Notre Dame University
Faculty of Natural and Applied Sciences
Department of Mathematics and Statistics

MAT 113
INTENSIVE CALCULUS

EXAM 1

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Duration: 65 minutes

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Section: A

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Grade: $\frac{99}{100}$

Please note that you have 9 exercises and 8 pages

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1) **(11 points)** Calculate the following limits:

a) $\lim\limits_{y\to 3}\dfrac{y^2-2y-3}{-y^3+5y^2-6y}$ $= \dfrac{(3)^2-2(3)-3}{-(3)^3+5(3)^2-6(3)} = \boxed{\dfrac{0^{\pm}}{0^{\pm}}}$ Ind. Form

$$\Rightarrow \lim_{y\to 3}\frac{(y-3)(y+1)}{-y(y-3)(y-2)} = \frac{3+1}{-3(3-2)} = \boxed{\frac{-4}{3}}$$

b) $\lim\limits_{x\to 1}\dfrac{\sqrt{17-x}-4}{\sqrt{x}-1}$ $= \dfrac{\sqrt{17-1}-4}{\sqrt{1}-1} = \boxed{\dfrac{0^{\pm}}{0^{\pm}}}$ Ind Form.

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt{17-x}-4}{\sqrt{x}-1}\times\left(\frac{\sqrt{17-x}+4}{\sqrt{17-x}+4}\right)$$

$$= \lim_{x\to 1}\frac{17-x-16}{(\sqrt{x}-1)(\sqrt{17-x}+4)} = \lim_{x\to 1}\frac{1-x}{(\sqrt{x}-1)(\sqrt{17-x}+4)}$$

$$= \lim_{x\to 1} -\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{17-x}+4)}$$

$$= \boxed{-\frac{1}{4}}$$

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2) **(11 points)** Consider the function $f(x)=\dfrac{x^2-1}{x^2-5x+4}$

Can $f(x)$ be extended to become continuous at $x=1$? At $x=4$? Give reasons for your answers. Define the continuous extension at the value where it exists.

$$f(x) = \frac{x^2-1}{x^2-5x+4}$$

$$f(1) = \frac{1-1}{1-5+4} = \boxed{\frac{0^\pm}{0^\pm}}$$ Ind form, so f is discontinuous at $x=1$.

$$\lim_{x\to 1} f(x) = \cancel{\lim_{x\to 1}\frac{x-1}{(x-4)(x-1)}} \quad \lim_{x\to 1}\frac{x^2-1}{x^2-5x+4} = \frac{0^\pm}{0^\pm}$$ Ind form

$$\Rightarrow \lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-4)(x-1)} = \lim_{x\to 1}\frac{x+1}{x-4} = \boxed{-\frac{2}{3}}$$

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So we can extand the function $f(x)$ to make it continuous at $x=1$, and the extended function

$$f(x)\begin{cases} f(x) & ; x\neq 1 \\ \lim_{x\to 1} f(x) & ; x=1 \end{cases} \Rightarrow \begin{cases} \dfrac{x^2-1}{x^2-5x+4} & ; x\neq 1 \\ -\dfrac{2}{3} & ; x=1 \end{cases}$$

$$* \quad f(4) = \frac{(4)^2-1}{(4)^2-5(4)+4} = \boxed{\frac{15}{0}}$$ Ind form

$$\lim_{x\to 4}\frac{x^2-1}{x^2-5x+4} = \lim_{x\to 4}\frac{x+1}{x-4} = \frac{5}{0^\pm} = \pm\infty$$, so we cannot extend it on $x=4$.

3) **(5 points)** Show that $f(x)=x^3-4x+4$ has a root in the interval [-3; 0]

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* f is a polominal, so it is defin and continuous on its ~~do~~ interval $[-3;0]$

* $f(-3) = (-3)^3 - 4(-3) + 4 = -11 < 0$

  $f(0) = 4 > 0$

then $f(x)$ has at least one root α between -3 and 0, where $f(\alpha) = 0$.

