Theorem 1.4.4 (product of invertible matrices is invertible)

let A & B be 2 square invertible matrices having same size $n\times n$.
then AB is an invertible $n\times n$ matrix where $(AB)^{-1}=B^{-1}A^{-1}$

In General

$A_1, A_2, \dots, A_k$ k $n\times n$ invertible matrices then $A_1 \cdot A_2 \cdots A_k$ is an invertible $n\times n$ matrix where $(A_1 \cdot A_2 \cdots A_k)^{-1}=A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$

Pf (for A & B)

$$(AB)\cdot(B^{-1}A^{-1})\overset{\text{ass}}{=}A(BB^{-1})A^{-1}=(A\cdot I)A^{-1}=AA^{-1}=I.$$

$\Rightarrow$ AB is invertible where $(AB)^{-1}=B^{-1}A^{-1}$

example

$$A=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} \qquad B=\begin{pmatrix}3 & 2\\ 2 & 2\end{pmatrix} \Rightarrow AB=\begin{pmatrix}7 & 6\\ 9 & 8\end{pmatrix}$$

$$(AB)^{-1}=\frac{1}{56-54}\begin{pmatrix}8 & -6\\ -9 & 7\end{pmatrix}=\begin{pmatrix}4 & -3\\ -9/2 & 7/2\end{pmatrix}$$

now $A^{-1}=\begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix}$ $B^{-1}=\frac{1}{2}\begin{pmatrix}2 & -2\\ -2 & 3\end{pmatrix}=\begin{pmatrix}1 & -1\\ -1 & 3/2\end{pmatrix}$ & $B^{-1}A^{-1}=\begin{pmatrix}4 & -3\\ -9/2 & 7/2\end{pmatrix}=(AB)^{-1}$

definition 1.4.3

let A be a square matrix we define

(a) $A^0=I$

(b) $A^n=\underbrace{A\cdot A\cdots A}_{n\text{ times}}$ $(n\geq 0)$

(c) $A^{-n}=(A^{-1})^n=\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n\text{ times}}$ $(n\geq 0)$

properties

$$A^r\cdot A^s=\underbrace{A\cdots A}_{r\text{ times}}\ \underbrace{A\cdot A\cdots A}_{s\text{ times}}=\underbrace{A\cdot A\cdots A}_{r+s}=A^{r+s}.$$

$$(A^r)^s=\underbrace{A^r A^r\cdots A^r}_{s\text{ times}}=\underbrace{\underbrace{(A\cdots A)}_{r\text{ times}}(A\cdots A)\cdots(A\cdots A)}_{rs\text{ times}}=A^{rs}.$$

Theorem 1.4.5

let A be an invertible matrix then

(a) A^{-1} is invertible with $(A^{-1})^{-1} = A$

(b) A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ (denoted by A^{-n})

(c) $\forall k \neq 0$ kA is invertible & $(kA)^{-1} = \frac{1}{k} A^{-1}$

(d) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

pf

(a) to show A^{-1} invertible enough to find a matrix B having same size of A^{-1} such that $A^{-1}B = I$.

let $B = A$ (since we know that A is inv ie $A^{-1} \exists$ where $AA^{-1} = I = A^{-1}A$)

A is a square matrix having same size of A^{-1} / $A^{-1}A = I \Rightarrow A^{-1}$ invertible with $(A^{-1})^{-1} = A$.

(b) A invertible $\Rightarrow A^n$ product of invertible matrices by th 1.4.4. A^n is invertible where $(A^n)^{-1} = (A.A\cdots A)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{\text{starting by last one}} = (A^{-1})^n$.

(c) A is invertible $\Rightarrow A^{-1} \exists$ where $AA^{-1} = I = A^{-1}A$.

notice that $(kA).(\frac{1}{k}A^{-1}) = (k\frac{1}{k})(AA^{-1}) = 1.I = I$.

let $B = \frac{1}{k}A^{-1} \Rightarrow \exists B = \frac{1}{k}A^{-1}$ same size of kA / $(kA)B = I$.

