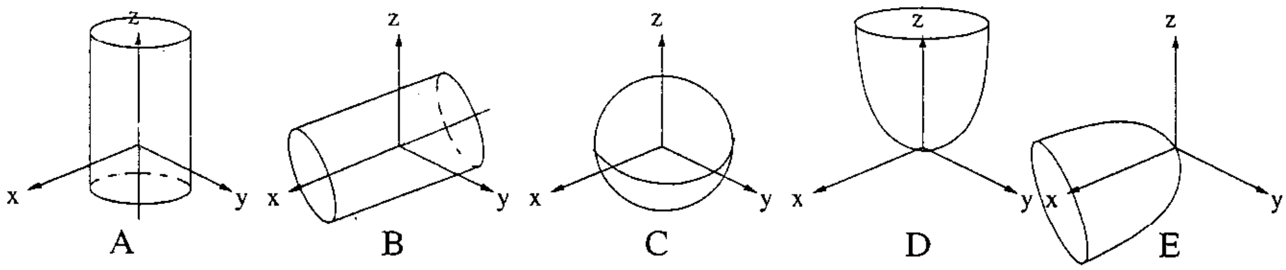
Fall 2004-2005

Math 201 F'04-05 Quiz II Solutions

Professor Khuri-Makdisi

1. Multiple-choice questions (1a–1f): **CIRCLE YOUR ANSWERS.** NO PARTIAL CREDIT, NO PENALTY FOR GUESSING. You do not have to justify your answers for the multiple-choice questions. Each of questions 1a–1f is worth 2 points.

(i) Identify each equation below with ONE of figures A–E below.



Circle your answers:

1a)	$x^2+y^2+z^2=1$	A	B	C	D	E
1b)	$x^2+y^2=1$	A	B	C	D	E
1c)	$y^2+z^2=x$	A	B	C	D	E

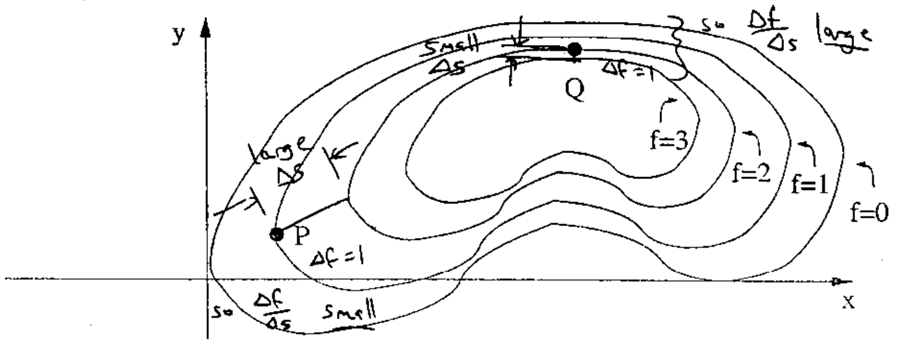
(ii) Given a function $f(x,y)$ satisfying $f(1,2)=6$, $\vec{\nabla} f\Big|_{(1,2)} = (3,-2)$. Evaluate (in scratchwork) an approximation for each of $f(1.01, 2.02)$ and $f(0.98, 1.99)$. Then use your approximation to circle the correct answer in each question below:

[scratch: $f(1+\Delta x, 2+\Delta y) \approx f(1,2) + \vec{\nabla} f|_{(1,2)} \cdot (\Delta x, \Delta y) = 6 + 3\Delta x - 2\Delta y$.] here $\Delta x = 0.01$, $\Delta y = 0.02$ here $\Delta x = -0.02$, $\Delta y = -0.01$

1d) The LARGEST number is :	$f(1,2)$	$f(1.01, 2.02)$	$f(0.98, 1.99)$
1e) The SMALLEST number is :	$f(1,2)$	$f(1.01, 2.02)$	$f(0.98, 1.99)$
	$= 6$	$\approx 6-0.01$	$\approx 6-0.04$

Note: comparing values in this way is not actually justified unless we can also estimate the error in the approximation! But don't let this worry you for the purposes of this exercise.

