Theorem 1.5.4

Let A and B be two square matrices having the same size $n \times n$ such that $BA = I$ (or $AB = I$) then A is invertible with inverse B
not necessarily both

proof

Theorem 1.5.5

Let A & B be two square matrices having the same size n×n such that A.B is invertible then ⟹ A & B are invertible matrices

proof

section 1.6

Determinants

All the matrices considered in this section are square matrices

definition 1.6.1

A determinant function is a real valued function of a square matrix variable

i.e. $\det : M \longrightarrow \mathbb{R}$ (set of square matrices)

$A \longrightarrow \det(A)$ the determinant of a square matrix A is denoted by $\det(A)$ or $|A|$

definition 1.6.2 (Determinant of a 2x2 matrix)

The determinant of a 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denoted by $|A|$ or $\det(A)$ is given by the formula:

$$|A| = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(a 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\Leftrightarrow ad - bc \neq 0$) section 1.4

example

let $A = \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$ find $|A|$ & $|B|$

answer

$|A| = 3(2) - (1)(-2) = 6 + 2 = 8$ & $|B| = 3(2) - 4(5) = 6 - 20 = -14$

definition 1.6.3 (Minors & cofactors)

(a) If A is a square matrix we define the minor of the entry a_{ij} denoted by M_{ij} to be the determinant of the submatrix that remains after the ith row & jth column are deleted from A

(b) The cofactor of the entry a_{ij} denoted by C_{ij} is given by the formula $C_{ij} = (-1)^{i+j} M_{ij}$

example

If $A=\begin{pmatrix}2 & 1 & -1\\ 3 & 1 & 4\\ 5 & -3 & 3\end{pmatrix}$ find M_{11}, M_{23}, C_{23} & C_{31}

Answer

$M_{11}=\begin{vmatrix}1 & 4\\ -3 & 3\end{vmatrix}=1(3)-(4)(-3)=3+12=15$ $\quad M_{23}=\begin{vmatrix}2 & 1\\ 5 & -3\end{vmatrix}=2(-3)-5(1)=-11$

$C_{23}=(-1)^{2+3}M_{23}=(-1)(-11)=11$ $\quad C_{31}=(-1)^{3+1}M_{31}=M_{31}=\begin{vmatrix}1 & -1\\ 1 & 4\end{vmatrix}=1(4)-1(-1)=5$

definition 1.6.4 (determinant of 3x3 and any nxn matrix along 1st row)

The determinant of the nxn matrix A computed by cofactor expansion along the 1st row is given by the formula

$$\det(A)=a_{11}C_{11}+a_{12}C_{12}+\cdots+a_{1n}C_{1n} \quad \text{where } A=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

$\Rightarrow$ if $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ is any 3x3 matrix then determinant of A computed by cofactor expansion along the 1st row is given by the formula $\det(A)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$

example

$A=\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & 1\\ -1 & 0 & -1\end{pmatrix}$ $\quad B=\begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$

find $|A|$ & $|B|$

answer

NB

Cofactors & Minors differ only in signs
if C is the matrix of cofactors of A with entries a_{ij} and corresponding cofactors C_{ij} & minors M_{ij} then

$$C = \begin{pmatrix} +M_{11} & -M_{12} & +M_{13} & - \cdots & + \cdots \\ -M_{21} & +M_{22} & -M_{23} & + \cdots & - \cdots \\ + & - & + & - \cdots & \\ - & + & - & + \cdots & - \end{pmatrix}$$

alternating signs
the identity in any cofactor matrix starts with the sign +(minor) since $C_{11} = +M_{11}$ and then for the other cofactors we use alternating signs

Theorem 1.6.1

Let A be any $n \times n$ matrix then the determinant of A can be computed by cofactor expansion along any row or any column

along the ith row: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

along the jth col: $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

example

find the determinant of $B = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$ along 2nd col & 3rd row

answer

Smart choice (example)

- find det(A) where $A=\begin{pmatrix}1&0&0&-1\\3&1&2&2\\1&0&-2&1\\2&0&0&1\end{pmatrix}$

