Math 201 — Fall 2005–06
Calculus and Analytic Geometry III, sections 5–8
Quiz 2, December 1 — Duration: 1 hour

Dec. 2, 2005

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

YOUR NAME: Solutions to

YOUR AUB ID#: Professor Makdisi's Quiz 2
Math 201, Fall 2005

PLEASE CIRCLE YOUR SECTION:

Section 5	Section 6	Section 7	Section 8
Recitation Tu 11	Recitation Tu 12:30	Recitation Tu 2	Recitation Tu 3:30
Ms. Jaber	Ms. Jaber	Ms. Jaber	Professor Makdisi

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points.
3. You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Open book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points.
Take a minute to look at all the questions, THEN
solve each problem on its corresponding page INSIDE the booklet.

1. Given the function $f(x,y,z) = x^2 + z^2 - y$. Let S be the level set of f given by $S = \{(x,y,z) \mid x^2 + z^2 - y = 5\}$.

a) (6 pts) Find the equation of the tangent plane to S at the point $P_0(2,0,1)$.

b) (6 pts) Draw a rough picture of S. (Hint: S is a paraboloid. Your drawing should clearly indicate the x, y, and z axes and the coordinates of the vertex [i.e., summit] of S.)

2. Given the parametrized curve $P(t) = (\frac{t^3}{3}, t^2, 2t)$.

a) (2 pts) Find the velocity vector $\vec{v}(t)$.

b) (5 pts) Find the arclength of the curve between the points $Q_1 = (0,0,0)$ and $Q_2 = (\frac{8}{3}, 4, 4)$.

c) (5 pts) Given $P\big|_{t=1} = (\frac{1}{3}, 1, 2)$, we know that there exist certain constants a, b, c giving us an approximation $P\big|_{t=1+\Delta t} \approx (\frac{1}{3} + a\Delta t, 1 + b\Delta t, 2 + c\Delta t)$. Find a, b, and c.

3. a) (5 pts) Make a table of values, and use it to sketch the curve C given in polar coordinates by $r = 1 + \cos 2\theta$.

b) (7 pts) Define $f(x) = \begin{cases} 0, & \text{if } -\pi \le x < -\pi/2 \\ x, & \text{if } -\pi/2 \le x \le \pi/2 \\ 0, & \text{if } \pi/2 < x \le \pi \end{cases}$ [we extend $f(x)$ to be periodic with period 2π]. In the Fourier series $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, find ONLY the coefficient b_3.

4. We are given a function $f(x,y)$ which satisfies:

$$f(1,2) = 10, \qquad \vec{\nabla} f \Big|_{(1,2)} = (3,4), \qquad \vec{\nabla} f \Big|_{(3,4)} = (5,6), \qquad \vec{\nabla} f \Big|_{(5,6)} = (1,2).$$

a) (4 pts) Find the directional derivative of f at the point $P_0(5,6)$ in the direction of the vector $\vec{v} = (3,4)$.

b) (4 pts) Find an approximate value for $f(1.02, 1.99)$.

c) (4 pts) Find the partial derivative $\frac{\partial}{\partial s}\left[f(s^2+t^2, 3st)\right]_{(s,t)=(1,2)}$. (In other words, we have substituted $x = s^2 + t^2$ and $y = 3st$.)

5. Consider an elliptic disk R in the plane, given by $R = \{(x,y) \mid x^2 + 2y^2 \le 12\}$. Find the maximum and minimum of the function $f(x,y) = xy + 10y$ on the region R. Indicate both the maximum/minimum values and the points where they are attained.

6. a) (4 pts) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does NOT exist.

b) (4 pts) Show that $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2} = 0$.

c) (4 pts) Find the value of $\lim_{(x,y)\to(0,0)} \frac{e^{x^2/2} - \cos y}{x^2+y^2}$. Hint: use Taylor series and $O(\cdot)$ notation. Even if you have not done part (b), you may use the result from there as well as the similar result $\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2} = 0$.

1. Given the function $f(x,y,z) = x^2 + z^2 - y$. Let S be the level set of f given by $S = \{(x,y,z) \mid x^2 + z^2 - y = 5\}$.

a) (6 pts) Find the equation of the tangent plane to S at the point $P_0(2,0,1)$.

