# MTH 201. HOMEWORK I SOLUTION.

## Section 7.6:

9. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

13. $\lim\limits_{x \to 1^-} \sin^{-1} x = \frac{\pi}{2}$

16. $\lim\limits_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

22. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

23. $y = \sin^{-1}\sqrt{2}t \;\Rightarrow\; \frac{dy}{dt} = \frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}t\right)^2}} = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

33. $y = \ln\left(\tan^{-1} x\right) \;\Rightarrow\; \frac{dy}{dx} = \frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1} x} = \frac{1}{\left(\tan^{-1} x\right)\left(1+x^2\right)}$

42. $y = \ln\left(x^2+4\right) - x\tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - x\left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right] = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2} = -\tan^{-1}\left(\frac{x}{2}\right)$

43. $\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + C$

49. $\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}} = \left[4\sin^{-1}\frac{s}{2}\right]_0^1 = 4\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right) = 4\left(\frac{\pi}{6} - 0\right) = \frac{2\pi}{3}$

55. $\int \frac{3\,dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2(r-1)$ and $du = 2\,dr$
$= \frac{3}{2}\sin^{-1} u + C = \frac{3}{2}\sin^{-1} 2(r-1) + C$

63. $\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1+e^{2x}} = \int_1^{\sqrt{3}} \frac{du}{1+u^2}$, where $u = e^x$ and $du = e^x\,dx$; $x = 0 \;\Rightarrow\; u = 1$, $x = \ln\sqrt{3} \;\Rightarrow\; u = \sqrt{3}$
$= \left[\tan^{-1} u\right]_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

65. $\int \frac{y\,dy}{\sqrt{1-y^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = y^2$ and $du = 2y\,dy$
$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} y^2 + C$

77. $\int \frac{x^2+2x-1}{x^2+9}dx = \int\left(1 + \frac{2x-10}{x^2+9}\right)dx = \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx$; $\int \frac{2x}{x^2+9}dx = \int \frac{1}{u}du$ where $u = x^2+9$
$\Rightarrow du = 2x\,dx \Rightarrow \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx = x + \ln(x^2+9) - \frac{10}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

# Section 7.7:

11. (a) $\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$
    (b) $\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sin x = \cosh^2 x + \sinh^2 x$

23. $y = (x^2+1)\operatorname{sech}(\ln x) = (x^2+1)\left(\frac{2}{e^{\ln x}+e^{-\ln x}}\right) = (x^2+1)\left(\frac{2}{x+x^{-1}}\right) = (x^2+1)\left(\frac{2x}{x^2+1}\right) = 2x \Rightarrow \frac{dy}{dx} = 2$

37. (a) If $y = \tan^{-1}(\sinh x) + C$, then $\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$, which verifies the formula
    (b) If $y = \sin^{-1}(\tanh x) + C$, then $\frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x$, which verifies the formula

42. $\int \sinh \frac{x}{5}\, dx = 5\int \sinh u\, du$, where $u = \frac{x}{5}$ and $du = \frac{1}{5}\, dx$
    $= 5\cosh u + C = 5\cosh \frac{x}{5} + C$

45. $\int \tanh \frac{x}{7}\, dx = 7\int \frac{\sinh u}{\cosh u}\, du$, where $u = \frac{x}{7}$ and $du = \frac{1}{7}\, dx$
    $= 7\ln|\cosh u| + C_1 = 7\ln\left|\cosh \frac{x}{7}\right| + C_1 = 7\ln\left|\frac{e^{x/7}+e^{-x/7}}{2}\right| + C_1 = 7\ln\left|e^{x/7}+e^{-x/7}\right| - 7\ln 2 + C_1$
    $= 7\ln\left|e^{x/7}+e^{-x/7}\right| + C$

47. $\int \operatorname{sech}^2\left(x-\frac{1}{2}\right) dx = \int \operatorname{sech}^2 u\, du$, where $u = \left(x-\frac{1}{2}\right)$ and $du = dx$
    $= \tanh u + C = \tanh\left(x-\frac{1}{2}\right) + C$

57. $\int_1^2 \frac{\cosh(\ln t)}{t}\, dt = \int_0^{\ln 2} \cosh u\, du = [\sinh u]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - 0 = \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$, where $u = \ln t$, $du = \frac{1}{t}\, dt$, the lower limit is $\ln 1 = 0$ and the upper limit is $\ln 2$

