1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2}$

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^4}$

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^3+y^3}$

6) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y}$

7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}$

8) Is it possible to define $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$ at $(0,0)$ so that $f(x,y)$ is continuous???

9) Determine whether:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at the origin

10) Very Important:

Give an example of a function $f(x,y)$ such that

$f_x(0,0) = f_y(0,0) = 0$, but $f$ is not continuous at $(0,0)$

Hence proving the "existence of partial derivatives does not ensure continuity theorem"

{Shaggya 2012}

Note: this exercise is enlisted as a required proof for exam 2

11) If $z$ is implicitly defined as a function of $x$ and $y$

$$x^2+y^2-z^2=3$$

find the gradient, Then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

12) If $z=\ln\sqrt{x^2+y^2}$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1$

13) If $z=\int_x^y e^{\sin(t)}\,dt$ ; find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2}$ Let $y=mx$

$$\to \frac{2x\,m^2x^2}{x^2+m^2x^2} \to \frac{2m^2}{1+m^2}x \to \lim_{x\to 0}\frac{2m^2}{1+m^2}x = 0$$

. however this is <u>not</u> enough to prove $2xy^2/x^2+y^2 \to 0$

thus $\left|\dfrac{2xy^2}{x^2+y^2}\right| \le \dfrac{2\left(\sqrt{x^2+y^2}\right)(x^2+y^2)}{x^2+y^2} = 2\sqrt{x^2+y^2}$

$= \boxed{0}$

where $|x| \le \sqrt{x^2+y^2}$

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2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^4}$ Let $y=mx$

$$\to \frac{2x\,m^2x^2}{x^2+m^4x^4} \to \frac{2m^2x}{1+m^4x^2} \to \lim_{x\to 0} = \frac{0}{1+0} = \boxed{0}$$

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3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ let $y=mx$

$$\frac{x^2-m^2x^2}{x^2+m^2x^2} \to \frac{1-m^2}{1+m^2}$$ dependent on $m$ $\therefore$ limit doesn't exist

3) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ Let $y = mx$

$\to \frac{2x m^2 x^2}{x^2+m^4x^4} \to \frac{2m^2x}{1+m^4x^2} \to \lim_{x\to 0} \frac{0}{1+0} = \boxed{0}$

<u>But</u> if we choose another path say $x = y^2$

$\to \frac{2y^4}{y^4+y^4} \to \frac{2\cancel{y^4}}{2\cancel{y^4}} = \boxed{1}$ $\boxed{0 \neq 1}$

$\therefore$ limit doesn't exist

4) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

notice $|x| \leq \sqrt{x^2+y^2}$

$\to \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{\cancel{\sqrt{x^2+y^2}}\; y}{\cancel{\sqrt{x^2+y^2}}}\right| = |y| \to \boxed{0}$

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \boxed{0}$

5) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$ Let $y=mx$

$\to \frac{x^2m^2x^2}{x^3+m^3x^3} \to \frac{x\,m^2}{1+m^3} \to \frac{0}{1+m^2} \to \boxed{0}$

Take 2nd path Let $y=-xe^x$

$\frac{x^2(-xe^x)^2}{x^3+(-xe^x)^3} \to \frac{-x^4e^{2x}}{x^3-x^3e^{3x}} \to \frac{xe^{2x}}{e^{3x}-1} \to \boxed{\frac{1}{3}}$

By L'Hopital Law

$0 \neq \frac{1}{3}$ $\therefore$ limit doesnt exist

6) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y}$ Let $y=mx$

$\to \frac{x^3+m^3x^3}{x^2+mx} \to \frac{x^2(1+m^2)}{x+m} \to \boxed{0}$

Let $y=-x^2e^x$

$\to \frac{x^3-x^6e^{3x}}{x^2-x^3e^x} \to \frac{x-x^4e^{3x}}{1-e^x}$ By H.L

$\to \frac{x-x^4e^{3x}}{1-e^x} = \lim_{x\to0} \frac{1-3x^4e^{3x}-4x^3e^{3x}}{-e^x} = \boxed{-1}$

$\boxed{0 \neq -1}$ $\therefore$ lim $\neq$ exist

7) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}$ it is obvious that

as we know $|x| \leq \sqrt{x^2+y^2}$

i.e. $x^2 \leq x^2+y^2$

so $\left|\dfrac{x^2y^2}{x^2+y^2}\right| \leq \dfrac{\cancel{(x^2+y^2)}\,y^2}{\cancel{(x^2+y^2)}} = y^2 = \boxed{0}$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = 0$

8) What is meant is can we change the form of the equation thus changing the shape of the curve such that it becomes continuous at $(0,0)$ ?