The gradient $\vec{\nabla} f|_{P_0}$ is orthogonal to the tangent plane at P_0.

$\vec{\nabla} f = (2x, -1, 2z)$ so $\vec{\nabla} f|_{(2,0,1)} = (4,-1,2)$.

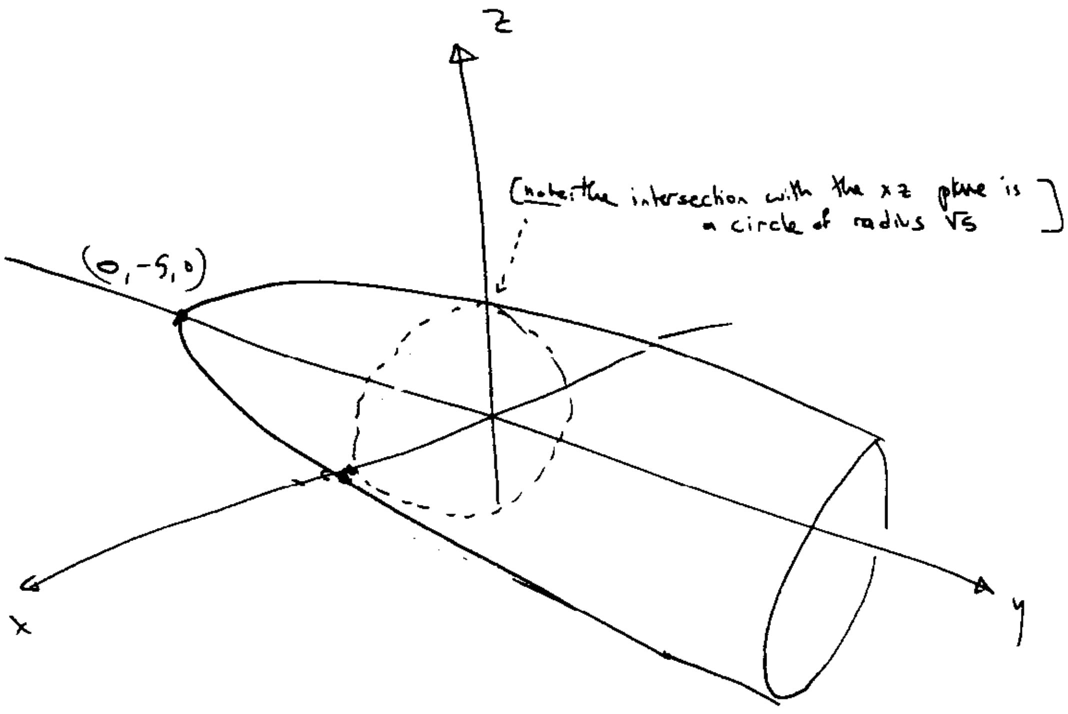
$Q(x,y,z) \in$ tangent plane $\iff \overrightarrow{P_0Q} \perp \vec{\nabla} f|_{P_0}$

$\iff (x-2, y, z-1) \cdot (4,-1,2) = 0$

$\iff \boxed{4(x-2) - y + 2(z-1) = 0}$

b) (6 pts) Draw a rough picture of S. (Hint: S is a paraboloid. Your drawing should clearly indicate the x, y, and z axes and the coordinates of the vertex [i.e., summit] of S.)

$x^2 + z^2 - y = 5 \iff y = x^2 + z^2 - 5$



2. Given the parametrized curve $P(t) = (\frac{t^3}{3}, t^2, 2t)$.

a) (2 pts) Find the velocity vector $\vec{v}(t)$.

$\vec{r} = \vec{OP} = (t^3/3, t^2, 2t)$ ← this is $(x(t), y(t), z(t))$

$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = (t^2, 2t, 2)}$ ← this is $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$

b) (5 pts) Find the arclength of the curve between the points $Q_1 = (0,0,0)$ and $Q_2 = (\frac{8}{3}, 4, 4)$.

