Fall 2008-2009	**Quiz 1**	**Date: October 27**
Prof: M. Egeileh	**Math 201 - Sections 17 to 20**	**Duration: 1 hour**

Exercise 1 For each of the following series, say (with justification) whether it is absolutely convergent, conditionally convergent, or divergent : *(10 pts each)*

a) $\displaystyle\sum_{n\geqslant 1} \frac{e^{n\sqrt{n}}}{n^n}$ b) $\displaystyle\sum_{n\geqslant 0} \frac{\cos(n!\pi^n)}{e^n}$ c) $\displaystyle\sum_{n\geqslant 2} \frac{(-1)^n}{\sqrt{n}(\ln n)^{2008}}$ d) $\displaystyle\sum_{n\geqslant 1} \frac{\ln(n)}{n^p}$, where $p > 1$

Exercise 2 $(u_n)_{n\in\mathbb{N}}$ being a sequence of non-negative real numbers, are the following statements true or false? Justify your answer. *(4 pts each)*

1. If $\displaystyle\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, then $\displaystyle\lim_{n\to\infty} u_n = 0$.
2. If $\displaystyle\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then $\displaystyle\lim_{n\to\infty} u_n = 0$.
3. If $\displaystyle\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then $\displaystyle\lim_{n\to\infty} u_n \neq 0$.
4. If $\displaystyle\lim_{n\to\infty} n^{1.0001} u_n = 0$, then the series $\displaystyle\sum_{n\geqslant 0} u_n$ is convergent.
5. If $\displaystyle\lim_{n\to\infty} n^{0.9999} u_n = +\infty$, then the series $\displaystyle\sum_{n\geqslant 0} u_n$ is divergent.

Exercise 3

1. We consider the series $\displaystyle\sum_{n\geqslant 0} (-1)^n \frac{x^{n+1}}{n+1}$, where x is a real number. Is it an alternating series? Explain. *(4 pts)*
2. Find the radius of convergence, as well as the interval of convergence of the power series $\displaystyle\sum_{n\geqslant 0} (-1)^n \frac{x^{n+1}}{n+1}$. *(10 pts)*
3. We consider the function f defined by $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ for every real number x satisfying $|x| < 1$. Justify briefly the existence of $f'(x)$ when $-1 < x < 1$, and give the Maclaurin series of $f'(x)$. *(7 pts)*
4. Compute the sum of the Maclaurin series you found in the preceding question, and deduce that $f(x) = \ln(1+x)$ whenever $|x| < 1$. *(6 pts)*
5. Is the series $\displaystyle\sum_{n\geqslant 1} \left(\ln(1 + \frac{\sqrt{n}}{n}) - \frac{2\sqrt{n} - 1}{2n} + \frac{\sqrt{n}}{3n^2} \right)$ convergent? Justify. *(6 pts)*

Exercise 4

1. Pretend you don't know that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e$; prove that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \leqslant 3$. *(4 pts)*
2. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers, converging to a rational number $\frac{p}{q}$. Prove that $(u_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence**, that is : for every positive real number ε, there exists a positive integer n_0 such that $|u_n - u_m| < \varepsilon$ whenever $n > n_0$ and $m > n_0$. *(2 pts)*
3. Is the converse true? In other terms, if $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of rational numbers, does it converge necessarily to a rational number? If not, give a counter-example. *Hint : In this question, don't pretend that you don't know... (1 pt)*

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Exercise 1

a) $\sum_{n\geq 1} \frac{e^{n\sqrt{n}}}{n^n}$. We have $\frac{e^{n\sqrt{n}}}{n^n} = \frac{e^{n\sqrt{n}}}{e^{n\ln n}} = e^{n\sqrt{n} - n\ln n}$

$$= e^{n\sqrt{n}\left(1-\frac{\ln n}{\sqrt{n}}\right)} \xrightarrow[n\to\infty]{} +\infty$$

By n^{th} term test, the series diverges.

b) $\sum_{n\geq 0} \frac{\cos(n!\pi^n)}{e^n}$. We have $\left|\frac{\cos(n!\pi^n)}{e^n}\right| \leq \frac{1}{e^n}$

Since $\sum_{n\geq 0}\frac{1}{e^n}$ is convergent (geometric series with ratio $\frac{1}{e} < 1$), $\sum_{n\geq 0}\frac{\cos(n!\pi^n)}{e^n}$ is absolutely convergent (direct comparison test).

c) $\sum_{n\geq 2} \frac{(-1)^n}{\sqrt{n}(\ln n)^{2008}}$. We have $n\left|\frac{(-1)^n}{\sqrt{n}(\ln n)^{2008}}\right| = \frac{\sqrt{n}}{(\ln n)^{2008}}$

$$= \left(\frac{n^{\frac{1}{4016}}}{\ln n}\right)^{2008} \xrightarrow[n\to\infty]{} +\infty$$

which implies that $\frac{1}{n} \lll \left|\frac{(-1)^n}{\sqrt{n}(\ln n)^{2008}}\right|$.

