Math 201 — Fall 2004–05

Calculus and Analytic Geometry III, sections 5–8
Quiz 1, October 28 — Duration: 1 hour

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

YOUR NAME: ______________________

YOUR AUB ID#: ______________________

PLEASE CIRCLE YOUR SECTION:

Section 5	Section 6	Section 7	Section 8
Recitation M 1	Recitation Tu 12:30	Recitation Tu 2	Recitation Tu 3:30
Professor Makdisi	Mr. Khatchadourian	Mr. Khatchadourian	Mr. Khatchadourian

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points.
3. You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. No calculators, books, or notes allowed. Turn OFF and put away any cell phones.

GOOD LUCK!

Sample solutions to last year's Quiz I (2004-2005) with professor Makdisi

1. (4 pts for each part, total 12 pts) Which of the following series converge, and why?

$$\text{a)} \sum_{n=0}^{\infty} \frac{2^n + n}{2^n + n^2} \qquad \text{b)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.9}} \qquad \text{c)} \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$$

a) $a_n = \dfrac{2^n+n}{2^n+n^2} \cdot \dfrac{\frac{1}{2^n}}{\frac{1}{2^n}} = \dfrac{1+\frac{n}{2^n}}{1+\frac{n^2}{2^n}} \longrightarrow \dfrac{1+0}{1+0} = 1$

because $n \lll 2^n$ & $n^2 \lll 2^n$ for n large (polynomial growth $\lll$ exponential growth).

$\therefore$ $a_n \not\to 0$ so $\sum a_n$ diverges by the nth term test.

b) here $a_n = (-1)^n u_n$ where $u_n = \frac{1}{n^{0.9}} > 0$.
We note that $n \to \infty \Rightarrow u_n \longrightarrow 0$ (since $n^{0.9} \to \infty$)
and n increases $\Rightarrow$ $n^{0.9}$ increases $\Rightarrow$ u_n decreases.
So this is an alternating series that converges by the alternating series test.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$ can be compared to the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.8}}$ as follows: for large n, $(\ln n) \lll n^{0.1}$ (log growth $\lll$ poly growth). So $\dfrac{(\ln n)^2}{n^2} \lll \dfrac{(n^{0.1})^2}{n^2} = \dfrac{1}{n^{1.8}}$ as claimed. But now $\sum \frac{1}{n^{1.8}}$ converges (p-series with $p = 1.8 > 1$), so $\displaystyle\sum_1^{\infty} \frac{(\ln n)^2}{n^2}$ CONVERGES by the direct comparison test.

2. (4 pts for each part, total 12 pts) Find the limit of each sequence below:

a) $\lim\limits_{n\to\infty} \dfrac{10^n \sin n}{n!}$ b) $\lim\limits_{n\to\infty} \cos^{-1}(n^2(0.1)^n)$ c) $\lim\limits_{n\to\infty} \dfrac{\ln(1+\frac{2}{n^2})}{\cos(\frac{10}{n})-1}$

a) $|\sin n| \leq 1$ so $\dfrac{-10^n}{n!} \leq \dfrac{10^n \sin n}{n!} \leq \dfrac{10^n}{n!}$

But $\dfrac{10^n}{n!} \to 0$ (expo. growth <<< factorial growth)

and $-\left(\dfrac{10^n}{n!}\right) \to -0 = 0$, so by the sandwich theorem
(same limit)

$\dfrac{10^n \sin n}{n!} \longrightarrow \boxed{0}$

b) here $n^2(0.1)^n = \dfrac{n^2}{10^n} \longrightarrow 0$ since poly growth <<< expo. growth.

Now $f(x) = \cos^{-1} x$ is continuous at $x=0$, so

$\lim\limits_{n\to\infty} \cos^{-1}\left(\dfrac{n^2}{10^n}\right) = \cos^{-1}\left(\lim \dfrac{n^2}{10^n}\right) = \cos^{-1} 0 = \boxed{\pi/2}$.

c) use Taylor expansion: $\boxed{\text{for } x \approx 0}$ $\ln(1+x) = x - \frac{x^2}{2} + \cdots$, $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$

$\ln\left(1+\frac{2}{n^2}\right) = \frac{2}{n^2} + O\left(\frac{1}{n^4}\right)$ (note $n \to \infty$ means $\frac{1}{n} \approx 0$)

$\cos\left(\frac{10}{n}\right) - 1 = \cancel{1} - \dfrac{(10/n)^2}{2} + O\left(\frac{1}{n^4}\right) \cancel{-1} = -\dfrac{50}{n^2} + O\left(\frac{1}{n^4}\right)$

So $\dfrac{\ln(1+\frac{2}{n^2})}{\cos(\frac{10}{n})-1} = \dfrac{\frac{2}{n^2} + O(\frac{1}{n^4})}{-\frac{50}{n^2} + O(\frac{1}{n^4})} \cdot \dfrac{n^2}{n^2} = \dfrac{2 + O(\frac{1}{n^2})}{-50 + O(\frac{1}{n^2})} \longrightarrow \dfrac{2+0}{-50+0}$ as $n \to \infty$.