Q_1 corresponds to $t_1 = 0$ $\left((\frac{t^3}{3}, t^2, 2t)\big|_{t=0} = (0,0,0) \right)$

Q_2 corresponds to $t_2 = 2$ $\left((\frac{t^3}{3}, t^2, 2t)\big|_{t=2} = (8/3, 4, 4) \right)$

the arclength from Q_1 to Q_2 is $\boxed{\int_{t=t_1}^{t_2} |\vec{v}|\, dt}$

$$= \int_{t=0}^{2} \sqrt{(t^2)^2 + (2t)^2 + 2^2}\, dt = \int_{t=0}^{2} \underbrace{\sqrt{t^4 + 4t^2 + 4}}_{\text{this is } \sqrt{(t^2+2)^2}}\, dt = \int_{t=0}^{2} (t^2+2)\, dt$$

$$= \left[\frac{t^3}{3} + 2t\right]_{t=0}^{2} = \frac{8}{3} + 4 - 0 = \boxed{\frac{20}{3}}$$

c) (5 pts) Given $P\big|_{t=1} = (\frac{1}{3}, 1, 2)$, we know that there exist certain constants a, b, c giving us an approximation $P\big|_{t=1+\Delta t} \approx (\frac{1}{3} + a\Delta t, 1 + b\Delta t, 2 + c\Delta t)$. Find a, b, and c.

put $P_0 = P|_{t=1}$

$P_1 = P|_{t=1+\Delta t}$

then $\vec{P_0P_1} = \Delta\vec{r} \approx \Delta t \cdot \vec{v}\big|_{t=1} = \Delta t(1, 2, 2) = (\Delta t, 2\Delta t, 2\Delta t)$

Thus if $P_0 = (\frac{1}{3}, 1, 2)$, we get $P_1 \approx (\frac{1}{3} + \Delta t, 1 + 2\Delta t, 2 + 2\Delta t)$

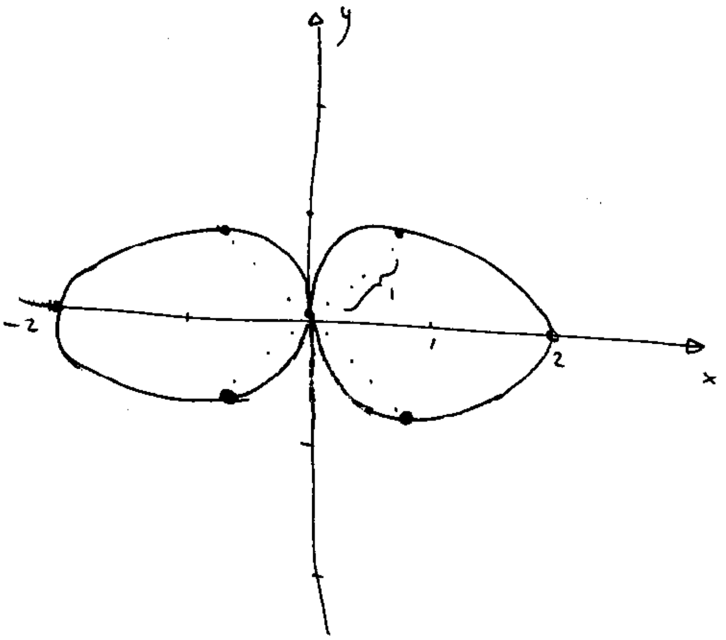
$\boxed{\text{so } a = 1,\ b = 2,\ c = 2}$

<u>Note</u> this can be done in other ways, eg. $x(t) = t^3/3$

$\Rightarrow x(1+\Delta t) \approx x(1) + \frac{dx}{dt}\Big|_{t=1} \cdot \Delta t = \frac{1}{3} + 1 \cdot \Delta t$, etc...

3. a) (5 pts) Make a table of values, and use it to sketch the curve C given in polar coordinates by $r = 1 + \cos 2\theta$.

