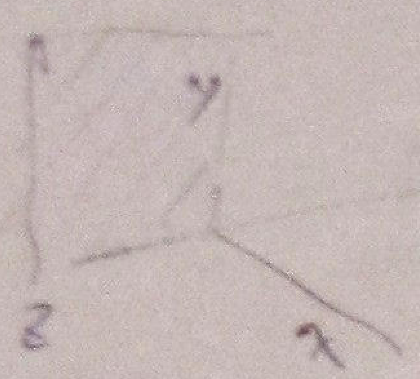
I. (10 *Points*) Find the standard matrix for the stated composition of linear operators: On R^3, a rotation of 90° about the x-axis, followed by a projection on the yz-plane. What is the image of the vector (1;1;3)?

∘ rotation of 90°:

std matrix: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$



∘ orthog. projection on (yz):

$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10

∘ composed transformation:

$S = BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

∘ image of (1; 1; 3): $(w_1; w_2; w_3)$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

∴ The image is: (0; -3; 1)

II. (30 *Points*)

a) Express the equations: $\begin{cases} y_1 = x_1 - x_2 + x_3 \\ y_2 = 3x_1 + x_2 - 4x_3 \\ y_3 = -2x_1 - 2x_2 + 3x_3 \end{cases}$ and $\begin{cases} z_1 = 4y_1 - y_2 + y_3 \\ z_2 = -3y_1 + 5y_2 - y_3 \end{cases}$ in the matrix forms $Y = AX$ and $Z = BY$ respectively. Then use these to obtain a direct relationship $Z = CX$ between Z and X.

b) Use the equation $Z = CX$ obtained in (a) to express z_1 and z_2 in terms of x_1, x_2 and x_3.

c) Check the result in (b) by directly substituting the equations of y_1, y_2 and y_3 into the equations for z_1 and z_2.

a)
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -4 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leftarrow (Y = AX)$$

5

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -3 & 5 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leftarrow (Z = BY)$$

and so
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -3 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -4 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = CX$$

5

so
$$\boxed{Z = \begin{bmatrix} -1 & -7 & 11 \\ 14 & 10 & -26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

$$\left(\text{i.e: } C = \begin{bmatrix} -1 & -7 & 11 \\ 14 & 10 & -26 \end{bmatrix} \right)$$

$Z = CX$ is obtained.

continued on back side

b) $Z = CX$

so $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 & -7 & 11 \\ 14 & 10 & -26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1x_1 & -7x_2 & 11x_3 \\ 14x_1 & 10x_2 & -26x_3 \end{bmatrix}$

5 $\therefore \begin{cases} z_1 = -x_1 - 7x_2 + 11x_3 \\ z_2 = 14x_1 + 10x_2 - 26x_3 \end{cases}$ ✓

c) $z_1 = 4y_1 - y_2 + y_3 \quad \therefore \quad z_1 = 4(x_1 - x_2 + x_3) - (3x_1 + x_2 - 4x_3) + (-2x_1 - 2x_3 + 3x_3)$

so $z_1 = -x_1 - 7x_2 + 11x_3$ True

5

$z_2 = -3y_1 + 5y_2 - y_3 \quad \therefore \quad z_2 = -3(x_1 - x_2 + x_3) + 5(3x_1 + x_2 - 4x_3) - (-2x_1 - 2x_2 + 3x_3)$

so $z_2 = 14x_1 + 10x_2 - 26x_3$ True

✓

III. (20 *Points*) Let $\vec{u}=(1;-1;4)$, $\vec{v}=(3;2;0)$, $\vec{w}=(4;-5;-1)$. Find a vector $\vec{x}$ such that:

$$\begin{cases} \vec{x}.\vec{u}=10 \\ \vec{x}.\vec{v}=1 \\ \vec{x}.\vec{w}=7 \end{cases}$$

$\vec{x}(x_1;x_2;x_3)$

$$\begin{cases} 1x_1 - 1x_2 + 4x_3 = 10 \\ 3x_1 + 2x_2 + 0x_3 = 1 \\ 4x_1 - 5x_2 - 1x_3 = 7 \end{cases}$$

$$\therefore \begin{bmatrix} 1 & -1 & 4 & 10 \\ 3 & 2 & 0 & 1 \\ 4 & -5 & -1 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 4 & 10 \\ 0 & 5 & -12 & -29 \\ 0 & -1 & -17 & -33 \end{bmatrix}$$

