Introduction to Differential Geometry

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Exam 2 — 5 April 2019

Exercise 1. 30 P.

Let $h\colon \mathbb{R}^3 \to \mathbb{R}$ be defined as $h(x,y,z) = x^2 + y^2 + z^4 - 1$.

(a) Show that $S := h^{-1}(0)$ is a surface.

(b) Is S orientable? If so, determine the Gauß map $N\colon S \to S^2$.

(c) Let $\ell\colon S \to \mathbb{R}$ be given by $\ell(x,y,z) = z$. Find all points $p \in S$ where $d_p\ell = 0$.

Exercise 2. 40 P.

Let $\varphi\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a surface S given by

$$\varphi(t,s) := \begin{pmatrix} -s + \frac{1}{3}s^3 - t^2 s \\ t - \frac{1}{3}t^3 + ts^2 \\ s^2 - t^2 \end{pmatrix}$$

where

$$N(\varphi(t,s)) = \frac{1}{1+t^2+s^2} \cdot \begin{pmatrix} 2s \\ 2t \\ 1 - t^2 - s^2 \end{pmatrix}$$

defines an orientation on S. You do not need to verify this.

(a) Determine the matrix of the first fundamental form.

(b) Determine the Gaußian curvature and the mean curvature of S.

(c) Does there exist a conformal map from the plane to S? Does there exist a local isometry from the plane to S? Does there exist a local isometry from any sphere to S?

Exercise 3. 30 P.

Let $\varphi\colon U \to S$ be a parametrization of a surface with variables $(t,s) \in U \subseteq \mathbb{R}^2$. Let E, F, G resp. e, f, g denote the entries of the matrix of the first resp. the second fundamental form.

(a) State the definition of the Christoffel symbols.

(b) Assume that $F = 0$, $G = 1$ and that $E = h(s)$, i.e. the entry E depends only on the second variable. Express all the Christoffel symbols using h and its derivative.