I. (20 Points) Answer by <u>True</u> or <u>False</u>. To obtain complete grade, justify your answer by giving a logical argument or a counterexample. <u>Do not guess; the number of wrong answers will be subtracted from the number of right answers, with the possibility of a negative score!</u>

1) If A and B are invertible matrices and $XA = B$, then $X = A^{-1}B$ (…F…)
2) If A is a (3×3) matrix and $\det(A) \neq 0$, then $\det[\operatorname{adj}(A)] = (\det A)^2$ (…T…)
3) For the system given below, a necessary and sufficient condition for the existence of a solution is: $3a = c - b$ (…T…).

$$\begin{cases} x + y + 2z = a \\ -2x - z = b \\ x + 3y + 5z = c \end{cases}$$

4) The matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ is invertible for any real number "k" (…T….).
5) The matrix $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ is in reduced-row echelon form (...F...)
6) If the matrix B is obtained by multiplying a column of a matrix A by 9, then $\det(B) = 9\det(A)$ (…T….).
7) The system $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$ has infinitely many solutions for "k=2" (…T….).
8) If the matrix A is invertible, then $\det(A^TA) \geq 0$(…F…).
9) Suppose A and B are invertible matrices and X is a matrix such that $AXB = A + B$, then $X = A^{-1} + B^{-1}$ (…T…..).
10) A diagonal matrix with a "0" on its main diagonal is not invertible (…T…).

1) $XA = B$, $XAA^{-1} = BA^{-1} \Rightarrow X = BA^{-1}$

2) $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) \Rightarrow \operatorname{adj}(A) = \det(A)\,A^{-1} \Rightarrow \det[\operatorname{adj}(A)] = [\det(A)]^3 \frac{1}{\det A} = [\det A]^2$

3) $(\times 2)\begin{pmatrix} 1 & 1 & 2 & a \\ -2 & 0 & -1 & b \\ 1 & 3 & 5 & c \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & a \\ 0 & 2 & 3 & b+2a \\ 0 & 2 & 3 & c-a \end{pmatrix} (\times\tfrac{1}{2}) \begin{pmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 3/2 & \frac{b+2a}{2} \\ 0 & 2 & 3 & c-a \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 3/2 & \frac{b+2a}{2} \\ 0 & 0 & 0 & c-a-b-2a \end{pmatrix} \Rightarrow -3a - b + c = 0$ (existence of a solution)

$\boxed{3a = c - b}$

4) $\begin{bmatrix} K-2 & 3 \\ -3 & K-2 \end{bmatrix}$ is invertible if its determinant is $\neq 0$

$\Longleftrightarrow (K-2)^2 + 9 \neq 0$

$K^2 - 4K + 4 + 9 \neq 0$

$K^2 - 4K + 13 \neq 0$; As $\Delta < 0 \Rightarrow$ det cannot be "0" for real "K".

II. (10 Points) Suppose $\overrightarrow{u_1}$, $\overrightarrow{u_2}$ and $\overrightarrow{u_3}$ are three nonzero vectors such that $\overrightarrow{u_1} \perp \overrightarrow{u_2}$, $\overrightarrow{u_1} \perp \overrightarrow{u_3}$ and $\overrightarrow{u_2} \perp \overrightarrow{u_3}$ (i.e they are orthogonal to each other), and suppose that $\vec{w} = a\overrightarrow{u_1} + b\overrightarrow{u_2} + c\overrightarrow{u_3}$

a) Show that: $a = \dfrac{\vec{w}.\overrightarrow{u_1}}{\left\|\overrightarrow{u_1}\right\|^2}$, $b = \dfrac{\vec{w}.\overrightarrow{u_2}}{\left\|\overrightarrow{u_2}\right\|^2}$, $c = \dfrac{\vec{w}.\overrightarrow{u_3}}{\left\|\overrightarrow{u_3}\right\|^2}$

b) If $\overrightarrow{u_1} = (1;-2;3)$, $\overrightarrow{u_2} = (1;2;1)$ and $\overrightarrow{u_3} = (-8;2;4)$, show that these three vectors are orthogonal to each other and by using the result of part (a) find a, b and c such that $\vec{w} = (13;-4;7) = a\overrightarrow{u_1} + b\overrightarrow{u_2} + c\overrightarrow{u_3}$

a) $\vec{w}.\vec{u_1} = \left(a\vec{u_1} + b\vec{u_2} + c\vec{u_3}\right).\vec{u_1} = a\,\vec{u_1}.\vec{u_1} + b\vec{u_2}.\vec{u_1} + c\vec{u_3}.\vec{u_1}$
$= a\|\vec{u_1}\|^2 + 0 + 0$