$\therefore$ the limit is $\dfrac{-2}{50} = \boxed{-\dfrac{1}{25}}$.

3. (3 pts for each of a–d, total 12 pts)

a, b, and c) Find ONLY the radius of convergence R for the following three series. Do NOT test the endpoints.

$$\text{a)} \sum_{n=0}^{\infty} \frac{(3x-2)^n}{5^{n+1}} \qquad \text{b)} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} x^n \qquad \text{c)} \sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^{n^2} x^n$$

d) Find the sum of the series (a) in terms of x. What is the interval of convergence?

a) write as $\sum_n \frac{3^n (x-\frac{2}{3})^n}{5^{n+1}}$. Here $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}(x-\frac{2}{3})^{n+1}}{5^{n+2}} \cdot \frac{5^{n+1}}{3^n(x-\frac{2}{3})^n}\right|$

$= \frac{3|x-\frac{2}{3}|}{5} \xrightarrow[n\to\infty]{as} \rho = \frac{3|x-\frac{2}{3}|}{5}$. This converges for $\rho<1$

so $\frac{3|x-\frac{2}{3}|}{5} < 1$ so $\boxed{|x-\frac{2}{3}| < 5/3}$. The radius is $\boxed{R = 5/3}$.

b) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{(2n+2)!}\; x^{n+1}}{(n+1)!} \cdot \frac{n!}{\sqrt{(2n)!}\; x^n}\right|$

$= \sqrt{\frac{(2n+2)!}{(2n)!}} \cdot \frac{n!}{(n+1)!} \cdot |x| = \frac{\sqrt{(2n+2)(2n+1)}}{n+1} \cdot |x|$ $\frac{\sqrt{\frac{1}{n^2}}}{\frac{1}{n}}$

$= \frac{\sqrt{(2+\frac{2}{n})(2+\frac{1}{n})}}{1+\frac{1}{n}} |x| \longrightarrow \frac{\sqrt{2.2}}{1}|x| = 2|x| = \rho.$

here $\rho<1 \Leftrightarrow |x| < \frac{1}{2}$ so $\boxed{R = \frac{1}{2}}$.

c) Use the root test: $\sqrt[n]{|a_n|} = \left|\left(1+\frac{2}{n}\right)^{n^2} x^n\right|^{\frac{1}{n}}$

$= \left|\left(1+\frac{2}{n}\right)^n x\right| \longrightarrow e^2|x|$ since $\lim_{n\to\infty}\left(1+\frac{b}{n}\right)^n = e^b$.

so $\rho = e^2|x|$ & $\rho<1 \Leftrightarrow |x| < \frac{1}{e^2}$ so $\boxed{R = \frac{1}{e^2}}$.

d) See p.7 of extrasheets for the continuation of this problem

d) ~~See p.7 of extra sheets~~

4. (6 pts for each part, total 12 pts)

a) Find the second-order Taylor polynomial $P_2(x)$ of the function $f(x) = \sqrt{x}$, at the center $a = 100$.

b) For $90 \le x \le 110$, estimate the error $|f(x) - P_2(x)|$. Your answer should have the form $|f(x) - P_2(x)| \le a$, where a is an explicit constant. You do not have to simplify your expression for a.

a) $f(x) = x^{1/2} \Longrightarrow f(100) = 10$

$f'(x) = \frac{1}{2}x^{-1/2} \Longrightarrow f'(100) = \frac{1}{2} \cdot \frac{1}{10} = \frac{1}{20}$

$f''(x) = -\frac{1}{4}x^{-3/2} \Longrightarrow f''(100) = -\frac{1}{4} \cdot \frac{1}{10^3} = -\frac{1}{4000}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ (useful for part b)

$$P_2(x) = f(100) + f'(100)(x-100) + f''(100)\frac{(x-100)^2}{2!}$$

$$= 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2$$ (note the 2!)

[It is also correct to write: $P_2(100+\Delta x) = 10 + \frac{1}{20}\Delta x - \frac{1}{8000}(\Delta x)^2$]

b) here $-10 \le x - 100 \le 10$ so $|\Delta x| \le 10$.

the |error| is $|R_2(x)| = \left|\frac{f'''(c)}{3!}(\Delta x)^3\right|$

where c is between 100 and x, so in particular $\boxed{90 \le c \le 110}$

Thus $|R_2(x)| = \left|\frac{\frac{3}{8}c^{-5/2}}{6}\right| \cdot |\Delta x|^3$

$$= \frac{1}{16\, c^{5/2}} |\Delta x|^3 \le \boxed{\frac{1}{16 \cdot (90)^{5/2}} \cdot (10)^3}$$