Answer

We can find the det(A) by cofactor expansion along any row or column but notice how much we save time in finding the determinant of the matrix A if we choose the 2nd column (ie the col or row with the most of zeros) instead of choosing 1st row or 1st col etc---

$$|A| = \underbrace{-0\left|\quad\right|}_{0} + 1\begin{vmatrix}1&0&-1\\1&-2&1\\2&0&1\end{vmatrix} \underbrace{-0\left|\quad\right|}_{0} \underbrace{+0\left|\quad\right|}_{0}$$

$$\Rightarrow |A| = \begin{vmatrix}1&0&-1\\1&-2&1\\2&0&1\end{vmatrix} \underset{\text{cof expansion along 2nd col}}{=} \underbrace{-0\left|\quad\right|}_{0} + (-2)\begin{vmatrix}1&-1\\2&1\end{vmatrix} \underbrace{-0\left|\quad\right|}_{0} = -2(1+2) = -6$$

definition 1.6.6 (triangular & diagonal matrix)

(a) A square matrix is called an upper triangular matrix iff all its entries below the main diagonal are zeros

ie $a_{ij}=0 \ \forall i>j$

(b) A square matrix is called a lower triangular matrix iff all its entries above the main diagonal are zeros

ie $a_{ij}=0 \ \forall i<j$

(c) A square matrix is called a diagonal matrix iff all the entries off the main diagonal are zeros.

ie $a_{ij}=0 \ \forall i\neq j$

examples

$\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\0&a_{22}&a_{23}&a_{24}\\0&0&a_{33}&a_{34}\\0&0&0&a_{44}\end{pmatrix}$ upper Δ matrix - $\begin{pmatrix}a_{11}&0&0&0\\a_{21}&a_{22}&0&0\\a_{31}&a_{32}&a_{33}&0\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix}$ low Δ matrix $\begin{pmatrix}a&0&0&0\\0&b&0&0\\0&0&c&0\\0&0&0&d\end{pmatrix}$ diag matrix

Theorem 1.6.2

If A is an $n \times n$ triangular matrix then the determinant of A is the product of the entries on the main diagonal

(This is also true for diagonal matrix since diag matrix is upper & lower Δ matrix at the same time)

proof

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & a_{nn} \end{pmatrix} \text{ upper } \Delta \text{ matrix}$$

$\det(A)$ cf expansion along 1st col or last row:

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ 0 & 0 & & a_{nn} \end{vmatrix} \underset{\text{along 1st col}}{=} a_{11} a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ 0 & \cdots & & a_{nn} \end{vmatrix} \underset{\text{etc}}{=} a_{11} a_{22} a_{33} \begin{vmatrix} a_{44} & \cdots & a_{4n} \\ 0 & a_{55} \cdots & a_{5n} \\ 0 & 0 \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} \cdots a_{n-2\,n-2} \begin{vmatrix} a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & a_{nn} \end{vmatrix} = a_{11} a_{22} a_{33} \cdots a_{nn}$$ (product entries on the main diagonal)

Theorem 1.6.2

let A be a square matrix

(a) if A has a row or column of zeros then $\det(A) = 0$

(b) $\det(A) = \det(A^T)$ (since det can be found along any row or column)

Theorem 1.6.3 (properties of determinant)

let A be a square matrix

(a) If two rows or columns of A are interchanged to produce a matrix B then $\det(B) = -\det(A)$

(b) If a row or column of A is multiplied by a constant k to produce a matrix B then $\det(B) = k \det(A)$

(c) If a multiple of one row or column of A is added to another to produce B then $\det(B) = \det(A)$

example

$$A = \begin{pmatrix} 0 & 1 & 3 & -1 \\ 2 & 4 & -6 & 1 \\ 0 & 3 & 9 & 2 \\ -2 & -4 & 1 & -3 \end{pmatrix}$$

Theorem 1.6.4

Let A & B be $n\times n$ matrices and $k\in\mathbb{R}$, then:

(a) $\det(AB)=\det(A).\det(B)$

(b) $\det(kA)=k^n\det(A)$

(c) If A has 2 proportional rows (or columns) then $\det(A)=0$

example

let $A=\begin{pmatrix}1 & 2\\ 3 & -2\end{pmatrix}$ $B=\begin{pmatrix}1 & -1\\ 1 & 4\end{pmatrix}$ find $\det(AB)$

1st way

$\det(A.B)=\det(A).\det(B)=((-2)-6)(4-(-1))=(-8)(5)=-40$

2nd way

$A.B=\begin{pmatrix}3 & 7\\ 1 & -11\end{pmatrix}\Longrightarrow \det(A.B)=-33-7=-40$

Remark 1

let A & B be two $n\times n$ matrices

$\det(A.B)=\det(A).\det(B)$ But!!! $\det(A+B)\neq\det(A)+\det(B)$

see previous example

$A+B=\begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix}$ $\det(A+B)=4-4=0\neq-8+5=3=\det(A)+\det(B)$

