Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 113
Intensive Calculus
Exam #1
Monday November 16, 2009
Duration: 55 minutes

Name: ____________________

Section: ____________________

Instructor: ____________________

Grade: ____________________

Directions:

1. Write neatly and clearly.
2. Only scientific calculators are allowed.
3. Your mobile must be turned off and unseen

Please note that you have 6 exercises and 6 pages

1) (16 points) Calculate the following limits:

a) $\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{x}-2} = \dfrac{4-4}{\sqrt{4}-2} = \dfrac{0}{0}$: Indetermined

$\lim\limits_{x\to 4} \dfrac{(x-4).(\sqrt{x}+2)}{(\sqrt{x}-2).(\sqrt{x}+2)} = \lim\limits_{x\to 4} \dfrac{x\sqrt{x}+2x-4\sqrt{x}-8}{x-4}$

$= \lim\limits_{x\to 4} \dfrac{\sqrt{x}(x-4)+2(x-4)}{(x-4)} = \lim\limits_{x\to 4} \dfrac{(x-4)(\sqrt{x}+2)}{(x-4)}$

$= \lim\limits_{x\to 4} (\sqrt{x}+2) = 4$.

b) $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x^3}$ **(Use the Sandwich Theorem)** $= 0 \cdot \sin\left(\dfrac{1}{0}\right)$: Indetermined.

We have $-1 \leq \sin\left(\dfrac{1}{x^3}\right) \leq 1$;

$\Rightarrow \quad -x^2 \leq x^2 \cdot \sin\left(\dfrac{1}{x^3}\right) \leq x^2$;

So $\lim\limits_{x\to 0}(-x^2) = 0$; and $\lim\limits_{x\to 0}(x^2) = 0$,

So by the sandwich theorem, $\lim\limits_{x\to 0} x^2 \cdot \sin\left(\dfrac{1}{x^3}\right) = 0$.

2) (16 points) Let f the function be defined by

$$f(x)=\begin{cases} 3-x & \text{if } x<1 \\ 4 & \text{if } x=1 \\ x^2+1 & \text{if } x>1 \end{cases}$$

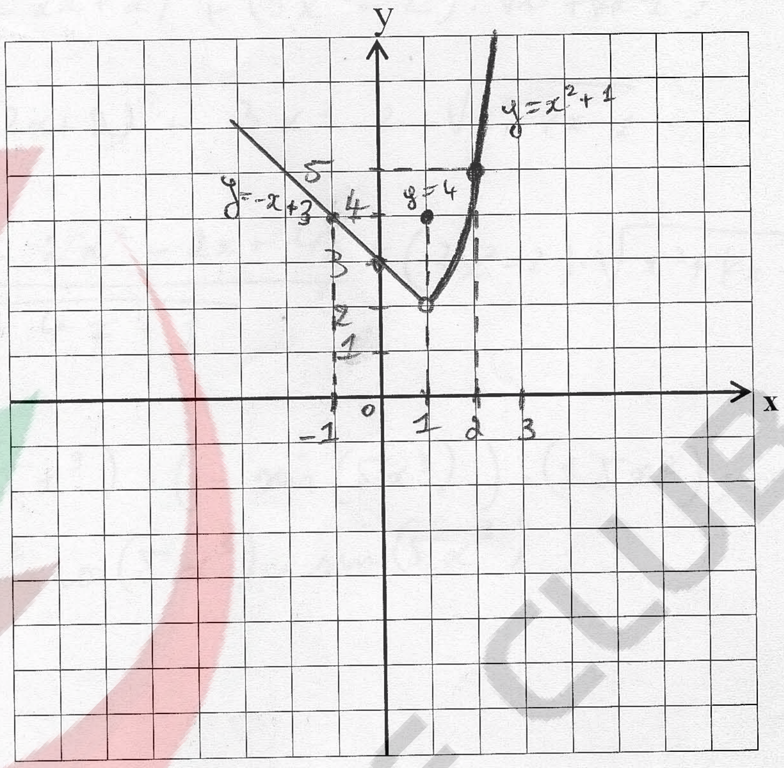
a) Sketch the graph of f.

. for $x<1$; $y=3-x$;

y	4	3	5
x	-1	0	-2

- for $x>1$, $y=x^2+1$

y	5	10
x	2	3



b) Is f continuous at $x=1$? Explain your answer.

. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = 2$;

But, $f(1)=4$;

So $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) \neq f(1)$;

So this piecewised function does not satisfy the 3rd property of the continuity test, so f is discontinuous at $x=1$.

3) **(24 points)** Differentiate the following functions:

a) $f(x)=\sqrt{x^2+4x-1}\left(x^3-2x+2\right)$

$u=\sqrt{x^2+4x-1}$

$v=x^3-2x+2$.

