# Summary (section 2.3)

let $S = \{v_1, v_2, \dots, v_k\}$ be a set of $k$ vectors in $\mathbb{R}^n$.

we distinguish 3 cases.

## case 1

if $k > n$ then we can say directly that $S$ is linearly dependent. (By th. 2.3.1)

example: a set of 4 vectors in $\mathbb{R}^3$ is dependent

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$ is lin dependent in $\mathbb{R}^4$

since #vectors in $S = 5 > 4$ (order of $\mathbb{R}^4$) by th 2.3.1 $S$ is dependent.

## case 2.

if $k = n$. $S$ is independent $\iff \det(A) \neq 0$.

where $A = (v_1 | v_2 | \dots | v_n)$.

$S$ is dependent $\iff \det(A) = 0$

example show that $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is lin indep in $\mathbb{R}^3$.

notice that #vectors in $S = 3 =$ order of $\mathbb{R}^3$.

to show $S$ is indep enough to show $\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq 0$?

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1(0-1) - \underbrace{1(1-1) + 0(1-0)}_{0}$$
$$= -1 \neq 0$$

$\Rightarrow$ $S$ is indep.

## case 3.

if $k < n$ (#vectors in $S <$ order of $\mathbb{R}^n$)

we solve the system $c_1 v_1 + \dots + c_k v_k = 0$ i.e. we solve $\left(v_1 | \dots | v_k \,\middle|\, \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right)$.

$S$ is indep iff the only solution is the trivial $c_1 = \dots = c_k = 0$

$S$ is dependent iff the system has a nontrivial sol.

example

show that $S=\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right\}$ is indep in $\mathbb{R}^3$

# vectors in $S = 2 < 3 =$ order of $\mathbb{R}^3$

let $k_1v_1+k_2v_2=0$ $\Rightarrow$ $\left(\begin{array}{cc|c}1&2&0\\1&1&0\\1&0&0\end{array}\right)\to\left(\begin{array}{cc|c}1&1&0\\1&2&0\\1&0&0\end{array}\right)\to\left(\begin{array}{cc|c}1&1&0\\0&1&0\\0&0&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right)$

show that $k_1=k_2=0$ is the only solut

$\Rightarrow k_1=k_2=0$ is the only solut

proof of case 1 (if $k=n$)

$S$ is lin indep $\iff k_1v_1+\dots+k_nv_n=0$ has only the trivial solut

$\iff \underbrace{\left(v_1\middle|\dots\middle|v_n\right)}_{A}\underbrace{\begin{pmatrix}k_1\\k_2\\\vdots\\k_n\end{pmatrix}}_{k}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ has only the trivial sol.

$\iff Ak=0$ has only the trivial solut

$\iff \det(A)\neq 0$ (by the th.)

exercises (on sect 2.2)

1) show that any vector in $\mathbb{R}^3$ can be written as lin combinat of the vectors

$\underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{v_1},\underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{v_2},\underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{v_3},\underbrace{\begin{pmatrix}-1\\0\\0\end{pmatrix}}_{v_4}$.

2) show that not any vector in $\mathbb{R}^3$ is a lin combinat of the vectors

$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\2\end{pmatrix}$

3) show that not any vector in $P_2(x)$ is a lin combinat of the vectors

$p_1(x)=1+x$, $p_2(x)=1+x^2$, $p_3(x)=2+x+x^2$

answer

1) let $v=\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3$; $v$ lin comb of $v_1,v_2,v_3,v_4 \iff \exists k_1,k_2,k_3,k_4\in\mathbb{R}/$
$k_1v_1+k_2v_2+k_3v_3+k_4v_4=v$.

