MATH 215 Exam 2, April 5, 2019

1) a) h is polynomial, hence $\mathcal{C}^\infty$

$$J_{(x,y,z)}h = (2x, 2y, 4z^3)$$

vanishes only in $(0,0,0)$.

$$h^{-1}(0) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^4=1\}$$

does not contain $(0,0,0)$

$\Rightarrow S = h^{-1}(0)$ is a surface. 10 P

b) S is orientable, since it is a level surface

$$N(x,y,z) = \frac{J_{(x,y,z)}h}{\|J_{(x,y,z)}h\|} = \frac{(2x, 2y, 4z^3)}{\sqrt{4x^2+4y^2+16z^6}}$$ 5 P

c) $T_pS = (N(p))^\perp$

$d_p l = 0$ for $l: S \longrightarrow \mathbb{R}$ means that

$$J_pL \perp T_pS \Longleftrightarrow J_pL = \underbrace{\lambda(p)}_{\in\mathbb{R}} \cdot N(p)$$

where $L: \mathbb{R}^3 \longrightarrow \mathbb{R}$ s.t. $L|_S = l$.

Choose $L(x,y,z) = z$. $\Longrightarrow$

$$\begin{cases} (0,0,1) = \tilde{\lambda}\cdot(2x, 2y, 4z^3) \\ x^2+y^2+z^4 = 1 \end{cases}$$

$\tilde{\lambda} = 0$ contradicts $1 = \tilde{\lambda}\cdot 4z^3 \Longrightarrow$

$\tilde{\lambda} \neq 0$, $z \neq 0 \Longrightarrow x=0$, $y=0 \Longrightarrow z^4=1$

$\Longrightarrow z = \pm 1$, $\tilde{\lambda} = \frac{1}{4}\cdot(\pm 1)$

$\Longrightarrow p = (0,0,\pm 1)$ 15 P

2) a) $\varphi_t = \begin{pmatrix} -2ts \\ 1-t^2+s^2 \\ -2t \end{pmatrix}, \quad \varphi_s = \begin{pmatrix} -1+s^2-t^2 \\ +2ts \\ +2s \end{pmatrix}$

$$\langle \varphi_t, \varphi_t \rangle = E = (-2ts)^2 + (1-t^2+s^2)^2 + (-2t)^2$$

$$= 4t^2s^2 + 1 + t^4 + s^4 - 2t^2 + 2s^2 - 2t^2s^2 + 4t^2$$

$$= 1 + t^4 + s^4 + 2t^2 + 2s^2 + 2t^2s^2$$

$$= (1+t^2+s^2)^2$$

5 P

$$\langle \varphi_s, \varphi_s \rangle = G = 1 + s^4 + t^4 - 2s^2 + 2t^2 - 2s^2t^2 + 4t^2s^2 + 4s^2$$

$$= 1 + s^4 + t^4 + 2s^2 + 2t^2 + 2s^2t^2$$

$$= (1+t^2+s^2)^2$$

5 P

$$\langle \varphi_t, \varphi_s \rangle = F = -2ts \cdot (-1+s^2-t^2) + (1-t^2+s^2) \cdot 2ts$$

$$-4ts \quad = 0$$

5 P

$$\Longrightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix} = (1+t^2+s^2)^2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(We notice that $\det \begin{pmatrix} E & F \\ F & G \end{pmatrix} > 0$,

in particular φ is at least immersive.)

2) b)

second fundamental form $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$

$$= \begin{pmatrix} \langle \varphi_{tt}, N \rangle, & \langle \varphi_{ts}, N \rangle \\ \langle \varphi_{st}, N \rangle, & \langle \varphi_{ss}, N \rangle \end{pmatrix}$$

$$e = \frac{\left\langle \begin{pmatrix} -2s \\ -2t \\ -2 \end{pmatrix}, \begin{pmatrix} 2s \\ 2t \\ 1-t^2-s^2 \end{pmatrix} \right\rangle}{1+t^2+s^2} =$$

$$= \frac{-2 \cdot \left(2s^2 + 2t^2 + 1 - t^2 - s^2 \right)}{1+t^2+s^2} = -2 \quad 3P$$

$$f = \frac{\left\langle \begin{pmatrix} -2t \\ 2s \\ 0 \end{pmatrix}, \begin{pmatrix} 2s \\ 2t \\ 1-t^2-s^2 \end{pmatrix} \right\rangle}{1+t^2+s^2} = 0 \quad 3P$$

$$g = \frac{\left\langle \begin{pmatrix} 2s \\ 2t \\ 2 \end{pmatrix}, \begin{pmatrix} 2s \\ 2t \\ 1-t^2-s^2 \end{pmatrix} \right\rangle}{1+t^2+s^2} = -e = 2$$