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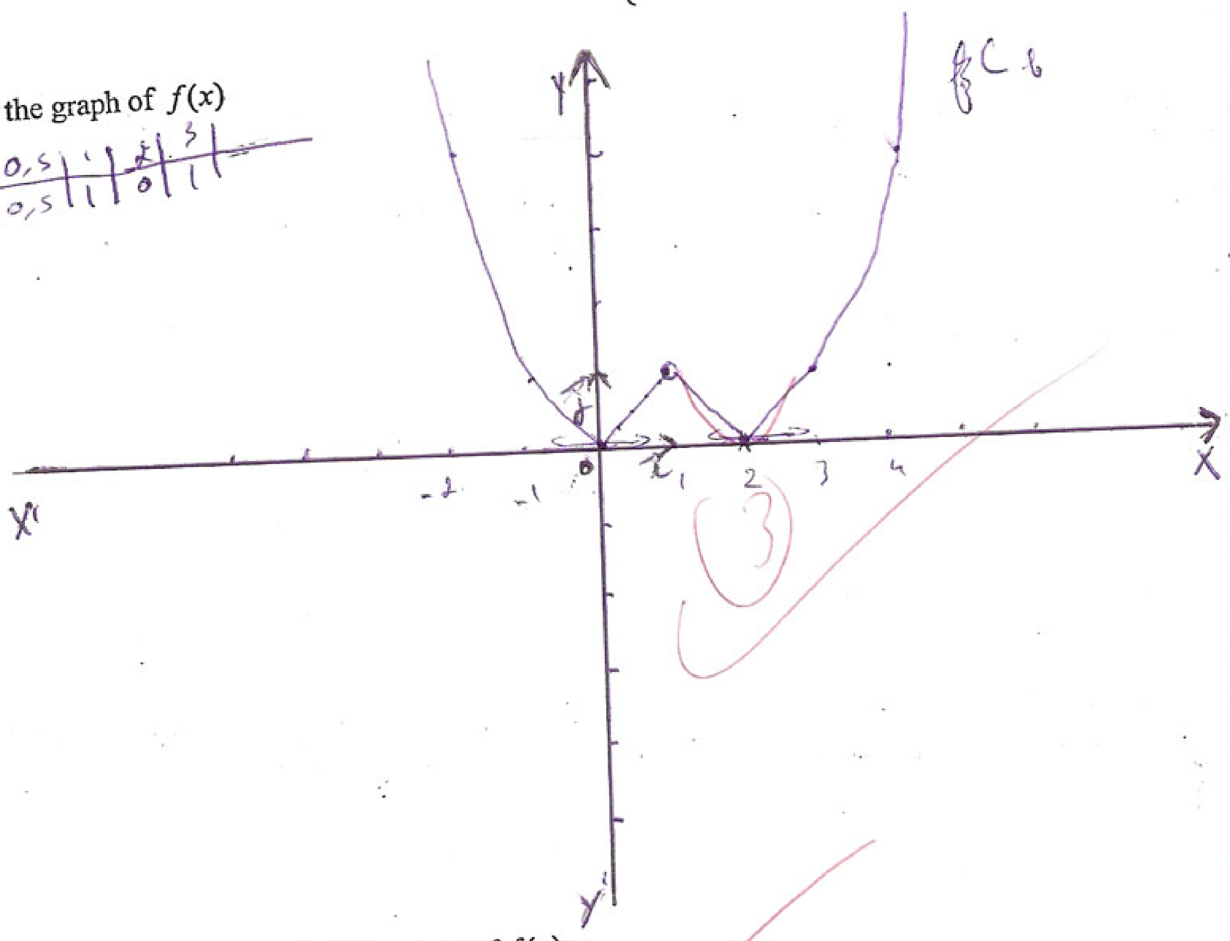
4) **(10 points)** Let $f(x)$ be the function defined by: $f(x)=\begin{cases} x^2 & x\le 0 \\ x & 0<x<1 \\ (x-2)^2 & x>1 \end{cases}$

a) Sketch the graph of $f(x)$

x	-2	-1	0	0,5	1	2	3
y	4	1	0	0,5	1	0	1



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b) Determine the points at which the curve of $f(x)$:

Is discontinuous? Justify your answer

f(x) is discontinuous at x = 1 because f(1) does not exist in the interval of f.

Is not differentiable? Justify your answer at x = 0, x = 1, ~~at x = 2~~.

f(x) is not differentiable at x = 1 and x = 2, f it is not diff. because there are a corner.

And at ~~x = 0~~ x = 1, f is not either continuous then for since it is not diff.

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Has a derivative equal to 0? Justify your answer

derivative equal to 0 => f'(x) = 0 => horizontal tang.

f'(x) = 0 at x = 0 and at x = 2.

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5) (12 points)

a) Using the definition of the derivative, find the derivative of $f(x)=3x^2-6x$

$$\lim_{h\to 0}\frac{f(h+x)-f(x)}{h} = \cancel{\lim_{h\to 0} 3(h+x)^2-6(h+x)-}$$

$$= \lim_{h\to 0}\frac{3(h+x)^2-6(h+x)-3x^2+6x}{h} \cancel{=\lim_{h\to 0} 3h^2+3x^2+6hx}$$

$$= \lim_{h\to 0}\frac{3h^2+3x^2+6hx-6h-6x-3x^2+6x}{h} = \lim_{h\to 0}\frac{h(3h+6x-6)}{h}$$

$$= \boxed{6x-6}$$

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b) Does the curve of $f(x)$ have any horizontal tangents? If yes, where?