Since $\sum_{n\geq 1}\frac{1}{n}$ is divergent (harmonic series), $\sum_{n\geq 2}\frac{(-1)^n}{\sqrt{n}(\ln n)^{2008}}$ is not absolutely convergent[(1)].

However, the series is conditionally convergent, since it converges by alternating series test ($\frac{1}{\sqrt{n}(\ln n)^{2008}}$ is positive, decreasing and converging to 0).

(1) by limit comparison test

d) $\sum_{n\geq 1} \frac{\ln(n)}{n^p}$, $p > 1$. We have $n^c \frac{\ln(n)}{n^p} = \frac{\ln(n)}{n^{p-c}}$.

We may choose c such that $\begin{cases} p-c > 0 \\ c > 1 \end{cases}$, for instance $c = \frac{1+p}{2}$. For this choice of c,

$n^c \cdot \frac{\ln(n)}{n^p} \xrightarrow[n\to\infty]{} 0$ therefore $\frac{\ln(n)}{n^p} <<< \frac{1}{n^c}$, and the convergence of $\sum_{n\geq 1} \frac{1}{n^c}$ (hyperharmonic series with $c > 1$) implies that of $\sum_{n\geq 1} \frac{\ln(n)}{n^p}$, by limit comparison test.

Exercise 2

1. True: if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, ratio test implies the convergence of the series $\sum_{n\geq 0} u_n$. By n^{th} term test, $\lim_{n\to\infty} u_n = 0$.

2. False: Define $u_n = n$ for all $n \in \mathbb{N}$
$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$, but $\lim_{n\to\infty} u_n \neq 0$

3. False: Define $u_n = \frac{1}{n}$ for all $n \in \mathbb{N}^*$ (and $u_0 = 3$)
$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$, but $\lim_{n\to\infty} u_n = 0$

4. True: if $\lim_{n\to\infty} n^{1.0001} u_n = 0$, $u_n <<< \frac{1}{n^{1.0001}}$
By limit comparison test, the convergence of $\sum_{n\geq 1} \frac{1}{n^{1.0001}}$ (hyperharmonic series, $1.0001 > 1$)
implies that of $\sum_{n\geq 0} u_n$.

5. True: if $\lim_{n\to\infty} n^{0.9999} u_n = +\infty$, $\frac{1}{n^{0.9999}} <<< u_n$
By limit comparison test, the divergence of $\sum_{n\geq 1} \frac{1}{n^{0.9999}}$ (hyperharmonic series, $0.9999 \leq 1$)
implies that of $\sum_{n\geq 0} u_n$.

Exercise 3

1. If $x > 0$, $\sum_{n\geq 0} (-1)^n \frac{x^{n+1}}{n+1}$ is an alternating series since $\frac{x^{n+1}}{n+1} > 0$ for all n.

If $x \leq 0$, $\sum_{n\geq 0} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n\geq 0} (-1)^n \frac{(-|x|)^{n+1}}{n+1}$

$= \sum_{n\geq 0} (-1)^n (-1)^{n+1} \frac{|x|^{n+1}}{n+1} = \sum_{n\geq 0} - \frac{|x|^{n+1}}{n+1}$

This is a series of nonpositive terms, therefore it is not an alternating series.

2. Consider $\sum_{n\geq 0} \left| (-1)^n \frac{x^{n+1}}{n+1} \right| = \sum_{n\geq 0} \frac{|x|^{n+1}}{n+1}$.

Define $v_n = \frac{|x|^{n+1}}{n+1}$ for all n.

Then $\frac{v_{n+1}}{v_n} = \frac{|x|^{n+2}}{n+2} \times \frac{n+1}{|x|^{n+1}} = \frac{n+1}{n+2}|x| \xrightarrow[n\to\infty]{} |x|$

By ratio test, if $|x| < 1$, $\sum_{n\geq 0} (-1)^n \frac{x^{n+1}}{n+1}$ is absolutely convergent, and if $|x| > 1$, $\sum_{n\geq 0} (-1)^n \frac{x^{n+1}}{n+1}$ is divergent. Thus, the radius of convergence of this power series is $R = 1$.

. For $x = 1$, the series is $\sum_{n\geq 0} \frac{(-1)^n}{n+1}$ and it is conditionally convergent since it converges by alternating series test ($\frac{1}{n+1}$ is positive, decreasing, and converging to 0), without being absolutely convergent.