3P

2) b) In the basis φ_t, φ_s:

$$d_{\varphi(t,s)} N = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \cdot \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

$$= \frac{-1}{(1+t^2+s^2)^2} \cdot \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$ 2 P

Gaußian curvature $\mathcal{K} = \det d_{\varphi(t,s)} N$ 1 P

$$= \frac{-4}{(1+t^2+s^2)^4}$$ 2 P

mean curvature $H = \frac{1}{2} \operatorname{tr} d_{\varphi(t,s)} N$ 1 P

$$= \frac{1}{2} \frac{-1}{(1+t^2+s^2)^2} \cdot (-2+2) = \underline{\underline{0}}$$ 2 P

c) φ is a conformal map $\mathbb{R}^2 \longrightarrow S$

since $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \lambda(t,s) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

with $\lambda(t,s) = (1+t^2+s^2)^2$ 3 P

There can't be a local isometry to a plane ($\mathcal{K}=0$) or to a sphere ($\mathcal{K} = r^{-2} > 0$), since local isometries preserve the first fundamental form, and by Gauß' Theorema egregium the Gaußian curvature $\mathcal{K}$ depends only on the first fundamental form. However, $\mathcal{K} < 0$ and non-constant, a contradiction. 5 P

3) a)

$$\varphi_{tt} = \Gamma_{11}^{1}\cdot\varphi_t + \Gamma_{11}^{2}\cdot\varphi_s + e\cdot N \qquad (*)\ 4P$$

$$\varphi_{ts} = \Gamma_{12}^{1}\cdot\varphi_t + \Gamma_{12}^{2}\cdot\varphi_s + f\cdot N \qquad (**)\ 4P$$

$$\varphi_{ss} = \Gamma_{22}^{1}\cdot\varphi_t + \Gamma_{22}^{2}\cdot\varphi_s + g\cdot N \qquad (***)\ 4P$$

b)

$$\langle *, \varphi_t\rangle = \Gamma_{11}^{1}\cdot E + \underbrace{\ldots}_{=0} = \Gamma_{11}^{1}\cdot \underbrace{h(s)}_{>0}$$

$$\| $$

$$\langle \varphi_{tt}, \varphi_t\rangle = \frac{1}{2}E_t = 0 \implies \underline{\underline{\Gamma_{11}^{1} = 0}} \qquad 3P$$

$$\langle *, \varphi_s\rangle = \Gamma_{11}^{2}\cdot G = \Gamma_{11}^{2}$$

$$\| $$

$$\langle \varphi_{tt}, \varphi_s\rangle = \underbrace{F_t}_{=0} - \frac{1}{2}\cdot E_s \implies \underline{\underline{\Gamma_{11}^{2} = -\frac{1}{2}h'}} \qquad 3P$$

$$\langle **, \varphi_t\rangle = \Gamma_{12}^{1}\cdot h(s)$$

$$\| $$

$$\langle \varphi_{ts}, \varphi_t\rangle = \frac{1}{2}E_s \implies \underline{\underline{\Gamma_{12}^{1} = \frac{1}{2}h'/h}} \qquad 3P$$

$$\langle **, \varphi_s\rangle = \Gamma_{12}^{2}$$

$$\| $$

$$\langle \varphi_{ts}, \varphi_s\rangle = \frac{1}{2}G_t = 0 \implies \underline{\underline{\Gamma_{12}^{2} = 0}} \qquad 3P$$

$$\langle ***, \varphi_t\rangle = \Gamma_{22}^{1}\cdot h(s)$$

$$\| $$

$$\langle \varphi_{ss}, \varphi_t\rangle = F_s - \frac{1}{2}G_t = 0 \implies \underline{\underline{\Gamma_{22}^{1} = 0}} \qquad 3P$$

$$\langle ***, \varphi_s\rangle = \Gamma_{22}^{2}$$

$$\| $$

$$\langle \varphi_{ss}, \varphi_s\rangle = \frac{1}{2}G_s = 0 \implies \underline{\underline{\Gamma_{22}^{2} = 0}} \qquad 3P$$