θ	2θ	$\cos 2\theta$	$r = 1+\cos 2\theta$
0	0	1	2
$\pi/4$	$\pi/2$	0	1
$\pi/2$	π	-1	0
$3\pi/4$	$3\pi/2$	0	1
π	2π	1	2
$5\pi/4$	$2\pi + \pi/2$	0	1
$3\pi/2$	$2\pi + \pi$	-1	0
$7\pi/4$	$2\pi + 3\pi/2$	0	1
2π	4π	1	2



b) (7 pts) Define $f(x) = \begin{cases} 0, & \text{if } -\pi \le x < -\pi/2 \\ x, & \text{if } -\pi/2 \le x \le \pi/2 \\ 0, & \text{if } \pi/2 < x \le \pi \end{cases}$ [we extend $f(x)$ to be periodic with period 2π]. In the Fourier series $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, find ONLY the coefficient b_3.

$$b_3 = \frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\sin 3x\,dx = \frac{1}{\pi}\left[\underbrace{\cancel{\int_{x=-\pi}^{-\pi/2} 0\cdot\sin 3x\,dx}}_{0} + \int_{x=-\pi/2}^{\pi/2} x\sin 3x\,dx + \underbrace{\cancel{\int_{x=\pi/2}^{\pi} 0\cdot\sin 3x\,dx}}_{0}\right]$$

$$= \frac{1}{\pi}\int_{x=-\pi/2}^{\pi/2} x\sin 3x\,dx = \frac{1}{\pi}\int_{x=-\pi/2}^{\pi/2} -\frac{x}{3}\,d(\cos 3x)$$

[now integrate by parts: $\int u\,dv = uv - \int v\,du$]
N.B. we took out the factor $-\frac{1}{3}$ first

$$= \frac{-1}{3\pi}\left[x\cos 3x - \int \cos 3x\,dx\right]_{x=-\pi/2}^{\pi/2}$$

$$= -\frac{1}{3\pi}\left[x\cos 3x - \frac{\sin 3x}{3}\right]_{x=-\pi/2}^{\pi/2}$$

$$= -\frac{1}{3\pi}\left[\frac{\pi}{2}\underbrace{\cos\frac{3\pi}{2}}_{0} - \frac{\overbrace{\sin\left(\frac{3\pi}{2}\right)}^{-1}}{3} - \left(-\frac{\pi}{2}\right)\underbrace{\cos\left(\frac{-3\pi}{2}\right)}_{0} + \frac{\overbrace{\sin\left(\frac{-3\pi}{2}\right)}^{+1}}{3}\right]$$

$$= -\frac{1}{3\pi}\left[0 + \frac{1}{3} - 0 + \frac{1}{3}\right) = \boxed{\frac{-2}{9\pi}}$$

4. We are given a function $f(x, y)$ which satisfies:

$$f(1,2) = 10, \qquad \vec{\nabla} f\Big|_{(1,2)} = (3,4), \qquad \vec{\nabla} f\Big|_{(3,4)} = (5,6), \qquad \vec{\nabla} f\Big|_{(5,6)} = (1,2).$$

a) (4 pts) Find the directional derivative of f at the point $P_0(5,6)$ in the direction of the vector $\vec{v} = (3,4)$.

$\vec{v}$ is not a unit vector, but $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{(3,4)}{\sqrt{3^2+4^2}} = \left(\frac{3}{5}, \frac{4}{5}\right)$ is a unit vector in the same direction as $\vec{v}$.

Then $D_{\vec{u}} f\Big|_{P_0} = \vec{\nabla} f\Big|_{\underset{(5,6)}{P_0}} \cdot \vec{u} = (1,2) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = 1 \cdot \frac{3}{5} + 2 \cdot \frac{4}{5}$

$= \boxed{\frac{11}{5}}$.

b) (4 pts) Find an approximate value for $f(1.02, 1.99)$.

if $P_1 = (1,2)$ $\qquad \Delta\vec{r} = (\Delta x, \Delta y)$

$P_2 = (1+\Delta x, 2+\Delta y)$

then $f(P_2) \approx f(P_1) + \vec{\nabla} f\Big|_{P_1} \cdot \Delta\vec{r}$.

In our case, $(\Delta x, \Delta y) = (0.02, -0.01)$ since $P_2 = (1.02, 1.99)$

and $\vec{\nabla} f\Big|_{P_1=(1,2)} = (3,4)$, so $f(P_2) \approx f(1,2) + (3,4) \cdot (0.02, -0.01)$

$= 10 + (3)(0.02) + (4)(-0.01)$ $\qquad$ (here $0.06 - 0.04 = 0.02$)

$= \boxed{10.02}$

[You can also do this by the equivalent method of saying $\Delta f \approx \frac{\partial f}{\partial x}\Big|_{P_1} \Delta x + \frac{\partial f}{\partial y}\Big|_{P_1} \Delta y$]

c) (4 pts) Find the partial derivative $\frac{\partial}{\partial s}\left[f(s^2+t^2, 3st)\right]_{(s,t)=(1,2)}$. (In other words, we have substituted $x = s^2 + t^2$ and $y = 3st$.)

