1-(10 points) Find the following limits:

1 pt   3 pt

a) $\lim_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x} = \frac{4-8+4}{8+20-28} = \frac{0}{0} = \lim_{x\to 2} \frac{(x-2)^2}{x(x^2+5x-14)}$

$= \lim_{x\to 2} \frac{(x-2)^2}{x(x-2)(x+7)} = \lim_{x\to 2} \frac{x-2}{x(x+7)} = \frac{0}{18} = 0$

1 pt

1 pt   3 pt

b) $\lim_{x\to 9} \frac{\sqrt{x+7}-4}{x-9} = \frac{4-4}{9-9} = \frac{0}{0} = \lim_{x\to 9} \frac{\sqrt{x+7}-4}{x-9}\left(\frac{\sqrt{x+7}+4}{\sqrt{x+7}+4}\right)$

$= \lim_{x\to 9} \frac{x+7-16}{(x-9)(\sqrt{x+7}+4)} = \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x+7}+4)}$

$= \lim_{x\to 9} \frac{1}{\sqrt{x+7}+4} = \frac{1}{4+4} = \frac{1}{8}$

1 pt

THE DEBATE CLUB

2-(10 points) Can $f(x)=\dfrac{x-2}{x^2-4}$ be extended to be continuous at x=2 or x= -2?

(Give reasons for your answers)

7

$* f(2)=\frac{0}{0} \Rightarrow f(2)$ is undefined $\Rightarrow f(x)$ is discont at $x=2$.

$\lim_{x\to 2} f(x)=\frac{0}{0}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$.

$\therefore$ f(x) can be extended to be cont at $x=2$.

and the extended function is $F(x)=\begin{cases} f(x)=\frac{x-2}{x^2-4} & ; x\neq 2 \\ \frac{1}{4} & ; x=2.\end{cases}$

3

$* f(-2)=\frac{-4}{0} \Rightarrow f(-2)$ is undefined $\Rightarrow f(x)$ is discont at $x=-2$

$\lim_{x\to -2} f(x)=\frac{-4}{0^{\pm}}=\mp\infty \Rightarrow \lim_{x\to -2} f(x)$ does not exist

$\therefore$ we can not extend f(x) to become continuous at $x=-2$.

3-(10 points) Show that $f(x)=x^3-2x+2$ has a root between -2 and 0.

2

$f(x)=x^3-2x+2$ is cont on $[-2,0]$

4

and $f(-2)=-8+4+2=-2<0$

$f(0)=2>0$

4

$\therefore$ f(x) takes the value zero for some $\alpha\in(-2,0)$

$\therefore$ there is a root $\alpha\in(-2,0)$ such that $f(\alpha)=0$.

THE DEBATE CLUB

4-(15 points) Find the derivatives of the following functions:

a) $f(x)=\dfrac{x^5+3x^3-2x+2}{x}$ $=(x^5+3x^3-2x+2)x^{-1}=x^4+3x^2-2+2x^{-1}$

$\therefore f'(x)=4x^3+6x-2x^{-2}=4x^3+6x-\dfrac{2}{x^2}$.

5 pts

b) $g(x)=\sin^2(x+\sqrt{x})$

$g'(x)=2\sin(x+\sqrt{x})\cdot\cos(x+\sqrt{x})\cdot\left(1+\dfrac{1}{2\sqrt{x}}\right)$

$=\left(\dfrac{4\sqrt{x}+2}{2\sqrt{x}}\right)\sin(x+\sqrt{x})\cdot\cos(x+\sqrt{x})$

$=\dfrac{2\sqrt{x}+1}{\sqrt{x}}\sin(x+\sqrt{x})\cdot\cos(x+\sqrt{x})$.

5 pts

c) $h(x)=(3+\dfrac{2}{x^2-1})^5$ $=(3+2(x^2-1)^{-1})^5$.

$h'(x)=5(3+2(x^2-1)^{-1})^4(0+2(-1)(x^2-1)^{-2})(2x)$

$=5(3+\dfrac{2}{x^2-1})^4\left(\dfrac{-4x}{(x^2-1)^2}\right)$

$=-\dfrac{20x}{(x^2-1)^2}(3+\dfrac{2}{x^2-1})^4$.