(iii) The level curves of a function $f(x,y)$ are shown below. Based on the picture, which of the two gradient vectors $\vec{\nabla} f\Big|_P$ and $\vec{\nabla} f\Big|_Q$ has a larger magnitude (i.e., length)? Circle your answer below.



1f) The LARGER magnitude is : $\left|\vec{\nabla} f\,|_P\right|$ $\left|\vec{\nabla} f\,|_Q\right|$.

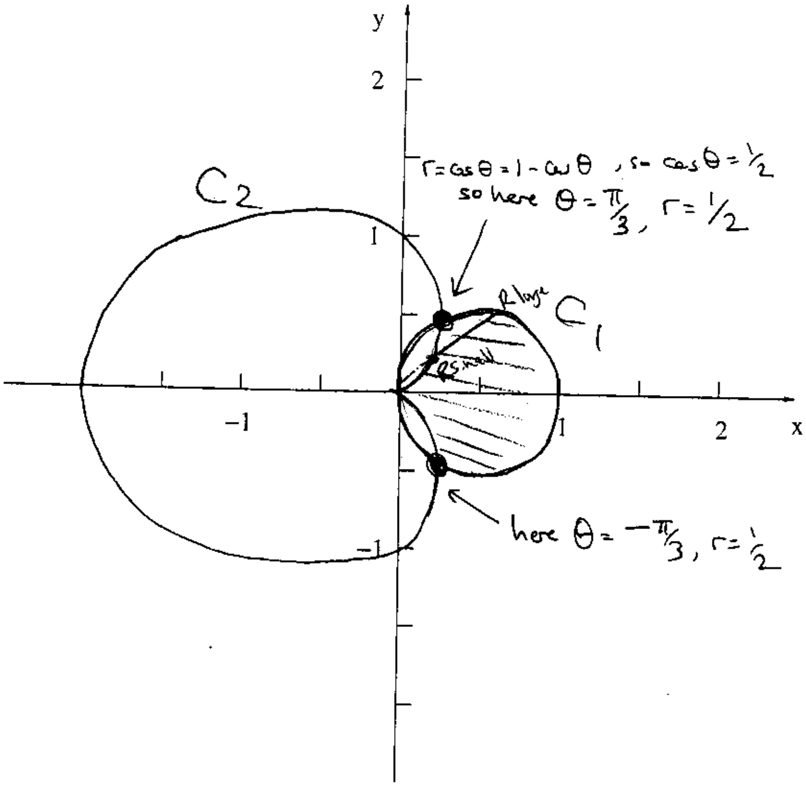
scratch: $\left|\vec{\nabla} f|_P\right| = D_{\vec{u}} f\Big|_P \approx \dfrac{\Delta f}{\Delta s}$ 1

for $\vec{u}$ in the direction of maximal increase; when you move in the direction of maximal increase.

2. Using polar coordinates, **sketch and clearly label** the curves C_1, C_2 given by

$$C_1 : r = \cos\theta, \qquad C_2 : r = 1 - \cos\theta$$

and **find the area** of the region that is **inside** C_1 **and outside** C_2.



table

θ	$\cos\theta$	$1-\cos\theta$
0	1	0
$\pi/4$	$\frac{1}{\sqrt{2}} \sim 0.7$	$1 - \frac{1}{\sqrt{2}} \sim 0.3$
$\pi/2$	0	1
$3\pi/4$	$-\frac{1}{\sqrt{2}} \sim -0.7$	$1 + \frac{1}{\sqrt{2}} \sim 1.7$
π	-1	2
$5\pi/4$	$-\frac{1}{\sqrt{2}} \sim -0.7$	~ 1.7
$3\pi/2$	0	1
$7\pi/4$	$\frac{1}{\sqrt{2}} \sim 0.7$	~ 0.3
2π	1	0