$\Rightarrow kA$ is invertible where $(kA)^{-1} = \frac{1}{k}A^{-1}$

(d) A invertible $\Rightarrow AA^{-1} = I$.

$\Rightarrow (AA^{-1})^T = I^T$

$\Rightarrow (A^{-1})^T A^T = I$.

$\Rightarrow$ there exists a matrix $\underbrace{(A^{-1})^T}_{B}$ / $\underbrace{(A^{-1})^T}_{B} . A^T = I \Rightarrow A^T$ invertible having same size of A^T with inverse $(A^T)^{-1} = (A^{-1})^T$

example

let $A=\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}$ find $(A^T)^{-1}$

answer

$(A^T)^{-1}$ can be found in 2 ways either by finding the inverse of A^T or by finding the transpose of A^{-1}

$A^T=\begin{pmatrix}3 & 5\\ 1 & 2\end{pmatrix}$ $\qquad (A^T)^{-1}=\frac{1}{6-5}\begin{pmatrix}2 & -5\\ -1 & 3\end{pmatrix}=\begin{pmatrix}2 & -5\\ -1 & 3\end{pmatrix}$

$(A^{-1})^T=\frac{1}{6-5}\begin{pmatrix}2 & -1\\ -5 & 3\end{pmatrix}^T=\begin{pmatrix}2 & -5\\ -1 & 3\end{pmatrix}$

exercise

let B be an $n\times n$ invertible matrix & A any $n\times n$ matrix / $A.B=BA$ show that A & B^{-1} commute

answer

given $AB=BA$

multp both sides to the left & to the right by B^{-1}

$B^{-1}(AB)B^{-1}=B^{-1}(BA)B^{-1}$

.ass

$\Rightarrow (B^{-1}A)\underbrace{(BB^{-1})}_{I}=\underbrace{(B^{-1}B)}_{I}(AB^{-1})$

$\Rightarrow (B^{-1}A)I=I.(AB^{-1})$

$\Rightarrow B^{-1}A=AB^{-1}$

section 1.5

Matrix equations

In this section we show how any linear system can be written in a matrix form by using a matrix and two vectors

Consider the following $m \times n$ linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

$$\Longleftrightarrow \underbrace{\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}}_{(*)} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

we replace the m equations in the system by an $m\times 1$ matrix since 2 matrices are equal iff corresp. entries are equal!

$$\Longleftrightarrow \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b.}$$

$(*)$ can be written as product of a matrix A and a vector x

Conclusion: Any $m\times n$ linear system can be written in a matrix form $\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$ (using a matrix A & 2 vectors x & b)

examples.

$$\begin{cases} x - 6y - 4z = -5 \\ 2x - 10y - 9z = -4 \\ -x + 6y + 5z = 3 \end{cases} \text{ has a matrix form } \begin{pmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$$

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + x_2 = 7 \\ 2x_1 + 3x_2 = 6 \end{cases} \text{ has a matrix form } \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 6 \end{pmatrix}$$

Theorem 1.5.1

Let A be an $n \times n$ invertible matrix then the linear system $Ax = b$ has the unique solution $x = A^{-1}b$ for any $n \times 1$ matrix b

pf

Consider the system $Ax = b$ $(*)$

since A is invertible $\Rightarrow A^{-1}$ exists

multp both sides of $(*)$ to the left by A^{-1}

$$(*) \Rightarrow A^{-1}(Ax) = A^{-1}b$$

$$\overset{\cdot\, \text{ass}}{\Rightarrow} (A^{-1}A)x = A^{-1}b$$

$$\Rightarrow I.x = A^{-1}b$$

$$\Rightarrow \boxed{x = A^{-1}b}$$

& since the inverse of a matrix is unique $\Rightarrow A^{-1}$ is unique

$\Rightarrow x = A^{-1}b$ is unique

Corollary 1.5.1

Let A be an $n \times n$ invertible matrix then the homogeneous linear system $Ax = 0$ has only the trivial solution $\boxed{x = 0}$ ($x = 0$ is the only solution)

pf

1st method (using th 1.5.1)