- For $x = -1$, the series is $\sum_{n \geq 0} -\frac{1}{n+1}$, which is divergent ($\frac{1}{n+1} \underset{\infty}{\sim} \frac{1}{n}$, and $\sum_{n \geq 1} \frac{1}{n}$ diverges).

Finally, the interval of convergence of $\sum_{n \geq 0} (-1)^n \frac{x^{n+1}}{n+1}$ is $]-1, 1]$.

3. We define $f :]-1, 1[\longrightarrow \mathbb{R}$
$$x \longmapsto f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

By the term-by-term differentiation theorem for power series, f is differentiable on $]-1, 1[$.
Therefore, $f'(x)$ exists for every $x \in]-1, 1[$.
The same theorem implies that the series $\sum_{n \geq 0} (-1)^n x^n$ is convergent for every $x \in]-1, 1[$, and $f'(x) = \sum_{n=0}^{+\infty} (-1)^n x^n$.
By unicity of the power series representation, $\sum_{n \geq 0} (-1)^n x^n$ is the Maclaurin series of $f'(x)$.

4. $f'(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$
for every $x \in]-1, 1[$. We deduce that
$f(x) = \ln(1+x) + C$ for every $x \in]-1, 1[$.
But since $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, $|x| < 1$,
we have $f(0) = 0$. Therefore
$0 = \ln(1+0) + C$, which implies $C = 0$.
Finally, $f(x) = \ln(1+x)$ whenever $|x| < 1$.

5. $\sum_{n\geq 1}\left[\ln\left(1+\frac{\sqrt{n}}{n}\right) - \frac{2\sqrt{n}-1}{2n} - \frac{\sqrt{n}}{3n^2}\right]$

$= \sum_{n\geq 1} \underbrace{\left[\ln\left(1+\frac{1}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}} + \frac{1}{2n} - \frac{1}{3n\sqrt{n}}\right]}_{u_n}$

We know that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

whenever $|x|<1$. Consequently, for $n \geqslant 2$,

$\ln\left(1+\frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} - \frac{1}{2n} + \frac{1}{3n\sqrt{n}} - \frac{1}{4n^2} + \cdots$

$\Rightarrow \quad u_n = -\frac{1}{4n^2} + \cdots$

$\Rightarrow \quad u_n \underset{\infty}{\sim} -\frac{1}{4n^2}$

$\Rightarrow \quad |u_n| \underset{\infty}{\sim} \frac{1}{4n^2}$

By limit comparison theorem, the convergence of $\sum_{n\geq 1}\frac{1}{n^2}$ implies the absolute convergence, and hence the convergence of $\sum_{n\geq 1} u_n$.

Exercise 4

1. $$n(n-1)\cdots 3\cdot 2 \geq \underbrace{2\cdot 2\cdots 2\cdot 2}_{(n-1)\text{ factors}}$$

$$\Rightarrow \quad n! \geq 2^{n-1}$$

$$\Rightarrow \quad \frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

$$\Rightarrow \quad \sum_{n=1}^{+\infty} \frac{1}{n!} \leq \sum_{n=1}^{+\infty} \frac{1}{2^{n-1}}$$

$$\Rightarrow \quad 1+\sum_{n=1}^{+\infty} \frac{1}{n!} \leq 1+\sum_{n=1}^{+\infty} \frac{1}{2^{n-1}}$$

$$\Rightarrow \quad \sum_{n=0}^{+\infty} \frac{1}{n!} \leq 1+\frac{1}{1-\frac{1}{2}} = 3$$

2. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}^*$ such that
$|u_k - \frac{p}{q}| < \varepsilon/2$ for every $k > n_0$.
In particular, for $n > n_0$ and $m > n_0$,
we have $|u_n - \frac{p}{q}| < \frac{\varepsilon}{2}$ and $|u_m - \frac{p}{q}| < \frac{\varepsilon}{2}$.

Then
$$|u_n - u_m| = \left|u_n - \frac{p}{q} + \frac{p}{q} - u_m\right|$$
$$\leq \left|u_n - \frac{p}{q}\right| + \left|u_m - \frac{p}{q}\right|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

3. The converse is false: if we take for example the sequence $(S_n)_{n\in\mathbb{N}}$ defined by $S_n = \sum_{k=0}^{n} \frac{1}{k!}$, it is a sequence of rational numbers, and $\lim_{n\to\infty} S_n = \sum_{n=1}^{+\infty} \frac{1}{n!} = e$, which is not a rational number. Since $(S_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, it is a Cauchy sequence of rational numbers, but it does not converge in $\mathbb{Q}$ since $e \notin \mathbb{Q}$.