The shaded region has $-\pi/3 \le \theta \le \pi/3$, $R_{small} = 1 - \cos\theta$, $R_{large} = \cos\theta$

$$\text{Area} = \int_{\theta=-\pi/3}^{\pi/3} \frac{1}{2}\left[R_{large}^2 - R_{small}^2\right] d\theta = \int_{\theta=-\pi/3}^{\pi/3} \frac{1}{2}\Big[\underbrace{\cos^2\theta - (1-\cos\theta)^2}_{\text{note this is } \cos^2\theta - 1 + 2\cos\theta - \cos^2\theta = -1 + 2\cos\theta}\Big] d\theta$$

$$= \int_{\theta=-\pi/3}^{\pi/3} \frac{1}{2}(-1 + 2\cos\theta)\, d\theta = \left[-\frac{\theta}{2} + \sin\theta\right]_{\theta=-\pi/3}^{\pi/3} = -\frac{\pi}{6} + \frac{\sqrt{3}}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{-\frac{\pi}{3} + \sqrt{3}}$$

3. Consider the function $f(x,y,z) = ye^{x^2z}$ and the point $P_0(-1,2,1)$.

a) Find the gradient $\vec{\nabla} f \Big|_{P_0}$.

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(2xzye^{x^2z}, e^{x^2z}, x^2ye^{x^2z} \right)$$

$$\boxed{\vec{\nabla} f \Big|_{(-1,2,1)} = (-4e, e, 2e)}$$

now substitute $x=-1$, $y=2$, $z=1$ to get answer

b) Find the equation of the tangent plane to the surface $ye^{x^2z} = 2e$ at the point P_0.

$Q(x,y,z) \in$ tangent plane $\iff \vec{P_0Q} \perp \vec{\nabla} f|_{P_0}$

$\iff (x+1, y-2, z-1) \perp (-4e, e, 2e)$

$\iff \boxed{-4e(x+1) + e(y-2) + 2e(z-1) = 0}$ (dot product is zero)

(you may, if you wish, simplify to $-4(x+1) + y-2 + 2(z-1) = 0$
or even to $-4x + y + 2z = 8$)

c) Find the directional derivative of f at P_0 in the direction of the vector $\vec{v} = (1,2,3)$.

The unit vector in the direction of $\vec{v}$ is

$$\vec{U} = \frac{\vec{v}}{|\vec{v}|} = \frac{(1,2,3)}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}(1,2,3).$$

Then $D_{\vec{U}} f \Big|_{P_0} = \vec{\nabla} f \Big|_{P_0} \cdot \vec{U}$

$$= (-4e, e, 2e) \cdot \frac{1}{\sqrt{14}}(1,2,3)$$

$$= \frac{1}{\sqrt{14}}(-4e + 2e + 6e) = \boxed{\frac{4e}{\sqrt{14}}}$$

4. Given the parametrized curve in space: $P(t) = (x(t), y(t), z(t)) = (e^t, e^{-t}, \sqrt{2} \cdot t)$.

a) Find the arclength of the part of the curve between the points $P_1 = (1,1,0)$ (corresponding to $t_1 = 0$) and $P_2 = (e, 1/e, \sqrt{2})$ (corresponding to $t_2 = 1$). Note: you can, and should, simplify the expression inside the integral to get rid of the square root.

$\vec{r}(t) = \vec{OP}(t) = (e^t, e^{-t}, \sqrt{2}t)$ so the Velocity is

$$\vec{v} = \frac{d\vec{r}}{dt} = (e^t, -e^{-t}, \sqrt{2})$$

and the speed is $|\vec{v}| = \sqrt{(e^t)^2 + (-e^{-t})^2 + 2} = \sqrt{e^{2t} + e^{-2t} + 2}$.