5 pts

THE DEBATE CLUB

5-(12 points) Use the definition to find the derivative for $f(x)=2x^2+5$, then find the equation of the tangent line to the curve of $f(x)$ at the point of abscissa x=2.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2+5-(2x^2+5)}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2+2hx+h^2)+5-2x^2-5}{h}$$

7 pts

$$= \lim_{h\to 0} \frac{\cancel{2x^2}+4hx+2h^2\cancel{+5}\cancel{-2x^2}\cancel{-5}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4x+2h)}{\cancel{h}} = \lim_{h\to 0} 4x+\cancel{2h}^{0} = 4x.$$

* pt of tangency = pt of contact = $(2, f(2)=13) = (2,13)$

m = slope of the tangent line = $f'(2) = 4(2) = 8$.

eq of the tangent line: $y-13 = 8(x-2)$

$y = 8x - 16 + 13$

$\boxed{y = 8x - 3}$.

5 pt

ints) Find the equation of the normal line to the curve $x^3 - 3xy + y^2 = -1$ at the point (1, 1).

$\frac{dy}{dx} = ??$

$(x^3 - 3xy + y^2)' = (-1)' \Rightarrow 3x^2 - 3(y + xy') + 2yy' = 0.$

$\Rightarrow 3x^2 - 3y - 3xy' + 2yy' = 0 \Rightarrow (2y - 3x)y' = 3y - 3x^2.$

8 pts $\Rightarrow y' = \frac{3y - 3x^2}{2y - 3x}$

m = slope of the tangent line $= \frac{3-3}{2-3} = 0 \Rightarrow$ horizontal

7 pts $\Rightarrow m'$ = slope of the normal line is undefined $\Rightarrow$ vertical

$\therefore$ The normal is the vertical line through (1,1)

$\Rightarrow$ normal is $\boxed{x = 1}$.

THE DEBATE CLUB

7-(15 points) Find the slope for the line tangent to the curve $x=\frac{1}{2}\tan t$, $y=\frac{1}{2}\sec t$ at $t=\frac{\pi}{3}$. Also, find the value of $\frac{d^2y}{dx^2}$ at this point.

$$* \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}\sec t \cdot \tan t}{\frac{1}{2}\sec^2 t} = \frac{\tan t}{\sec t} = \frac{\frac{\sin t}{\cos t}}{\frac{1}{\cos t}} = \sin t.$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \text{slope of the tangent line.}$$

$$* \quad \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = ???$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\cos t}{\frac{1}{2}\sec^2 t} = 2\,\frac{\cos t}{\frac{1}{\cos^2 t}} = 2\cos^3 t$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3}\right)^3 = 2\left(\frac{1}{2}\right)^3 = \frac{1}{4}.$$

THE DEBATE CLUB

8-(13 points) for what values of m, if any, is $f(x) = \begin{cases} \sin(2x), & x \le 0 \\ mx, & x \ge 0 \end{cases}$

a) Continuous at x=0?

b) Differentiable at x=0?

THE DEBATE CLUB

a) to be cont at $x=0 \Rightarrow f(0^-) = f(0^+) = f(0)$.

$f(0^-) = \lim_{x \to 0^-} f(x) = \sin(0^-) = 0 = f(0) = \sin(0)$

6 pts $f(0^+) = \lim_{x \to 0^+} f(x) = 0$

$\therefore f(x)$ is cont at 0 for any m.

b) $f'(x) = \begin{cases} 2\cos(2x) & ; x < 0 \\ m & ; x > 0 \end{cases}$

for $f(x)$ to be diff at $x=0 \Rightarrow f'(0^-) = f'(0^+)$.

$f'(0^-) = \lim_{x \to 0^-} f'(x) = 2\cos(0^-) = 2$

7 pts $f'(0^+) = \lim_{x \to 0^+} f'(x) = m$

$\Rightarrow m = 2$

$\therefore$ if $m=2$ then $f'(0^-) = f'(0^+) \Rightarrow f(x)$ is diff at $x=0$.

THE DEBATE CLUB