NOTRE DAME UNIVERSITY

MAT 113

FINAL EXAM

DATE: FRIDAY FEBRUARY 1, 2008
DURATION: 2 HOURS

NAME: --

ID NUMBER: --

SECTION: --

GRADE: --

1-(8 points) Find the following limits:

a) $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \frac{2-\sqrt{9-5}}{-3+3} = \frac{2-2}{0} = \frac{0}{0}$ } 1 pt

2 pts $\left\{ \begin{array}{l} = \lim_{x\to -3} \left(\frac{2-\sqrt{x^2-5}}{x+3}\right)\left(\frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}\right) = \lim_{x\to -3} \frac{4-(x^2-5)}{(x+3)(2+\sqrt{x^2-5})} \\ = \lim_{x\to -3} \frac{9-x^2}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x\to -3} \frac{(3-x)\cancel{(3+x)}}{\cancel{(x+3)}(2+\sqrt{x^2-5})} \end{array}\right.$

1 pt $\left\{ = \lim_{x\to -3} \frac{3-x}{2+\sqrt{x^2-5}} = \frac{3-(-3)}{2+\sqrt{9-5}} = \frac{6}{2+2} = \frac{6}{4} = \frac{3}{2}, \right.$

b) $\lim_{x\to 3} \frac{x^2+2x-15}{9-x^2} = \frac{9+6-15}{9-9} = \frac{0}{0}$ } 1 pt

2 pts $\left\{ = \lim_{x\to 3} \frac{(x+5)(x-3)}{(3-x)(3+x)} = \lim_{x\to 3} \frac{-\cancel{(3-x)}(x+5)}{\cancel{(3-x)}(3+x)} \right.$

1 pt $\left\{ = \lim_{x\to 3} \frac{-(x+5)}{(3+x)} = -\frac{8}{6} = -\frac{4}{3}. \right.$

2-(8 points) Let $f(x)=\dfrac{x-5}{x^2-3x-10}$

a) At which points does $f(x)$ fail to be continuous?

b) At which points, if any, is the discontinuity removable? Give reasons, and then extend the function to become continuous there.

Sol:

a) $f(x)=\dfrac{x-5}{(x-5)(x+2)}$

$\therefore$ $f(x)$ is discont at $x=-2$ and $x=5$ because $f(-2)$ and $f(5)$ are not defined.

(3 pts)

b) $\lim\limits_{x\to -2} f(x) = \dfrac{-7}{0^{\pm}} = \pm\infty \Rightarrow f(x)$ can not be extended to become cont at $x=-2$.

(2 pts)

$\lim\limits_{x\to 5} f(x) = \dfrac{0}{0} = \lim\limits_{x\to 5} \dfrac{1}{x+2} = \dfrac{1}{7}$

$\therefore$ we can extend $f(x)$ to become cont at $x=5$, and the extended function is $F(x)=\begin{cases} f(x) ; & x\neq 5 \\ \frac{1}{7} , & x=5 \end{cases}$

this means $F(x)=\begin{cases} \dfrac{x-5}{x^2-3x-10} ; & x\neq 5 \\ \dfrac{1}{7} ; & x=5 \end{cases}$ is cont at $x=5$.

(3 pts)

3-(13 points) Verify that the point A (-1, 3) is on the curve: $x^2y^2 = 9$ and then find

a) The equation of the line that is tangent to the curve at A.

1pt { If we replace A in the eq: $(-1)^2(3)^2 = 9$ ✓ $\Rightarrow A \in$ Curve.

slope at $(x,y) = \frac{dy}{dx} = ??$, we use the implicit differentiation.

4pts $(x^2y^2)' = (9)' \Rightarrow 2xy^2 + 2yy'(x^2) = 0 \Rightarrow 2yy'(x^2) = -2xy^2$

$\Rightarrow y' = \frac{-2xy^2}{2yx^2} \Rightarrow y' = \frac{dy}{dx} = -\frac{y}{x}$.

$\therefore$ slope at $(-1,3) = \left.\frac{dy}{dx}\right|_{(-1,3)} = -\frac{3}{-1} = 3$ = slope of the tangent line.

3pts eq: $y - 3 = 3(x-(-1)) \Rightarrow y = 3x + 3 + 3 \Rightarrow \boxed{y = 3x + 6}$ tangent line

b) The equation of the line that is normal to the curve at A.

2pts { slope of the normal line at A $= \frac{-1}{\text{slope of the tangent line}} = \frac{-1}{3}$

eq: $y - 3 = -\frac{1}{3}(x-(-1)) \Rightarrow y - 3 = -\frac{1}{3}(x+1)$

3pts $y = -\frac{1}{3}x - \frac{1}{3} + 3$.