$f'(x)=(u\cdot v)'=u'\cdot v+v'u$

$$f'(x)=\frac{2x+4}{2\sqrt{x^2+4x-1}}\cdot(x^3-2x+2)+(3x^2-2)\cdot\sqrt{x^2+4x-1};$$

$$f'(x)=\frac{x+2}{\sqrt{x^2+4x-1}}(x^3-2x+2)+3x^2-2\cdot\sqrt{x^2+4x-1}$$

$$f'(x)=\frac{(x^4+2x^3-2x^2-2x+4)}{\sqrt{x^2+4x-1}}+(3x^2-2)\cdot\sqrt{x^2+4x-1}$$

b) $g(x)=\cos^2\left(5x^3\right)$

$$g'(x)=2\cdot\cos(5x^3)\cdot(-\sin(5x^3))\cdot(15x^2);$$

$$g'(x)=-30x^2\cdot\cos(5x^3)\cdot\sin(5x^3).$$

c) $h(x)=\dfrac{\cot 2x}{1+\csc 2x}$

$$h'(x)=\frac{u'v-v'u}{v^2};$$

$$h'(x)=\frac{(-2\cdot\csc^2(2x))(1+\csc 2x)-(-2\csc 2x\cdot\cot 2x)(\cot$$

$$(1+\csc 2x)^2$$

4) **(16 points)** Use implicit differentiation to find an equation for the tangent line to the curve $(x^2+y^2)^3 = 8xy$ at the point $P(1,1)$.

Derive both sides with respect to x:

$3(x^2+y^2)^2 \cdot (2x+2yy') = 8(y+xy')$;

$3(x^2+y^2)^2 \cdot (2x+2yy') = 8y + 8xy'$;

$3x^4yy' + 3y^5y' + 6x^2y^3y' - 4xy' = 4y - 6x^3y^2 - 3xy^4 - 3x^5$

$\Rightarrow y'(-4x + 3x^4y + 3y^5 + 6x^2y^3) = 4y - 6x^3y^2 - 3xy^4 - 3x^5$.

$y' = \dfrac{4y - 6x^3y^2 - 3xy^4 - 3x^5}{3x^4y + 3y^5 + 6x^2y^3 - 4x}$; $y'(1;1) = \dfrac{-8}{8} = -1$;

So the slope of the tangent line to the curve at $P(1;1)$ is : "-1".

$\Rightarrow$ $y - y_P = y'_P(x - x_P)$;

$\Rightarrow$ $y - 1 = -1(x-1)$;

$\boxed{y = -x + 2}$ Tangent line at $P(1;1)$ to the curve.

5) **(10 points)** Consider $f(x) = \dfrac{2x^2-3x+1}{x-1}$. Can f be extended to become continuous at $x = 1$? If yes, define the continuous extension.

$f(x) = \dfrac{2x^2-3x+1}{x-1}$, So f can be extended to become continuous at $x=1$, $\lim\limits_{x\to 1^-} f(x)$ should be equal to $\lim\limits_{x\to 1^+} f(x)$

$\Rightarrow$ $\lim\limits_{x\to 1^-} f(x) = \dfrac{0}{0}$ Ind. $\Rightarrow \lim\limits_{x\to 1^-} \dfrac{2(x-1)(x-\frac{1}{2})}{(x-1)} = \dfrac{2}{2} = 1$; (exists).

$\lim\limits_{x\to 1^+} f(x) = \dfrac{0}{0}$ Ind; $\Rightarrow \lim\limits_{x\to 1^+} f(x) = 2\left(\dfrac{1}{2}\right) = 1$ (exists;

So $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1} f(x)$ exists, so then f can be extended to become continuous at $x=1$.

$\Rightarrow$ The continuous extension of f will turn to be:

$$F(x) = \begin{cases} f(x) = \dfrac{2x^2-3x+1}{x-1} \ ; & x \neq 1 \\ 1 \ ; & x = 1 \end{cases}$$

6) **(18 points)** Consider the curve C parametrized by $x = 1 - \cos t$ and $y = t - \sin t$ with $0 \le t \le 4\pi$.

a) Find $\frac{dy}{dx}$ and then $\frac{d^2y}{dx^2}$; $\cdot \ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-\cos t}{\sin t}$; $0 \le t \le 4\pi$;

$\cdot \ \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} \Rightarrow \frac{dy'}{dt} = \left(\frac{1-\cos t}{\sin t}\right)' = \frac{(\sin t \cdot \sin t) - (\cos t)(1-\cos t)}{\sin^2 t}$

$\frac{dy'}{dt} = \frac{\sin^2 t - \cos t + \cos^2 t}{\sin^2 t} = \frac{1-\cos t}{\sin^2 t}$;

We also had $\frac{dx}{dt} = \sin t \Rightarrow \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{d^2y}{dx^2} = \frac{\frac{1-\cos t}{\sin^2 t}}{\sin t} = \frac{1-\cos t}{\sin^3 t}$;

b) Find the points on C where the tangent lines are vertical.

Tangent lines are vertical $\Rightarrow$ slope of tangent $= \infty = y'$;

$\Rightarrow \frac{1-\cos t}{\sin t} = \infty \Rightarrow \boxed{\sin t = 0}$

$\sin t = 0$ for $t = 0$, and $t = \pi$;

rejected, since you get $\frac{0}{0}$

$x = 1 - \cos\pi$

$x = 1 + 1$

$\boxed{x = 2}$

and $y = \pi - \sin\pi$.

$y = \pi - 0$

$\boxed{y = \pi}$

$\Rightarrow$ So At the point $(2 ; \pi)$, we have vertical tangents on the curve C ;

$\Rightarrow$ But we have $0 \le t \le 4\pi$,

$\Rightarrow$ So for $t = 3\pi$ we have a vertical tangent.

$\Rightarrow$ $\Rightarrow \boxed{t = \pi}$ and $\boxed{t = 3\pi}$ at $(2 ; \pi)$.