$f'(x) = 6x - 6$

$6x - 6 = 0$

$6x = 6$

$\boxed{x = 1}$

at $x = 1$, $f(x)$ have an horizontal tangent

$f'(1) = 6(1) - 6 = 0$

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6) **(13 points)** Find the derivative of the following functions:

a) $f(x)=\dfrac{3x^5-8x^2+3x+4}{x^3}$ $= 3x^2 - 8x^{-1} + 3x^{-2} + 4x^{-3}$

$f'(x) = 6x + 8x^{-2} - 6x^{-3} - 12x^{-4}$

$f'(x) = 6x + \frac{8}{x^2} - \frac{6}{x^3} - \frac{12}{x^4}$

$\boxed{f'(x) = \frac{6x^4 + 8x^2 - 6x - 12}{x^4}}$

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b) $g(x)=(x+2\sqrt{x})^3+\sec^2(\frac{1}{x})$

$g'(x) = 3(x+2\sqrt{x})^2 \cdot (1 + \frac{2}{2\sqrt{x}}) + 2(\frac{-1}{x^2})\sec(\frac{1}{x}) \cdot \sec(\frac{1}{x}) \cdot \tan(\frac{1}{x})$

$g'(x) = 3(x+2\sqrt{x})^2 (\frac{\sqrt{x}+1}{\sqrt{x}}) - \frac{2}{x^2}\sec^2(\frac{1}{x}) \cdot \tan(\frac{1}{x})$

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7) **(10 points)** Consider the curve defined by the parametric equations: $\begin{cases} x = 2\cos t \\ y = \sin^2 t \end{cases}$

a) Find the slope for the line tangent to the curve at $t=0$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(\sin t)(\cos t)}{-2\sin t} = -\cos t$

$\left.\frac{dy}{dx}\right|_{t=0} = -\cos 0 = -1$

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b) Find the value of $\dfrac{d^2y}{dx^2}$ at the same point

$\frac{d^2y}{dx^2} = \frac{\sin t}{-2\sin t} = -\frac{1}{2}$

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8)(16 points) Find an equation for the normal line to the curve given by:

$\sin(xy)+xy=\frac{\pi}{2}+1$ at the point $(1;\frac{\pi}{2})$

$$\left(\sin(xy)+xy\right)'=\left(\frac{\pi}{2}+1\right)'$$

$$(y+xy')\cos(xy)+y+xy'=0$$

$$y\cos(xy)+xy'\cos(xy)+y+xy'=0$$

$$y'(x\cos(xy)+x)=-y\cos(xy)-y$$

$$y'=\frac{-y\cos(xy)-y}{x\cos(xy)+x}$$

$$y'(1;\pi/2)=\frac{-(\pi/2)\cos(\pi/2)-\pi/2}{1\cos(\pi/2)+1}=-\frac{\pi}{2}$$

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~~m'~~ slope of the normal $= m' = \frac{-1}{(-\frac{\pi}{2})} = \frac{2}{\pi}$

$$y-\frac{\pi}{2}=\frac{2}{\pi}(x-1)$$

$$y=\frac{2x}{\pi}-2+\frac{\pi}{2}$$

$$\boxed{y=\frac{2x}{\pi}+\frac{-2_1+\pi}{2}}$$

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9) **(12 points)** Consider the function $f(x) = 2x^2 - 8$ over the interval [-1; 3]

a) Find the absolute maximum and absolute minimum of $f(x)$ over the given interval.

$f(x) = 2x^2 - 8$ is continuous over its domain [-1; 3]

$f(-1) = -6$ at $x = -1 \to y = -6$

$f(3) = 10$ at $x = 3 \to y = 10$

$f'(x) = 4x$ is always defined on [-1; 3]

$f'(x) = 4x = 0$ when $x = 0$

$f(0) = -8$ at $x = 0 \to y = \underline{-8}$

Absolute maximum is 10 at $x = 3$

" minimum is -8 at $x = 0$

b) Explain why we can apply the Mean Value Theorem on $f(x)$ over the given interval then apply it.

we can apply the mean value theorem on $f(x)$ over the interval [-1; 3] because:

* f is continuous over [-1; 3]
* $f'(x) = 4x$ is defined over [-1; 3]

so there are some pt $\underline{c}$ that:

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

$$\Rightarrow 4c = \frac{F(3) - F(-1)}{3 + 1} = \frac{10 + 6}{4} = 4$$

$\boxed{c = 1}$