$\boxed{y = -\frac{1}{3}x + \frac{8}{3}}$

normal line.

4-(16 points) Find $\frac{dy}{dx}$ for each of the following functions:

a) $y=\sqrt{\tan(x^2+3)}$

4 pts

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan(x^2+3)}}\cdot\sec^2(x^2+3)\,(2x)=\frac{x\sec^2(x^2+3)}{\sqrt{\tan(x^2+3)}}.$$

b) $y=\sec(x\sqrt{x})$ $=\sec(x^{3/2})$

4 pts

$$\frac{dy}{dx}=\sec(x\sqrt{x})\cdot\tan(x\sqrt{x})\left(\frac{3}{2}x^{\frac{1}{2}}\right)=\frac{3}{2}\sqrt{x}\sec(x\sqrt{x})\cdot\tan(x\sqrt{x}).$$

c) $y=\sqrt{\dfrac{x^2+1}{x}}$

4 pts

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{x^2+1}{x}}}\cdot\left(\frac{2x(x)-x^2-1}{x^2}\right)=\frac{1}{2\sqrt{\frac{x^2+1}{x}}}\left(\frac{x^2-1}{x^2}\right).$$

$$=\frac{\sqrt{x}}{2\sqrt{x^2+1}}\left(\frac{x^2-1}{x^2}\right).$$

d) $y=\int_{\sec x}^{2}\frac{1}{t^2-1}dt$ $=-\int_{2}^{\sec x}\frac{1}{t^2-1}dt.$

4 pts

$$\frac{dy}{dx}=-\frac{1}{\sec^2x-1}\cdot\sec x\cdot\tan x=-\frac{\sec x\cdot\tan x}{\tan^2x}$$

$$=\frac{-\sec x}{\tan x}=-\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}=\frac{-1}{\sin x}=-\csc x.$$

5-(15 points) Consider the function $f(x)=-x^2-2x+3$ on the interval $[-4,0]$.

a) Find the absolute extremes of $f(x)$ on $[-4,0]$.

F(x) is cont on a closed interval $[-4,0] \Rightarrow$ Abs extremes at a critical points or end points.

2 pts { * Critical pts: $f'(x)=-2x-2=0 \Rightarrow x=-1 \Rightarrow (-1, f(-1)=\underline{4})$

$\therefore$ only Critical pt $(-1,4)$.

2 pts { * End pts: $(-4, f(-4)=-5) = (-4,\underline{-5})$

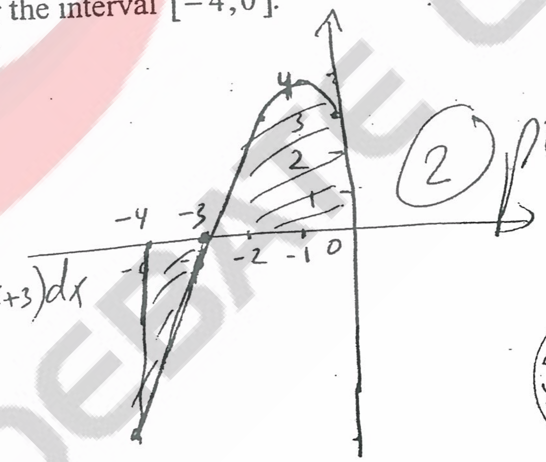
$(0, f(0)=3) = (0,3)$

$$\boxed{\begin{array}{l}\therefore \text{Abs min} = -5 \text{ at } x=-4 \\ \text{Abs max} = 4 \text{ at } x=-1.\end{array}}$$

2 pts

b) Sketch $f(x)$ on $[-4,0]$, then find the area of the region bounded between the curve of $f(x)$ and the X-axis over the interval $[-4,0]$.

x	-4	-3	-2	-1	0
$f(x)$	-5	0	3	4	3



2 pts

4 pts

$$A=\left|\int_{-4}^{-3}(-x^2-2x+3)\,dx\right|+\int_{-3}^{0}(-x^2-2x+3)\,dx$$

$$=\left|\left[-\frac{x^3}{3}-x^2+3x\right]_{-4}^{-3}\right|+\left[-\frac{x^3}{3}-x^2+3x\right]_{-3}^{0}$$