20

$$\Rightarrow \begin{bmatrix} 1 & -1 & 4 & 10 \\ 0 & 1 & -\frac{12}{5} & -\frac{29}{5} \\ 0 & 0 & -\frac{97}{5} & -\frac{194}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 4 & 10 \\ 0 & 1 & -\frac{12}{5} & -\frac{29}{5} \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore \vec{x}(1;-1;2)$$

IV. (20 *Points*) Show that $V = R^2$ is not a vector space over R with respect to the given operations; precise what are the axioms that fail?

$$\begin{cases}(a;b)+(c;d)=(a+c;b+d) & \textit{Addition}\\ k(a;b)=(a;kb) & \textit{Scalar Multiplication}\end{cases}$$

let u be (a;b) and v be (c;d)

1) $u \in V$, $v \in V$ $\quad u \oplus v = (a+c;\, b+d)$ ~~$\in (R;R)$~~ $\in V$ True

~~2) u, v, w ∈ V : u + v + w~~

2) $u \oplus v = (a+c;\, b+d)$; $v \oplus u = (c+a;\, d+b) = (a+c;\, b+d)$
$= u \oplus v$ True

3) $u, v, w \in V$: $w: (e;f)$

$(u \oplus v) \oplus w = ((a+c)+e;\, (b+d)+f) = (a+c+e;\, b+d+f)$
$= (a+(c+e);\, b+(d+f))$
$= u \oplus (v \oplus w)$

True

4) $\square$ is the zero object such that: $u + \square = u$
which is true for $u \oplus (0;0) = (a;b) \oplus (0;0)$
$= (a+0;\, b+0)$
$= (a+b) = u$

20

so $\square$ exists
& $\square = (0;0)$ True

5) the negative n of any object exists such that:
$u \oplus n = \square$ which is true for:
$(a;b) \oplus (-a;-b) = (a-a;\, b-b)$
$= (0;0)$
$= \square$

so n exists
& $n = (-a;-b)$ True

6) $u \in V$: $\quad ku = (a;kb) \in V$ True

7) $u \in V, v \in V$: $\quad K(u \oplus v) = K(a+c;\, b+d) = (a+c;\, K(b+d))$
$= (a+c;\, Kb + Kd)$

$Ku \oplus Kv = (a;Kb) \oplus (c;Kd) = (a+c;\, Kb+Kd)$

$\therefore K(u \oplus v) = Ku \oplus Kv$ TRUE

8) $(m+k)u = (m+k)(a;b) = (a; (m+k)b) = (a; mb+kb)$

$mu \oplus ku = m(a;b) \oplus k(a;b) = (a;mb) \oplus (a;kb)$

$= (a+a; mb+kb)$

$= (2a; mb+kb)$

so $(m+k)u \neq mu \oplus ku$

The axiom fails and V is NOT a vector space. ✓

9) $k(mu) = k(m(a;b)) = k(a;mb) = (a; kmb)$

$(km)u = (km)(a;b) = (a; kmb)$

$k(mu) = (km)u$ TRUE ✓

10) $1u = 1(a;b) = (a;1b) = (a;b) = u$

$1u = u$ TRUE ✓

V. (20 *Points*) Find the inverse transformation of the following transformation mapping vectors in R^3 into vectors in R^3:

$$T(x;y;z) = (3x+y;\ x-y+2z;\ x+y+z)$$

T: standard matrix: $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A}(C_{ij}(A))^T$$

$$C_{ij}(A) = \begin{bmatrix} \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & (+)1 & 2 \\ (-)1 & 3 & (-)2 \\ 2 & (-)6 & -4 \end{bmatrix}$$

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so $(C_{ij}(A))^T = \begin{bmatrix} -3 & -1 & 2 \\ +1 & 3 & -6 \\ 2 & -2 & -4 \end{bmatrix}$ ✓

and $\det(A) = 3\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 0$

$= -8$ ✓

so $A^{-1} = \frac{1}{-8}(C_{ij}A)^T = \begin{bmatrix} 3/8 & 1/8 & -2/8 \\ -1/8 & -3/8 & 6/8 \\ -2/8 & 2/8 & 4/8 \end{bmatrix}$

$\therefore$ T^{-1} is: $T^{-1}(w_1; w_2; w_3) = \left(\frac{3}{8}w_1 + \frac{1}{8}w_2 - \frac{1}{4}w_3 \quad -\frac{1}{8}w_1 - \frac{3}{8}w_2 + \frac{3}{4}w_3 \quad -1/4\, w_1 + 1/4\, w_2 + 1/2\, w_3\right)$