Please note that you have 7 questions and 7 pages

1) **(10 points)** Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$.

$y = \cot x$. $m_1 = m_2$ (parallel lines)

$y' = \csc^2 x$ $m_1 = y'_1 = -\csc^2 x$

$m_2 = y'_2 = -1$

To be parallel.

$m_1 = m_2$

$-\csc^2 x = -1$

$\csc^2 x = 1$

$\csc x = 1$ or $\csc x = -1$

$\frac{1}{\sin x} = 1$ $\frac{1}{\sin x} = -1$

$\sin x = 1$ $\sin x = -1$

$x = \pi/2$ $x = -\pi/2$.

2) (12 points) Find the derivative of each of the following functions:

a) $y = \tan\left(2x^2 - 3x\right)$

$y' = \sec^2(2x^2 - 3x)\,(4x - 3)$

$= 4x\sec^2(2x^2 - 3x) - 3\sec^2(2x^2 - 3x)$

12

b) $y = \dfrac{\sin x}{1 + \cos x}, \qquad 0 \le x \le \dfrac{\pi}{2}$

$y' = \dfrac{\cos x\,(1 + \cos x) - (-\sin x)(\sin x)}{(1 + \cos x)^2}$

$= \dfrac{\cos x + \cos^2 x - (-\sin^2 x)}{(1 + \cos x)^2}$

$\dfrac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$

3) **(18 points)** Given the parametrized curve: $\begin{cases} x = 1+2t \\ y = (t-1)^3 \end{cases}, t \in \mathbb{R}$.

a) Find the equation of the tangent line to the given curve at $t = 0$.

$\begin{cases} x_t = 1 \\ y_t = -1 \end{cases}$

$\begin{cases} x = 1 + 2t \text{ so } t = \frac{x-1}{2} \\ y = (t-1)^3 \end{cases}$

$y = \left(\frac{x-1}{2} - 1\right)^3$

$y' = 3\left(\frac{x-1}{2} - 1\right)^2 \left(\frac{2-0}{4}\right)$

$y' = 3\left(\frac{x-1}{2} - 1\right)^2 \left(\frac{1}{2}\right)$

$y'_{x_t} = 3\left(\frac{1-1}{2} - 1\right)^2 \left(\frac{1}{2}\right)$

$= 3(1)\left(\frac{1}{2}\right)$

$= \frac{3}{2}$

12

$y + 1 = \frac{3}{2}(x - 1)$

$y = \frac{3}{2}x - \frac{3}{2} - 1$

$y = \frac{3}{2}x - \frac{3-2}{2}$

$= \frac{3}{2}x - \frac{1}{2}$

b) Find $\frac{d^2y}{dx^2}$ at $t = 0$.

$y = \frac{3}{2}x - \frac{1}{2}$

$y' = 0 + \frac{3}{2}$

$d x^2$

$d y$

$d y^2$

$t = 0$

$x^2 = (1+2t)^2$

$= 1 + 2(1)(2t) + 4t^2$

$= 1 + 4t + 4t^2$

$x^{2\prime} = 4 + 8t$

$y' = 3(t-1)(1) = 3(t-1)^2$

$y^2 = (t-1)^3 = (t-1)^5$

$y^{2\prime} = 5(t-1)^4$

$\frac{9(t-1)^2}{4+8t}$ for $t = 0$

$\frac{9(0-1)^2}{} = \frac{9}{4}$

4) **(15 points)** Consider the curve: $x^2 + xy + y^2 = 7$.

a) Verify that the point (1, 2) is on the curve.

(1, 2) should verify the equation

$x^2 + xy + y^2 = 7$

$1 + 2 + 4 = 7$

$3 + 4 = 7$

$7 = 7$.

b) Find the equation of the tangent line to the given curve at point (1, 2).

$x^2 + xy + y^2 = 7.$ at $(1, 2)$

~~$2x$ $x^2 + xy$~~

$2x + (1 \times y) + (1 \times x) + 2y = 7$

$2x + y + x + 2y = 7$

$3y = 7 - 3x$

$y = \frac{7-3x}{3}$

$y = \frac{7}{3} - x$

so $m = \frac{7}{3} - 1$

$= \frac{7-3}{3} = \frac{4}{3}$

$y - 2 = \frac{4}{3}(x - 1)$

$y = \frac{4}{3}x - \frac{4}{3} + 2$.