Use the chain rule:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \vec{\nabla} f\Big|_{(x(s,t),\, y(s,t))} \cdot \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}\right)$$

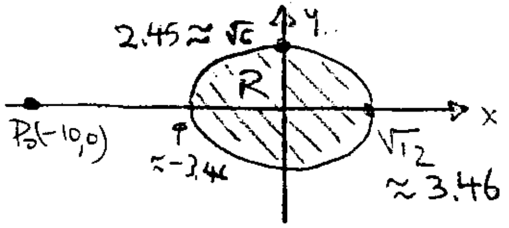
when $(s,t) = (1,2)$, then $(x,y) = (1^2+2^2, 3 \cdot 1 \cdot 2) = (5,6)$.

then $\vec{\nabla} f\Big|_{(x(1,2),\, y(1,2))} = \vec{\nabla} f\Big|_{(5,6)} = (1,2)$.

$$\left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}\right)\Big|_{(s,t)=(1,2)} = (2s, 3t)\Big|_{(s,t)=(1,2)} = (2,6)$$

answer: $\frac{\partial f}{\partial s}\Big|_{(s,t)=(1,2)} = \vec{\nabla} f\Big|_{\underset{(x,y)}{(5,6)}} \cdot \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}\right)\Big|_{(s,t)=(1,2)} = (1,2) \cdot (2,6) = \boxed{14}$

(12 pts.) 5. Consider an elliptic disk R in the plane, given by $R = \{(x, y) \mid x^2 + 2y^2 \le 12\}$. Find the maximum and minimum of the function $f(x, y) = xy + 10y$ on the region R. Indicate both the maximum/minimum values and the points where they are attained.



<u>Step 1</u> find critical point(s) of f & see if any belong to R. This amounts to checking the <u>interior</u> of R.

$\vec{\nabla} f = (f_x, f_y) = (y, x+10)$, so the critical pts satisfy $\begin{cases} y = 0 \\ x + 10 = 0 \end{cases}$.

There is only one critical point $P_0(-10,0)$ <u>BUT</u> $P_0 \notin R$ (look at figure or check $(-10)^2 + 2\cdot 0^2 > 12$ not $\le$)

so <u>we do not consider P_0</u>.

<u>Step 2</u> Since there are no critical points in R, the only possible max/min can be at the <u>boundary</u> of R, $x^2 + 2y^2 = 12$
(otherwise, we'd have to compare to values of f at critical pts in R)

boundary of R is just the ellipse not the inside

This is a <u>Lagrange</u> multiplier problem:

max/min $f(x,y) = xy + 10y$

subject to the <u>constraint</u> $g(x,y) = x^2 + 2y^2 = 12$. Note $\vec{\nabla} g = (2x, 4y)$; $\vec{\nabla} f = (y, x+10)$.

Lagrange multiplier equations: $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y) = 12 \end{cases} \iff \begin{cases} y = 2\lambda x & (a) \\ x + 10 = 4\lambda y & (b) \\ x^2 + 2y^2 = 12 & (c) \end{cases}$ eliminate λ: $\underbrace{2y^2}_{2y\cdot(a)} = 4\lambda xy = \underbrace{x^2 + 10x}_{x\cdot(b)}$

$\iff \begin{cases} 2y^2 = x^2 + 10x \\ x^2 + 2y^2 = 12 \end{cases}$ substitute $2y^2$ $\iff \begin{cases} 2y^2 = x^2 + 10x \\ x^2 + x^2 + 10x = 12 \end{cases}$.

The equation for x alone is $2x^2 + 10x - 12 = 0 \Rightarrow 2(x-1)(x+6) = 0$

so $\boxed{x = 1}$ or $\boxed{x = -6}$

$x = 1$: $\Downarrow$

$2y^2 = x^2 + 10x = 11$

$y = \pm\sqrt{\frac{11}{2}}$

$x = -6$: $\Downarrow$

$2y^2 = x^2 + 10x = 36 - 60 < 0$

so there are no possible values for y.