So the arclength is $\displaystyle\int_{t=t_1=0}^{t=t_2=1} |\vec{v}|\, dt = \int_{t=0}^{1} \sqrt{e^{2t} + e^{-2t} + 2}\, dt$

But note $e^{2t} = (e^t)^2$, $e^{-2t} = (e^{-t})^2$, $2 = 2(e^t)(e^{-t})$

So arclength $= \displaystyle\int_{t=0}^{1} \sqrt{(e^t)^2 + (e^{-t})^2 + 2(e^t)(e^{-t})}\, dt$

$$= \int_{t=0}^{1} \sqrt{(e^t + e^{-t})^2}\, dt$$

$$= \int_{t=0}^{1} (e^t + e^{-t})\, dt = \left[e^t - e^{-t}\right]_{t=0}^{1} = e - e^{-1} - 1 + 1 = \boxed{e - e^{-1}}$$

(or $e - \frac{1}{e}$.)

b) (UNRELATED to part (a)) Use the same curve $P(t)$ as above. Given a function $f(P) = f(x, y, z)$ with $\vec{\nabla} f \Big|_{(3,1/3,\sqrt{2}\ \ln 3)} = (1, 6, 4)$, find the derivative $\frac{d}{dt}\left[f(P(t))\right]\Big|_{t=\ln 3}$.

at $t = t_0 = \ln 3$, our position is $P_0 = P(\ln 3) = (e^{\ln 3}, e^{-\ln 3}, \sqrt{2}\ln 3) = (3, \frac{1}{3}, \sqrt{2}\ln 3)$.

and our velocity is $\vec{v}\Big|_{t=\ln 3} = (e^{\ln 3}, -e^{-\ln 3}, \sqrt{2})$ from above

$$= (3, -\tfrac{1}{3}, \sqrt{2}).$$

By the chain rule, $\displaystyle\frac{d}{dt}\left(f(P(t))\right)\Big|_{t=\ln 3} = \left[\vec{\nabla} f\Big|_{P(t)} \cdot \vec{v}\right]_{t=\ln 3 = t_0}$

$$= \vec{\nabla} f\Big|_{\underset{P_0}{P(t_0)}} \cdot \vec{v}\Big|_{t=\ln 3} = (1, 6, 4) \cdot (3, -\tfrac{1}{3}, \sqrt{2})$$

4 $= 3 - 2 + 4\sqrt{2} = \boxed{1 + 4\sqrt{2}}$.

5. a) Find the maximum and minimum values of the function $f(P) = f(x,y,z) = x+y+z$, under the **constraint** that the point $P(x,y,z)$ is restricted to lie on the ellipsoid $2x^2+y^2+z^2 = 1/2$.

This is a Lagrange multiplier problem with $f(x,y,z) = x+y+z$

$g(x,y,z) = 2x^2+y^2+z^2 = \frac{1}{2}$

Note $\vec{\nabla} f = (1,1,1)$ and $\vec{\nabla} g = (4x, 2y, 2z)$.

The system is $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y,z) = \frac{1}{2} \end{cases} \iff \begin{cases} 1 = 4\lambda x & (1) \\ 1 = 2\lambda y & (2) \\ 1 = 2\lambda z & (3) \\ 2x^2+y^2+z^2 = \frac{1}{2} & (4) \end{cases}$

now (1) & (2) $\Rightarrow y = 4\lambda x y = 2x$
(1) & (3) $\Rightarrow z = 4\lambda x z = 2z$

substitute into (4): $2x^2 + (2x)^2 + (2x)^2 = \frac{1}{2}$

so $2x^2+4x^2+4x^2 = \frac{1}{2}$ so $x^2 = \frac{1}{20}$, $x = \pm \frac{1}{\sqrt{20}} = \pm \frac{1}{2\sqrt{5}}$

$z = y = 2x = \pm \frac{1}{\sqrt{5}}$ with same sign

The possible points to try are $P_1\left(\frac{1}{2\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$P_2\left(\frac{-1}{2\sqrt{5}}, \frac{-1}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

$f(P_1) = \frac{1}{2\sqrt{5}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{5}{2\sqrt{5}} = \boxed{\frac{\sqrt{5}}{2}}$ largest value at P_1

$f(P_2) = -\frac{1}{2\sqrt{5}} - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = \boxed{-\sqrt{5}/2}$ smallest value is at P_2

b) (UNRELATED to part (a)) Find the critical points of the function $f(x,y) = x^3 - x + 2xy + y^2$, and classify each point as a local minimum, a local maximum, or a saddle point.