Cons $Ax = 0$ where A is invertible but when A is invertible Th 1.5.1 implies that $Ax = b$ has unique sol $x = A^{-1}b$ for any $n \times 1$ matrix b [illegible]

in particular for $b = 0$

$\Rightarrow Ax = 0$ has the unique sol $x = A^{-1}0 = 0$

2nd method (same pf of Th 1.5.1)

Consider $Ax = 0$ $(*)$

since A is invertible $\Rightarrow A^{-1} \exists$

multp both sides to the left by A^{-1}

$$(*) \Rightarrow A^{-1}(Ax) = A^{-1}0$$

$$\overset{\cdot\, \text{ass}}{\Rightarrow} (A^{-1}A)x = 0$$

$$\Rightarrow Ix = 0$$

$$\Rightarrow x = 0$$

example (on Th 1.5.1)

solve the system $AX=b$ by using A^{-1}

$$\begin{cases} 2x+y=1 \\ -4x+3y=2 \end{cases} \iff \underbrace{\begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

notice that $A=\begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix}$ is invertible since $ad-bc=6+4=10\neq 0$

where $A^{-1}=\frac{1}{10}\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}=\begin{pmatrix} \frac{3}{10} & -1/10 \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$

by Th 1.5.1 $X=\begin{pmatrix} x \\ y \end{pmatrix}=A^{-1}b=\begin{pmatrix} \frac{3}{10} & -1/10 \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}=\begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1/10 \\ 4/5 \end{pmatrix} \Rightarrow x=\frac{1}{10}$ & $y=4/5$

Counter example (on Corollary 1.5.1)

Find all vectors X solutions of $AX=0$ / $A=\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ & $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Answer

first observe that $X=0$ ie $\left(X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$ is a solution of $AX=0$

since $X=0$ is always a solution of any homogeneous system $AX=0$.

Now let's find the general solution of $AX=0$ by reducing the corresp. augmented matrix to row echelon form (the homog system $AX=0$ may have either exactly one sol or ∞ many sol)

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (*)$$

(Notice that the ref of the square matrix A contains a row of zeros)

$\Longrightarrow$ the rref of A is not $I \iff A$ is not invertible

That's why $AX=0$ has ∞ many solution (ie $X=0$ is not the only sol)

$$(*) \Rightarrow \begin{cases} x_1+2x_2+x_3=0 \\ x_2-x_3=0 \end{cases} \Rightarrow \text{for } x_3=t \begin{array}{l} \nearrow x_2=t \\ \searrow x_1=-2t-t=-3t \end{array}$$

$$\Rightarrow \begin{array}{l} x_1=-3t \\ x_2=t \\ x_3=t \end{array} \Rightarrow S=\left\{\begin{pmatrix} -3t \\ t \\ t \end{pmatrix} / t\in\mathbb{R}\right\}$$

(NB: the coeff matrix A is not row equivalent to I & hence A is not invertible
Th 1.5.1 is not applicable.)

Theorem 1.5.2 .

Given A an $m\times n$ matrix then

(a) the linear system $AX=b$ has either no solution or exactly one solution or ∞ many solutions

(b) the homogeneous system $AX=0$ has either exactly one solution $X=0$ or ∞ many solutions ($X=0$ & ∞ many other solutions)

Theorem 1.5.3 (very imp)

Let A be a square $n\times n$ matrix then the following are equivalent:

(a) A is invertible

(b) $AX=b$ has exactly one solution $X=A^{-1}b$ for any $n\times 1$ matrix b.

(c) $AX=0$ has only the trivial solution $X=0$

(d) the reduced row echelon form of A (rref of A) is I

proof (enough to show (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a))

(a)$\Rightarrow$(b) (Theorem 1.5.1) done

(b)$\Rightarrow$(c) (Corollary 1.5.1) done

(c)$\Rightarrow$(d) Given $AX=0$ has only the trivial solution

$$\Leftrightarrow \quad AX=0 \Longrightarrow X=0 \text{ ie } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}}_{I} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right)$$

after reducing the system to rref to get directly the sol $X=0$

$\Rightarrow$ the rref of A is I

(d)$\Rightarrow$(a) done (section 1.4 A invertible $\Leftrightarrow$ rref of A is I)