$\therefore$ This option of $x = -6$ does <u>NOT</u> correspond to any points on the boundary.
(You can also see this from the figure).

<u>Conclusion</u> the <u>only</u> points to consider are

$P_1\left(1, \sqrt{\frac{11}{2}}\right)$ and $P_2\left(1, -\sqrt{\frac{11}{2}}\right)$

$f(P_1) = 1\cdot\sqrt{\frac{11}{2}} + 10\cdot\sqrt{\frac{11}{2}} = \frac{11\sqrt{11}}{\sqrt{2}}$ larger,

$f(P_2) = 1\cdot\left(-\sqrt{\frac{11}{2}}\right) + 10\cdot\left(-\sqrt{\frac{11}{2}}\right) = -\frac{11\sqrt{11}}{\sqrt{2}}$ smaller

<u>So</u> the maximum value is $11\sqrt{\frac{11}{2}}$, attained at $P_1\left(1, \sqrt{\frac{11}{2}}\right)$
& the minimum value is $-11\sqrt{\frac{11}{2}}$, attained at $P_2\left(1, -\sqrt{\frac{11}{2}}\right)$.

6. a) (4 pts) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does NOT exist.

Use the 2-path test on the family of paths $P(t) = (x,y) = (t, at)$, a fixed, $t \to 0$, so $P(t) \to (0,0)$

$$\lim_{t\to 0} \frac{t^2}{t^2 + (at)^2} = \lim_{t\to 0} \frac{1}{1+a^2} = \frac{1}{1+a^2} \text{ depends on } a$$

so $\lim_{t\to 0} f(P(t))$ depends on the choice of path for different $P(t) \to (0,0)$

so $\lim_{(x,y)\to(0,0)} f(x,y)$ DOES NOT EXIST.

b) (4 pts) Show that $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2} = 0.$

Method I write as $x^2 \cdot \left(\frac{x^2}{x^2+y^2}\right)$ & note that $0 \le x^2 \le x^2+y^2$ $\Rightarrow 0 \le \frac{x^2}{x^2+y^2} \le 1$

so $x^2 \to 0$ and $\left(\frac{x^2}{x^2+y^2}\right)$ is bounded, hence the limit is 0.

Method II as above, but rewrite as $\underset{\downarrow\, 0}{0} \le \frac{x^4}{x^2+y^2} = x^2 \cdot \left(\frac{x^2}{x^2+y^2}\right) \le \underset{\downarrow\, 0}{x^2}$ & use sandwich theorem.

Method III use polar coordinates. Write $x = r\cos\theta$, $y = r\sin\theta$.

then $\frac{x^4}{x^2+y^2} = \frac{r^4\cos^4\theta}{r^2} = r^2\cos^4\theta$. Now either use $r^2 \to 0$, $|\cos^4\theta| \le 1$ bounded, or sandwich $0 \le r^2\cos^4\theta \le r^2$ & use the theorem.

c) (4 pts) Find the value of $\lim_{(x,y)\to(0,0)} \frac{e^{x^2/2} - \cos y}{x^2+y^2}$. Hint: use Taylor series and $O(\cdot)$ notation. Even if you have not done part (b), you may use the result from there as well as the similar result $\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2} = 0$.

$$e^{x^2/2} = 1 + \frac{x^2}{2} + \frac{(x^2/2)^2}{2!} + \dots = 1 + \frac{x^2}{2} + O(x^4)$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots = 1 - \frac{y^2}{2} + O(y^4)$$

(note: there is no difference between writing $+O(y^4)$ & $-O(y^4)$)

$$\frac{e^{x^2/2} - \cos y}{x^2+y^2} = \frac{\frac{x^2}{2} + \frac{y^2}{2} + O(x^4) + O(y^4)}{x^2+y^2}$$

$$= \frac{1}{2} + O\left(\frac{x^4}{x^2+y^2}\right) + O\left(\frac{y^4}{x^2+y^2}\right).$$

as $(x,y) \to (0,0)$ we know $\frac{x^4}{x^2+y^2} \to 0$, $\frac{y^4}{x^2+y^4} \to 0$

so our quantity $\to \frac{1}{2} + 0 + 0 = \boxed{\frac{1}{2}}$.