3 pts

$$=\left|9-9-9-\left(\frac{64}{3}-16-12\right)\right|+\left[0-(-9-9+9)\right]$$

$$=\left|-9-\left(\frac{64-84}{3}\right)\right|+[+9]=\left|-9+\frac{20}{3}\right|+9$$

$$=\left|-\frac{7}{3}\right|+9=\frac{7}{3}+9=\frac{34}{3}\ u^2.$$

6-(20 points) Evaluate each of the following integrals:

a) $\int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x} dx$ $= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} dx$

$= \int_0^{\frac{\pi}{4}} (\sec^2 x + \sec x \tan x) dx = \tan x + \sec x \Big]_0^{\frac{\pi}{4}}.$

$= \left[\tan\frac{\pi}{4} + \sec\frac{\pi}{4}\right] - \left[\tan 0 + \sec 0\right] = \left[1 + \sqrt{2}\right] - \left[0 + 1\right]$

$= \sqrt{2}.$

(5) pts

b) $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$ $= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c$

let $u = e^x \Rightarrow du = e^x\, dx$

$= \sin^{-1}(e^x) + c$

(5) pts

c) $\int_1^e \frac{dx}{x+x(\ln x)^2} = \int_1^e \frac{dx}{x(1+\ln^2 x)} = \int_{u(1)}^{u(e)} \frac{du}{1+u^2}$

let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$= \int_0^1 \frac{du}{1+u^2} = \tan^{-1}u \Big]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

(5) pts

d) $\int x^2 e^{3x}\,dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int x e^{3x}\,dx$

let $u = x^2 \Rightarrow du = 2x\,dx$

$dv = e^{3x}dx \Rightarrow v = \frac{e^{3x}}{3}$

let $u = x \Rightarrow du = dx$

$dv = e^{3x}dx \Rightarrow v = \frac{e^{3x}}{3}$

$$= \frac{x^2}{3}e^{3x} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx\right).$$

$$= \frac{x^2}{3}e^{3x} - \frac{2xe^{3x}}{9} + \frac{2}{9}\int e^{3x}dx.$$

$$= \frac{x^2}{3}e^{3x} - \frac{2xe^{3x}}{9} + \frac{2}{9}\frac{e^{3x}}{3} + C.$$

$$= \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right)e^{3x} + C.$$

(5) pts

7-(5 points) Find $f(\frac{1}{4})$ *if* $\int_0^x f(t)\,dt = \csc(2\pi x)$.

$\left(\int_0^x f(t)dt\right)' = (\csc(2\pi x))' \Rightarrow f(x) = -\csc(2\pi x)\cot(2\pi x)(2\pi)$

3 pts $\left\{ \Rightarrow f(x) = -2\pi \csc(2\pi x) \cdot \cot(2\pi x). \right.$

2 pts $\left[\Rightarrow f(\frac{1}{4}) = -2\pi \csc(\frac{\pi}{2}) \cdot \cot(\frac{\pi}{2}) = -2\pi(1)(0) = 0. \right.$

8-(8 points) Given $f(x) = \begin{cases} x^2 & , -1 \le x < 0 \\ -x^2 & , 0 \le x \le 1 \end{cases}$

a) Is $f(x)$ continuous at x=0.(explain)

b) Is $f(x)$ differentiable at x=0.(explain)

3 pts

a) $f(0) = -(0)^2 = 0$

$f(0^-) = (0^-)^2 = 0^+$

$f(0^+) = -(0^+)^2 = -0^+ = 0^-$

$\Rightarrow f(0^-) = f(0) = f(0^+)$ $(\lim_{x\to 0} f(x) = f(0) = 0)$.

$\Rightarrow f(x)$ is cont at $x=0$.

b) $f'(x) = \begin{cases} 2x ; & -1 \le x < 0 \\ -2x ; & 0 < x \le 1. \end{cases}$

5 pts

$f'(0^-) = 2(0^-) = 0^-$

$f'(0^+) = -2(0^+) = 0^-$

$\Rightarrow f'(0^-) = f'(0^+)$

$\therefore f'(0)$ exists

and $f'(0) = 0$

(left hand derivative = right hand derivative)

9-(7 points) Find the positions of the local extreme values of the function

$$f(x)=\int_0^x 9-t^2\ dt \text{. (Without finding } f(x)\text{)}$$

(3) pts

Since $f(x)$ is defined on an open interval $(-\infty, \infty)$

$\Rightarrow$ local extremes are at critical pts only.

$\therefore\ f'(x) = 9 - x^2 = 0 \Rightarrow (3-x)(3+x) = 0$

$\Rightarrow x' = -3 \qquad x'' = 3$

(4) pts

$- \quad \phi \quad + \quad \phi \quad -$

min $\qquad$ max

$\therefore$ there is a local min at $x = -3$.

a local max at $x = 3$.