we know $|x^3+y^3| \leq |x^3| + |y^3|$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \leq \frac{|x^3|}{|x^2+y^2|} + \frac{|y^3|}{|x^2+y^2|}$$

$$\frac{x^2\cdot|x|}{|x^2+y^2|} + \frac{y^2\,|y|}{|x^2+y^2|}$$

~~but $|x| \leq |x|$, $|y| \leq$~~

but $x^2 \leq x^2+y^2$

$y^2 \leq x^2+y^2$

$\Rightarrow |x|+|y| = \boxed{0}$

∴ yes we can redefine the function $f(x,y)$ to make it continuous at $(0,0)$

9) In general to prove $f$ is continuous at a point $(0,0)$ we prove:

$$\underbrace{f(0,0)} = \underbrace{\lim_{(x,y)\to(0,0)}}$$

If both are $=$ ∴ continuous

. we can translate $f(x,y) = \dfrac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$

as $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

. we are given $f(0,0)=0$, So we are asked to prove

that $\displaystyle\lim_{(x,y)\to(0,0)}$ to be $= 0$ since we need

. to prove $\underbrace{f(0,0)}_{0} = \displaystyle\lim_{(x,y)\to(0,0)}$ ,, ,

$y = mx \longrightarrow \dfrac{x^2 m}{x^2 + m^2 x^2} \longrightarrow \dfrac{m^2}{1+m^2}$ depends on $m$

then $\lim\limits_{(x,y)\to(0,0)}$ doesnt exist

thus $f(0,0) \neq \lim\limits_{(x,y)\to(0,0)}$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$

- add part b : Can we redefine f to be continuous at (0,0)

. Answer is ~~not~~ no because limit depends entirely on "m"

10) Theorem: existence of partial derivatives does not ensure continuity. The proof is this example:

. The easiest example is to use problem 9 where f is not continuous at (0,0). we need to prove that $f_x$ and $f_y$ are well defined meaning they give us values other than $\frac{1}{0}$ or $\frac{\infty}{\infty}$ or $\infty$

in the exam, resolve problem 9 and find $f_x(0,0)$
$f_y(0,0)$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y}$$

based on problem 9 we are given $f(0,0) = 0$

then $\lim_{\Delta x \to 0} f(\Delta x, 0) \to f(0,0)$ since $\Delta x \to 0$

then $f(\Delta x, 0) = f(0,0) = 0$

then $f_x(0,0) = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

$f_x$ and $f_y$ are well defined $\left\{ \text{not } \frac{\infty}{\infty} \text{ or } \infty \text{ or } \frac{1}{0} \right\}$

but $f$ is not continuous on $(0,0)$ refer to problem 9

∴ Theorem proved.

11) $x^2+y^2-z^2=3$

$\vec{\nabla} = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$

- $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z} = -\dfrac{2x}{-2z} = \boxed{\dfrac{x}{z}}$

- $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z} = \dfrac{-2y}{2z} = \boxed{-y/z}$

12) if $z = \ln\left(\sqrt{x^2+y^2}\right)$

$\dfrac{\partial z}{\partial x} = \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \div \sqrt{x^2+y^2} = \dfrac{x}{x^2+y^2}$

$\dfrac{\partial z}{\partial y} = \dfrac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} \div \sqrt{x^2+y^2} = \dfrac{y}{x^2+y^2}$

$$x\frac{\partial z}{\partial x} = \frac{x^2}{x^2+y^2}$$

$$y\frac{\partial z}{\partial y} = \frac{y^2}{x^2+y^2}$$

$$\Rightarrow \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = \boxed{1}$$

$$\therefore \left\{ x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}y = 1 \right\}$$

$y$

13) $z = \int_{x}^{y} e^{\sin(t)}\, dt$

• Recall: if $h(x) = \int_{k}^{g(x)} F(x)\, dx$

then $h'(x) = F(g(x)) \cdot g'(x)$

• $\frac{\partial z}{\partial y} = e^{\sin(y)} \cdot 1 = \boxed{e^{\sin(y)}}$

ct, varible ✓

$$\frac{\partial z}{\partial x} = \left[\int_{x}^{y} e^{\sin(t)}\, dt\right]' = \left[-\int_{y}^{x} e^{\sin(t)}\, dt\right]'$$

✓ varible, ct ✓

$$= -e^{\sin(x)} \cdot 1 = \boxed{-e^{\sin(x)}}$$

14) $z = x^2 - y^2$ and $x = 2$

The plane $x=2$ is ct $\rightarrow$ slope is $\frac{\partial z}{\partial y} = -2y$

at $(2,1,3)$, $\rightarrow -2(1) = -2$

$$\begin{cases} x = 2 \rightarrow cb \\ y = t+1 \\ z = -2t+3 \end{cases} \longrightarrow \text{for limit of } y \hookrightarrow \frac{\partial z}{\partial y} \text{ is } -2.$$

15) parall to $(x\,z)$ plan $\longrightarrow y = K \rightarrow y = 1$ at $(1,1,1)$

$z = 3x^2 + 4y^2 - 6$

then slope is $\frac{\partial z}{\partial x} = 6x \rightarrow 6(1) = \boxed{6}$

$$\begin{cases} x = t+1 \\ y = 1 \\ z = 6t+1 \end{cases}$$

14) Find the equation of the tangent line that results from the intersection of the surface $z = x^2 - y^2$ and the plane $x = 2$ at the point $(2, 1, 3)$

15) Find the equation of the tangent line that results from the intersection of the surface $z = 3x^2 + 4y^2 - 6$ ~~and~~ the ly planes through $(1, 1, 1)$ and parallel to $xz$