$\Rightarrow a = \dfrac{\vec{w}.\vec{u_1}}{\|\vec{u_1}\|^2}$

Same for "b" and "c"

b) $\vec{u_1}.\vec{u_2} = 0$
$\vec{u_1}.\vec{u_3} = 0$
$\vec{u_2}.\vec{u_3} = 0$

$$a = \frac{(13;-4;7).(1;-2;3)}{1+4+9} = \frac{13+8+21}{14} = 3$$

$$b = \frac{(13;-4;7).(1;2;1)}{1+4+1} = \frac{13-8+7}{6} = 2$$

$$c = \frac{(13;-4;7).(-8;2;4)}{64+4+16} = \frac{-104-8+28}{84} = -1$$

III. (20 Points) Solve the following system for "y" using Cramer's rule:

$$\begin{cases} x+y+z+w=2 \\ x+2y+3z+4w=2 \\ 2x+3y+5z+9w=2 \\ x+y+2z+7w=2 \end{cases}$$

$$y = \frac{D_y}{D}$$

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \end{vmatrix} = 1(18-7)-(12-3)$$
$$= 11-9 = 2$$

$$D_y = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 9 \\ 1 & 2 & 2 & 7 \end{vmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & -2 & 3 & 7 \\ 0 & 0 & 1 & 6 \end{bmatrix} = 1 \begin{vmatrix} 0 & 2 & 3 \\ -2 & 3 & 7 \\ 0 & 1 & 6 \end{vmatrix} = 18$$

$$\Longrightarrow y = \frac{18}{2} = 9.$$

IV. (10 Points) Let $\vec{u} = (3;2;1)$, $\vec{v} = (5;-3;4)$ and $\vec{w} = (1;6;-7)$. Compute the following:

a) $\|\vec{u}\|(\vec{v} \times \vec{w})$

b) $\|proj_{\vec{u}}\vec{v}\|$

c) $\|3\vec{u} - \vec{v} + \vec{w}\|$

d) $\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2$

e) $\vec{u} \times (\vec{v} - 2\vec{w})$

$$\boxed{\begin{aligned} \vec{v} - 2\vec{w} &= (5;-3;4) - (2;12;-14) \\ &= (3;-15;18) \\ \vec{u} \times (\vec{v} - 2\vec{w}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ 3 & -15 & 18 \end{vmatrix} = 51\vec{i} - 51\vec{j} - 51\vec{k} \end{aligned}}$$

a) $$\|\vec{u}\|(\vec{v} \times \vec{w}) = \sqrt{9+4+1} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -3 & 4 \\ 1 & 6 & -7 \end{vmatrix} = \sqrt{14}\left\{(21-24)\vec{i} - (-35-4)\vec{j} + (30+3)\vec{k}\right\}$$

$$= \sqrt{14}\left\{-3\vec{i} + 39\vec{j} + 33\vec{k}\right\}$$

$$= \boxed{-3\sqrt{14}\,\vec{i} + 39\sqrt{14}\,\vec{j} + 33\sqrt{14}\,\vec{k}}$$

b) $$\|proj_{\vec{u}}\vec{v}\| = \left\|\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\cdot\vec{u}\right\| = \left\|\frac{(3;2;1)\cdot(5;-3;4)}{14}(3;2;1)\right\|$$

$$= \left\|\left(\frac{39}{14}; \frac{26}{14}; \frac{13}{14}\right)\right\|$$

$$= \sqrt{\left(\frac{39}{14}\right)^2 + \left(\frac{26}{14}\right)^2 + \left(\frac{13}{14}\right)^2} = \sqrt{\frac{1521 + 1\ldots + 169}{196}}$$

c) $$\|3\vec{u} - \vec{v} + \vec{w}\| = \|(9;6;3) - (5;-3;4) + (1;6;-7)\|$$

$$= \|(5;15;-8)\|$$

$$= \sqrt{25+225+64} = \sqrt{314}$$

d) $$\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = \|(8;-1;5)\|^2 - \|(-2;5;-3)\|^2$$

$$= (64+1+25) - (4+25+9)$$

$$= 90 - 38$$

$$= 52$$

$\vec{u}\cdot\vec{v} = 15 - 6 + 4$
$= 13$

V. (20 Points) Consider the square matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$

a) Find the determinant of A: $|A|$

b) Find the adjoint of A: adj(A)

c) Verify that $A.(adjA) = |A| I$ where I is the identity matrix.

d) Use adj(A) to find A^{-1}

a) $|A| = 1(21-20) - 2(14-4) + 3(10-3)$

$= 2$

b) $\text{adj}(A) = \begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{bmatrix}^T$

$\text{adj}(A) = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$

c) $A.\text{adj}(A) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = |A| I$

d) $A^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -5 & 2 & 1 \\ 7/2 & -3/2 & -1/2 \end{bmatrix}$

VI. **(20 Points)**

a) Consider a skew symmetric $(n \times n)$ matrix A (i.e. $A^T = -A$), where n is odd. Show that det $(A) = 0$.

b) For an invertible $(n \times n)$ matrix A, show that $A.adj(A) = \det(A).I$, where I is the $(n \times n)$ identity matrix.

a) $A^T = -A$

$\det A^T = \det(-A) = \det(A)$

$= (-1)^n \det(A) = \det A$

(n odd) $\Rightarrow$ $-\det A = \det A \Rightarrow \boxed{\det A = 0}$

b) $A^{-1} = \dfrac{1}{\det A} adj(A) \Rightarrow adj(A) = (\det A)\, A^{-1}$

$A \,.\, adj(A) = (\det A)\, A \,.\, A^{-1}$

$\boxed{A \,.\, adj(A) = (\det A)\, I}$