(note $90 \le c \le 110$

$\Rightarrow (90)^{5/2} \le c^{5/2} \le (110)^{5/2}$

$\Rightarrow \frac{1}{(90)^{5/2}} \ge c^{-5/2} \ge \frac{1}{(110)^{5/2}}$)

(this is $\approx 8.1 \times 10^{-4}$, quite good accuracy over a range $90 \le x \le 110$ whose width is 20)

5. (6 pts each part, total 12 pts)

a) Using power series, express the integral

$$L = \int_{x=0}^{0.1} \frac{\tan^{-1} x^2}{x}\, dx$$

as a certain alternating series. For full credit, the answer should be written using Σ notation. You can get nearly full credit for just writing out the first four (nonzero) terms in the series.

b) Find (with justification, of course) a specific partial sum s_n for which the error satisfies $|s_n - L| < 10^{-11}$.

Note: in parts (a) and (b), you may use without proof the fact that your series satisfies the conditions of the alternating series estimation theorem.

$$\text{(so } \tan^{-1} x^2 = \Sigma (-1)^k \frac{x^{4k+2}}{2k+1}\text{)}$$

a) $$\tan^{-1} u = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+1}}{2k+1}$$

$$\frac{\tan^{-1} x^2}{x} = \frac{x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots}{x} = x - \frac{x^5}{3} + \frac{x^9}{5} - \frac{x^{13}}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{2k+1}$$

$$L = \int_0^{0.1} \frac{\tan^{-1} x^2}{x}\, dx = \left[\frac{x^2}{2} - \frac{x^6}{6 \cdot 3} + \frac{x^{10}}{10 \cdot 5} - \frac{x^{14}}{14 \cdot 7} + \cdots\right]_0^{0.1} = \boxed{\frac{(0.1)^2}{2} - \frac{(0.1)^6}{6 \cdot 3} + \frac{(0.1)^{10}}{10 \cdot 5} - \frac{(0.1)^{14}}{14 \cdot 7} + \cdots}$$

$$= \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)(4k+2)}\right]_0^{0.1} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k (0.1)^{4k+2}}{(2k+1)(4k+2)}}$$

full credit (6/6 pts)

part cred (5/6 pts)

b) we want to sandwich L between S_k and $S_{k+1} = S_k + \frac{(-1)^k (0.1)^{4k+2}}{(2k+1)(4k+2)}$ *. Looking at the expansion with the first few terms, we see that we can sandwich L between

$$\frac{(0.1)^2}{2} - \frac{(0.1)^6}{(6)(3)} \quad \text{and} \quad \frac{(0.1)^2}{2} - \frac{(0.1)^6}{6 \cdot 3} + \frac{(0.1)^{10}}{10 \cdot 5}$$

So $$\boxed{L \approx \frac{(0.1)^2}{2} - \frac{(0.1)^6}{(6)(3)}}$$ with $|\text{error}| < \frac{(0.1)^{10}}{(10)(5)} = \frac{10^{-10}}{(10)(5)} = \frac{1}{5} \times 10^{-11} < 10^{-11}$!

* since $|L - S_k| \leq |S_{k+1} - S_k| = \frac{(0.1)^{4k+2}}{(2k+1)(4k+2)}$.

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6. (4 pts for part a, 8 pts for part b, total 12 pts)

a) Show that the series $\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges.

b) Estimate the error that we obtain if we use the partial sum s_5 to approximate the value of the series.

a) $a_n = \frac{2^n}{1+3^n} < \frac{2^n}{3^n}$, so $\sum_{n=1}^{\infty} a_n$ converges by comparison with the geometric series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ (geometric with $r = \frac{2}{3} < 1$)

b) $s_5 = \frac{2^1}{1+3^1} + \frac{2^2}{1+3^2} + \cdots + \frac{2^5}{1+3^5}$

$$L = \sum_{n=1}^{\infty} \frac{2^n}{1+3^n} = \frac{2^1}{1+3^1} + \frac{2^2}{1+3^2} + \cdots + \frac{2^5}{1+3^5} + \frac{2^6}{1+3^6} + \frac{2^7}{1+3^7} + \cdots$$

$$L - s_5 = \frac{2^6}{1+3^6} + \frac{2^7}{1+3^7} + \cdots$$

so $0 < L - s_5 < \frac{2^6}{3^6} + \frac{2^7}{3^7} + \frac{2^8}{3^8} + \cdots$

$$= \frac{2^6}{3^6}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$$

$$\boxed{|\text{error}| = |L - s_5| \leq \frac{2^6}{3^6}\left(\frac{1}{1-\frac{2}{3}}\right) \quad \text{by summing the geometric series}}$$

you can [redacted] also simplify to

$$|L - s_5| < \frac{2^6}{3^5} \quad (\approx 0.263)$$

(Note: $|L - s_n| < \frac{2^{n+1}}{3^n}$ by a similar argument, so for larger n the error is $< 2 \cdot \left(\frac{2}{3}\right)^n$ which becomes small exponentially fast.)