# Section 8.4:

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3)$; $y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4}$; $y = 3 \Rightarrow A = \frac{3}{4}$;
    $\int_4^8 \frac{y\, dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$
    $= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 \quad \frac{\ln 15}{2}$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1)+B$
    $= Ax + (A+B) \Rightarrow A = 3, A+B = 2 \Rightarrow A = 3, B = -1$; $\int_0^1 \frac{x^3\, dx}{x^2+2x+1}$
    $= \int_0^1 (x-2)\, dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$
    $= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$
$x = -1 \Rightarrow C = \frac{1}{4}$; $x = 1 \Rightarrow D = \frac{1}{4}$; coefficient of $x^3 = A + B \Rightarrow A + B = 0$; constant $= A - B + C + D$
$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2}$; thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4}$; $\int \frac{dx}{(x^2-1)^2}$
$= \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s + 2$
$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$
$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$
$+ E(s^2+1)$
$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{array}{r} A + C = 0 \\ -3A + B - 2C + D = 0 \\ 3A - 3B + 2C - D + E = 0 \\ -A + 3B - 2C + D = 2 \\ -B + C - D + E = 2 \end{array}\right\} \text{summing all equations} \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0$; $C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;
$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$

27. $\frac{x^2-x+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow x^2 - x + 2 = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$
$\Rightarrow A + B = 1,\ A - B + C = -1,\ A - C = 2 \Rightarrow$ adding eq(2) and eq(3) $\Rightarrow 2A - B = 1$, add this equation to eq(1)
$\Rightarrow 3A = 2 \Rightarrow A = \frac{2}{3} \Rightarrow B = 1 - A = \frac{1}{3} \Rightarrow C = -1 - A + B = -\frac{4}{3}$; $\int \frac{x^2-x+2}{x^3-1}dx = \int \left(\frac{2/3}{x-1} + \frac{(1/3)x - 4/3}{x^2+x+1}\right)dx$
$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{x-4}{(x+\frac{1}{2})^2+\frac{3}{4}}dx \quad \left[u = x + \frac{1}{2} \Rightarrow u - \frac{1}{2} = x \Rightarrow du = dx\right]$
$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u-\frac{9}{2}}{u^2+\frac{3}{4}}du = \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}du - \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}du$
$= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln\left|\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right| - \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C = \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

29. $\frac{x^2}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx + D)(x-1)(x+1)$
$= (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x - A + B - D \Rightarrow A + B + C = 0,\ -A + B + D = 1,$
$A + B - C = 0,\ -A + B - D = 0 \Rightarrow$ adding eq(1) to eq (3) gives $2A + 2B = 0$, adding eq(2) to eq(4) gives $-2A + 2B = 1$, adding these two equations gives $4B = 1 \Rightarrow B = \frac{1}{4}$, using $2A + 2B = 0 \Rightarrow A = -\frac{1}{4}$, using $-A + B - D = 0 \Rightarrow D = \frac{1}{2}$, and using $A + B - C = 0 \Rightarrow C = 0$; $\int \frac{x^2}{x^4-1}dx = \int \left(\frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{x^2+1}\right)dx$
$= -\frac{1}{4}\int \frac{1}{x+1}dx + \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx = -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + \frac{1}{2}\tan^{-1}x + C = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x + C$

34. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}$; $\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1)$;
$x = -1 \Rightarrow A = -\frac{1}{2}$; $x = 1 \Rightarrow B = \frac{1}{2}$; $\int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

39. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y]\int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

45. $\int \frac{1}{x^{3/2}-\sqrt{x}}dx = \int \frac{1}{\sqrt{x}(x-1)}dx \left[\text{Let } u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}dx\right] \to \int \frac{2}{u^2-1}du;$
$\frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0, -A + B = 2$

$\Rightarrow B = 1 \Rightarrow A = -1; \int \frac{2}{u^2-1}du = \int\left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = -\int \frac{1}{u+1}du + \int \frac{1}{u-1}du = -\ln|u+1| + \ln|u-1| + C$
$= \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$

47. $\int \frac{\sqrt{x+1}}{x}\,dx \left[\text{Let } x+1 = u^2 \Rightarrow dx = 2u\,du\right] \to \int \frac{u}{u^2-1}\,2u\,du = \int \frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du$
$= 2\int du + \int \frac{2}{u^2-1}du; \frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$
$-A + B = 2 \Rightarrow B = 1 \Rightarrow A = -1; 2\int du + \int \frac{2}{u^2-1}du = 2u + \int\left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = 2u - \int \frac{1}{u+1}du + \int \frac{1}{u-1}du$
$= 2u - \ln|u+1| + \ln|u-1| + C = 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$