$y = \frac{4}{3}x - \frac{4+6}{3}$ so $y = \frac{4}{3}x - \frac{10}{3}$

c) Find the equation of the normal line to the curve at point (1, 2).

the curve is defined for all $\mathbb{R}$.

for $x = 0$ $y = \sqrt{7}$ $(0 ; \sqrt{7})$

$y = ax + b$

$\sqrt{7} = 0 + b$ $\boxed{b = \sqrt{7}}$.

$x - 1$ $2 = a + b$

$y = (2 - \sqrt{7})x + \sqrt{7}$

~~$2 + \sqrt{7} = a$~~

$\sqrt{7} - 2 = -a$

$\boxed{a = 2 - \sqrt{7}}$

5) (20 points) Indicate and identify the kind of all extreme values (absolute and local) of the function: $y = f(x) = \frac{x^4}{4} - \frac{x^2}{2} + 2$, where $x \in [-2, 2]$.

$$f(x) = \frac{x^4}{4} - \frac{x^2}{2} + 2 = \frac{x^4 - 2x^2 + 8}{4}$$

~~f'(x) =~~

$$f'(x) = \frac{(4x^3 - 4x)4 - 0}{16}$$

$$f'(x) = 0$$

$$(4x^3 - 4x)4 = 0$$

$$16x^3 - 16x = 0$$

$$x(16x^2 - 16) = 0$$

$$x = 0 \qquad 16x^2 - 16 = 0$$

$$16x^2 = 16$$

$$x^2 = 1$$

$$x = \pm\sqrt{1}$$

x	$-\infty$	-2		-1		0		1		2		∞
f'			$-$		$+$		$-$		$+$		$+$	
f	∞	4		$\frac{7}{4}$		2		$\frac{7}{4}$		4		∞

absolute min.   min   max.   absolute max.

6) **(10 points)** Is $y = x^3$ a solution to the differential equation $x^2y'' + xy' - 9y = 0$? Explain.

$\cancel{y' =}$ $y' = 3x^2$

$y'' = 6x$

$x^2(6x) + x(3x^2) - 9(x^3) = 0$

$6x^3 + 3x^3 - 9x^3 = 0$

$9x^3 - 9x^3 = 0$

So it is a solution

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7) **(15 points)** A particle $p(x,y)$ moves along the curve $y = x^{\frac{3}{2}}$ in the first quadrant in such a way that its distance from the origin $D = \sqrt{x^2 + y^2}$ increases at the rate of $11\,cm$ per second. Find $\frac{dx}{dy}$ when $x = 3\,cm$.

$D = \sqrt{x^2 + y^2}$

$D^2 = x^2 + y^2$

$x^2 = D^2 - y^2$

$x = \sqrt{D^2 - y^2}$

$\frac{dD}{dt} = \frac{2x}{2\sqrt{x^2+y^2}} \frac{dx}{dt} + \frac{2y}{2\sqrt{x^2+y^2}} \frac{dy}{dt}$ for $x = 3$, $y = x^{3/2}$, $y = 3^{3/2} = 3\sqrt{3}$

$11 = \frac{3}{\sqrt{9+27}} dx + \frac{3\sqrt{3}}{\sqrt{9+27}} dy$

$\Rightarrow 11 = \frac{3}{6} dx + \frac{3\sqrt{3}}{6} dy$ $\quad dy = \frac{3}{2} x^{1/2} dx$

$66 = 3dx + 3\sqrt{3}\, dy$

$22 = dx + \sqrt{3}\, dy$

$\boxed{26 = dx + \sqrt{3}\, dy}$ $\Rightarrow 22 = dx + \sqrt{3} \cdot \frac{3\sqrt{3}}{2} dx = dx + \frac{9}{2} dx = dx\left(1+\frac{9}{2}\right) = \frac{11}{2}$

$dy = \frac{3}{2} x^{1/2} dx$

$\boxed{dy = \frac{3\sqrt{3}}{2} dx}$

$dx = \frac{22}{11} \times 2 = 4$

$dy = \frac{3\sqrt{3}}{2} \cdot 4 = 6\sqrt{3}$

$\frac{dx}{dy} = \frac{4}{6\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{3 \times 3} = \frac{2\sqrt{3}}{9}$