$\vec{\nabla} f = (\underbrace{3x^2-1+2y}_{f_x}, \underbrace{2x+2y}_{f_y})$.

$P(x,y)$ is a critical point $\iff \vec{\nabla} f|_P = \vec{0} \iff \begin{cases} 3x^2-1+2y = 0 \\ 2x+2y = 0 \end{cases} \iff \begin{cases} 3x^2-1+2(-x) = 0 \\ y = -x \end{cases}$

$\iff \begin{cases} 3x^2-2x-1 = 0 \\ y = -x \end{cases}$

but $3x^2-2x-1 = (3x+1)(x-1)$ has roots (or use quadratic formula)

$x_1 = -\frac{1}{3}$, $x_2 = 1$

$\Downarrow$ $y_1 = +\frac{1}{3}$, $\Downarrow$ $y_2 = -1$

So the critical points are

$\boxed{P_1\left(-\frac{1}{3}, \frac{1}{3}\right)}$, $\boxed{P_2(1,-1)}$.

Hessian determinant: $\begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & 2 \\ 2 & 2 \end{pmatrix}$

At P_1: $\Delta = \begin{vmatrix} -2 & 2 \\ 2 & 2 \end{vmatrix} = -4-4 = -8 < 0$ $\therefore$ $\boxed{P_1 \text{ is a saddle point}}$

at P_2: $\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 2 \end{vmatrix} = 12-4 = 8 > 0$ and $f_{xx}|_{P_2} = 6 > 0$ $\therefore$ $\boxed{P_2 \text{ is a local minimum}}$

6. a) Using the two-path test, show that $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4}$ does NOT exist.

Take a path $P(t) = (t, at)$ for fixed a (i.e. $x=t$, $y=at$)

$t \to 0 \Rightarrow P(t) \to (0,0)$.

You must point this out

Then $\left.\frac{x^3y}{x^4+y^4}\right|_{P(t)} = \frac{t^3 \cdot at}{t^4+a^4t^4} = \frac{a}{1+a^4}$ after cancelling $\frac{t^4}{t^4}$

so $\lim_{t\to 0} \left.\frac{x^3y}{x^4+y^4}\right|_{P(t)} = \frac{a}{1+a^4}$ which depends on the choice of a

i.e. on the choice of path, as $P(t) \to (0,0)$

So $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4}$ does not exist.

b) (Challenging) Show that $\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^4} = 0$.

CAUTION it is NOT enough to check that the limit exists on straight-line paths $P(t)$ like those in part a. You must actively show that $\left|\frac{x^3y^2}{x^4+y^4}\right|$ becomes small for all $P(x,y)$ close to the origin.

This amounts to showing that the numerator becomes small faster than the denominator.

I.e. show the denominator is not too small.

Method I. Note $(x^2-y^2)^2 \geq 0$, so

$x^4+y^4-2x^2y^2 \geq 0$, so $\boxed{x^4+y^4 \geq 2x^2y^2}$.

Then $\left|\frac{x^3y^2}{x^4+y^4}\right| = \frac{|x^3y^2|}{x^4+y^4} \leq \frac{|x^3y^2|}{2x^2y^2} = \frac{|x|}{2}$

i.e. $-\frac{x}{2} \leq \left(\frac{x^3y^2}{x^4+y^4}\right) \leq \frac{x}{2}$

$-\frac{x}{2} \to 0$ as $(x,y)\to(0,0)$; $\frac{x}{2} \to 0$ as $(x,y)\to(0,0)$

since $0=0$ (the same limit on each end), by the sandwich theorem

$\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^4} = 0$.

