right)\left(5280\,\frac{\text{ft}}{\text{mi}}\right)\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)(50\,\text{ft})(2)$$

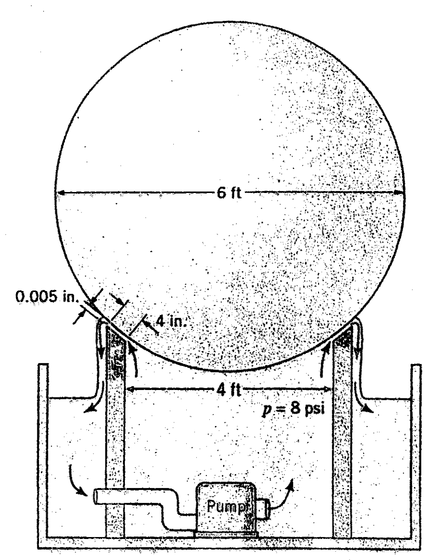
$$= \underline{\underline{34{,}900\,\text{lb}}}$$

(b) At 30 mph

$$|F_y| = 3 \times (F_y \text{ @ } 10\,\text{mph}) = \underline{\underline{105{,}000\,\text{lb}}}$$

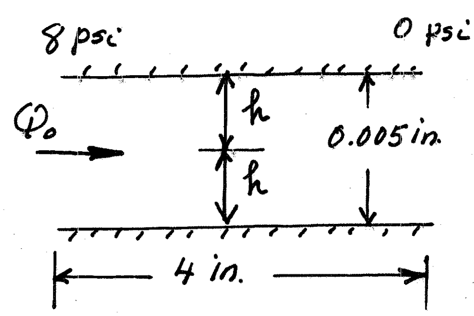
6.104

6.104 (See "10 tons on 8 psi," Section 6.9.1.) A massive, precisely machined, 6-ft-diameter granite sphere rests upon a 4-ft-diameter cylindrical pedestal as shown in Fig. P6.104. When the pump is turned on and the water pressure within the pedestal reaches 8 psi, the sphere rises off the pedestal, creating a 0.005-in. gap through which the water flows. The sphere can then be rotated about any axis with minimal friction. **(a)** Estimate the pump flowrate, Q_0, required to accomplish this. Assume the flow in the gap between the sphere and the pedestal is essentially viscous flow between fixed, parallel plates. **(b)** Describe what would happen if the pump flowrate were increased to $2Q_0$.



■ FIGURE P6.104

(a) $q = \dfrac{2h^3 \Delta p}{3\mu \ell}$ where $q = \dfrac{\text{flowrate}}{\text{unit width}}$ (Eq. 6.136)



width $= \pi\ (4\text{ft})$

$$h = \frac{0.005\text{ in.}}{2} = 0.0025\text{ in.} = 2.08\times 10^{-4}\text{ ft}$$

$$q = \frac{(2)(2.08\times10^{-4}\text{ ft})^3\left(8\,\frac{\text{lb}}{\text{in.}^2}\right)\left(144\,\frac{\text{in.}^2}{\text{ft}^2}\right)}{3\left(2.34\times10^{-5}\,\frac{\text{lb}\cdot\text{s}}{\text{ft}^2}\right)\left(\frac{4\text{ in.}}{12\text{ in./ft}}\right)}$$

$$= 8.86\times10^{-4}\,\frac{\text{ft}^3}{\text{s}} \text{ per unit width}$$

Thus,

$$Q_0 = \left(8.86\times10^{-4}\,\frac{\text{ft}^2}{\text{s}}\right)(4\pi\text{ ft}) = 0.0111\,\frac{\text{ft}^3}{\text{s}} \left(4.98\,\frac{\text{gallons}}{\text{min}}\right)$$

(b) Since 8 psi supports the sphere it is expected that this pressure remains approximately the same as the flowrate increases. To maintain this pressure the distance h would have to increase as Q_0 (or q) is increased. Thus, from Eq. 6.136,

$$\frac{q_{new}}{q_{old}} = \left(\frac{h_{new}}{h_{old}}\right)^3$$

$$2 = \left(\frac{h_{new}}{h_{old}}\right)^3$$

$$h_{new} = (2)^{1/3}(0.0025\text{ in.}) = 0.00315\text{ in.}$$

Thus, the gap width would increase to approximately 0.00630 in.