Remark 2

Let A, B & C be 3 $n\times n$ matrices that differ only in a single row or column saying the ith row or jth col where

(ith row of C) = (ith row of A) + (ith row of B)

or (jth col of C) = (jth col of A) + (jth col of B)

Then $\det(C)=\det(A)+\det(B)$

$$A=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}\quad B=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ b_{i1} & b_{i2} & \cdots & b_{in}\\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}\quad C=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ (a_{i1}+b_{i1}) & (a_{i2}+b_{i2}) & & (a_{in}+b_{in})\\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

$$\det(C)=(a_{i1}+b_{i1})C_{i1}+\cdots+(a_{in}+b_{in})C_{in}=(a_{i1}C_{i1}+\cdots+a_{in}C_{in})+(b_{i1}C_{i1}+\cdots+b_{in}C_{in})$$
$$=\det(A)+\det(B)$$

Theorem 1.6.5

A square matrix A is invertible if & only if $\det(A) \neq 0$

proof

Corollary 1.6.1

let A be an invertible matrix then $\det(A^{-1}) = \frac{1}{\det(A)}$

proof

Theorem 1.6.6 (Cramer's rule)

Let $Ax=b$ be a linear system of n linear equations & n unknowns such that $\det(A)\neq 0$ (ie A invertible) then the system has a unique solution $x=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$ where $x_i=\dfrac{\det(A_i)}{\det(A)}$ $\forall i=1,\dots,n$.

& A_i is the matrix obtained by replacing the entries of the ith column of A by the entries of the constant matrix b

$$A_i=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1i-1} & b_1 & a_{1i+1} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2i-1} & b_2 & a_{2i+1} & \cdots & a_{2n}\\ \\ a_{n1} & a_{n2} & \cdots & a_{ni-1} & b_n & a_{ni+1} & \cdots & a_{nn}\end{pmatrix}$$

(ith col)

example

solve the system $\begin{cases}x-y-2z=3\\-x+2y+3z=1\\2x-2y-2z=-2\end{cases}$ using Cramer's rule

answer

$$\underbrace{\begin{pmatrix}1&-1&-2\\-1&2&3\\2&-2&-2\end{pmatrix}}_{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}\qquad \det(A)=1(-4+6)+1(2-6)-2(2-4)=2-4+4=2$$

let $A_1=\begin{pmatrix}3&-1&-2\\1&2&3\\-2&-2&-2\end{pmatrix}$ $A_2=\begin{pmatrix}1&3&-2\\-1&1&3\\2&-2&-2\end{pmatrix}$ $A_3=\begin{pmatrix}1&-1&3\\-1&2&1\\2&-2&-2\end{pmatrix}$

$|A_1|=3(2)+4-4=6$ $\quad|A_2|=4-12-2(0)=16$ $\quad|A_3|=1(-2)+1(0)+3(-2)=-8$

$\Rightarrow x=\dfrac{|A_1|}{|A|}=3$ $\quad y=\dfrac{|A_2|}{|A|}=8$ $\quad z=\dfrac{|A_3|}{|A|}=-4$

Theorem 1.6.7

let A be a square matrix then the following are equivalent

(a) A is invertible

(b) $Ax=b$ has a unique solution $x=A^{-1}b$ for any $n\times 1$ matrix b

(c) $Ax=0$ has only the trivial solution

(d) the reduced row echelon form of A is I (ie A is row equivalent to I)

(e) the determinant of the matrix A: $\det(A)\neq 0$

Supp exercises

[1] Determine Conditions on b_1, b_2 & b_3 for the following systems to be Consistent

system 1
$$\begin{cases} x_1 + x_2 + 2x_3 = b_1 \\ x_1 \qquad + x_3 = b_2 \\ 2x_1 + x_2 + 3x_3 = b_3 \end{cases}$$

system 2
$$\begin{cases} x_1 + 2x_2 + 3x_3 = b_1 \\ 2x_1 + 5x_2 + 3x_3 = b_2 \\ x_1 \qquad + 8x_3 = b_3 \end{cases}$$

[2] Determine Conditions on k for the following system to have:

(a) unique sol

(b) No solution

(c) ∞ many soluts

$$\begin{cases} kx + y + z = 1 \\ x + ky + z = 1 \\ x + y + kz = 1 \end{cases}$$

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determine Conditions on k for the following system to have

$$\begin{cases} kx + y + z = 0 \\ x + ky + z = 0 \\ x + y + kz = 0 \end{cases}$$

(a) a unique solution

(b) a one-parameter family of solutions

(c) a two-parameter family of solutions