$\iff \exists k_1,k_2,k_3,k_4\in\mathbb{R}/$
$k_1\begin{pmatrix}1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}0\\1\\1\end{pmatrix}+k_3\begin{pmatrix}1\\1\\0\end{pmatrix}+k_4\begin{pmatrix}-1\\0\\0\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$

we obtain the system $\begin{cases}k_1+k_3-k_4=a\\k_1+k_2+k_3=b\\k_1+k_2=c\end{cases}$

$\iff$ the system $\left(\begin{array}{cccc|c}1&0&1&-1&a\\1&1&1&0&b\\1&1&0&0&c\end{array}\right)$ is consistent

study consistency

$\left(\begin{array}{cccc|c}1&0&1&-1&a\\1&1&1&0&b\\1&1&0&0&c\end{array}\right)\to\left(\begin{array}{cccc|c}1&0&1&-1&a\\0&1&0&1&b-a\\0&1&-1&1&c-a\end{array}\right)\to\left(\begin{array}{cccc|c}1&0&1&-1&a\\0&1&0&1&b-a\\0&0&-1&0&c-b\end{array}\right)$

notice that the system is consistent without cond on $a,b,c$

$\Rightarrow$ any $v=\begin{pmatrix}a\\b\\c\end{pmatrix}$ lin comb of the vectors $v_1,\dots,v_4$

2) let $v=\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3$ ; $v$ lin Comb of $v_1, v_2$ & $v_3 \Longleftrightarrow \exists k_1, k_2, k_3 \in \mathbb{R} / k_1v_1+k_2v_2+k_3v_3 = v$

$\Longleftrightarrow \exists k_1, k_2, k_3 \in \mathbb{R} / k_1\begin{pmatrix}1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}0\\1\\1\end{pmatrix}+k_3\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$\Longleftrightarrow$ the system $\left(\begin{array}{ccc|c}1&0&1&a\\1&1&2&b\\1&1&2&c\end{array}\right)$ is consistent

study consistency

$$\left(\begin{array}{ccc|c}1&0&1&a\\1&1&2&b\\1&1&2&c\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&a\\0&1&1&b-a\\0&1&1&c-a\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&a\\0&1&1&b-a\\0&0&0&c-b\end{array}\right)$$

The system is consistent only when $c-b=0 \Longleftrightarrow c=b$

$\Rightarrow$ not any vector in $\mathbb{R}^3$ is a lin Combinatn of the vectors $v_1, v_2, v_3$.

only the vectors of the form $\begin{pmatrix}a\\b\\b\end{pmatrix}$ can be written as lin Comb of $v_1, v_2, v_3$.

for example $\underbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}}_{\in\mathbb{R}^3}$ can't be written as lin Comb of $v_1, v_2, v_3$.

since $c=3\neq 2=b$

3) let $p(x)=ax^2+bx+c \in P_2$

$p(x)$ lin Comb of $p_1(x), p_2(x), p_3(x) \Longleftrightarrow \exists k_1, k_2, k_3 \in \mathbb{R} / k_1p_1(x)+k_2p_2(x)+k_3p_3(x)=p(x)$

$\Longleftrightarrow \exists k_1, k_2, k_3 \in \mathbb{R} /$

$\Longleftrightarrow k_1(1+x)+k_2(1+x^2)+k_3(2+x+x^2)=ax^2+bx+c$

$\exists k_1, k_2, k_3 \in \mathbb{R} /$

$\Longleftrightarrow (k_2+k_3)x^2+(k_1+k_3)x+(k_1+k_2+2k_3)=ax^2+bx+c$

$\Longleftrightarrow$ the system $\begin{cases}k_2+k_3=a\\k_1+k_3=b\\k_1+k_2+k_3=c\end{cases}$ is consistent

study consistency

$$\left(\begin{array}{ccc|c}0&1&2&a\\1&0&1&b\\1&1&1&c\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&2&c\\1&0&1&b\\0&1&1&a\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&c\\0&-1&-1&b-c\\0&1&1&a\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&c\\0&-1&-1&b-c\\0&0&0&a+b-c\end{array}\right)$$

The system is consistent only when $a+b-c=0$
i.e. $c=a+b$

$\Rightarrow$ not any $p(x)$ is a lin Comb. of $p_1(x)$, $p_2(x)$ & $p_3(x)$ only polynomial $p(x)$ of the form $p(x)=ax^2+bx+(a+b)$

for ex $p(x)=x^2$ is not a lin Comb of $p_1, p_2, p_3$.