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Method II. Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{x^3y^2}{x^4+y^4} = \frac{r^5 \cos^3\theta \sin^2\theta}{r^4(\cos^4\theta + \sin^4\theta)} = r \cdot \left(\frac{\cos^3\theta \sin^2\theta}{\cos^4\theta + \sin^4\theta}\right)$$

$$= r \cdot \frac{\cos^3\theta \sin^2\theta}{(\cos^2\theta + \sin^2\theta)^2 - \underbrace{2\cos^2\theta\sin^2\theta}_{\text{this is } \frac{1}{2}\cdot 4\cos^2\theta\sin^2\theta}} = r \cdot \frac{\cos^3\theta \sin^2\theta}{1 - \frac{1}{2}\sin^2 2\theta}$$

now notice $|\cos^3\theta| \le 1$, $|\sin^2\theta| \le 1$, $\boxed{1 - \frac{1}{2}\sin^2 2\theta \geq 1 - \frac{1}{2} = \frac{1}{2}}$ ⟸ VERY IMPORTANT

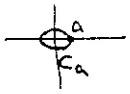
So $$\left|\frac{x^3y^2}{x^4+y^4}\right| = r \cdot \left|\frac{\cos^3\theta\sin^2\theta}{1 - \frac{1}{2}\sin^2 2\theta}\right| \leq r \cdot \frac{1\cdot 1}{\frac{1}{2}} = 2r$$

so again use the sandwich theorem for $\underset{\downarrow \atop 0}{-2r} \le \frac{x^3y^2}{x^4+y^4} \le \underset{\downarrow \atop 0}{2r}$.

Method III. Write $u = x^2$, $v = y^2$ & note $(x,y) \to (0,0) \iff (u,v) \to (0,0)$

So evaluate $\lim_{(u,v)\to(0,0)} \frac{u^{3/2}\, v}{u^2+v^2}$ which is easy by polar coordinates for u & v.

Method IV. To show that x^4+y^4 is not too small in terms of $r^2 = x^2+y^2$, fix a value of $r = a$ & consider the small circle $C_a : x^2+y^2 = a^2$.



Then use Lagrange multipliers to MINIMIZE $f(x,y) = x^4+y^4$ subject to the CONSTRAINT $g(x,y) = x^2+y^2 = a^2$.

So $$\begin{cases} \vec{\nabla} f = (4x^3, 4y^3) = \lambda \vec{\nabla} g = (2\lambda x, 2\lambda y) \\ g(x,y) = a^2 \end{cases} \iff \begin{cases} 4x^3 = 2\lambda x \\ 4y^3 = 2\lambda y \\ x^2+y^2 = a^2 \end{cases} \Rightarrow 4x^3y = 4y^3x \Rightarrow$$ $x=0$ ($y = \pm a$) or $y = 0$ (so $x = \pm a$) or $x = \pm y$ (so $x, y = \pm a/\sqrt{2}$)

so you consider the points
$(0, \pm a) \longrightarrow f = a^4$
$(\pm a, 0) \longrightarrow f = a^4$
$(\pm \frac{a}{\sqrt{2}}, \pm\frac{a}{\sqrt{2}}) \longrightarrow f = \frac{a^4}{4} + \frac{a^4}{4} = \frac{a^4}{2}$
so min value of f is $a^4/2$ where $a^2 = x^2+y^2$.

So in terms of polar coordinates, go back to $x^2+y^2 = r^2$, so $x^4+y^4 \geq r^4/2$

and $$\left|\frac{x^3y^2}{x^4+y^4}\right| \leq \frac{|x^3y^2|}{(r^4/2)} = \frac{r}{2}\,|\cos^3\theta \sin^2\theta| \leq \frac{r}{2},$$ so use the sandwich theorem again.