MATH 201. Homework 6 Solution

Section 10.8:

1. $f(x) = e^{2x}$, $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, $f'''(x) = 8e^{2x}$; $f(0) = e^{2(0)} = 1$, $f'(0) = 2$, $f''(0) = 4$, $f'''(0) = 8 \Rightarrow P_0(x) = 1$, $P_1(x) = 1 + 2x$, $P_2(x) = 1 + x + 2x^2$, $P_3(x) = 1 + x + 2x^2 + \frac{4}{3}x^3$

2. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$; $f(0) = \sin 0 = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$ $\Rightarrow P_0(x) = 0$, $P_1(x) = x$, $P_2(x) = x$, $P_3(x) = x - \frac{1}{6}x^3$

3. $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$; $f(1) = \ln 1 = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2 \Rightarrow P_0(x) = 0$, $P_1(x) = (x-1)$, $P_2(x) = (x-1) - \frac{1}{2}(x-1)^2$, $P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

4. $f(x) = \ln(1+x)$, $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$; $f(0) = \ln 1 = 0$, $f'(0) = \frac{1}{1} = 1$, $f''(0) = -(1)^{-2} = -1$, $f'''(0) = 2(1)^{-3} = 2 \Rightarrow P_0(x) = 0$, $P_1(x) = x$, $P_2(x) = x - \frac{x^2}{2}$, $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$

5. $f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$; $f(2) = \frac{1}{2}$, $f'(2) = -\frac{1}{4}$, $f''(2) = \frac{1}{4}$, $f'''(x) = -\frac{3}{8}$ $\Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{1}{4}(x-2)$, $P_2(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$, $P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

6. $f(x) = (x+2)^{-1}$, $f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$, $f'''(x) = -6(x+2)^{-4}$; $f(0) = (2)^{-1} = \frac{1}{2}$, $f'(0) = -(2)^{-2} = -\frac{1}{4}$, $f''(0) = 2(2)^{-3} = \frac{1}{4}$, $f'''(0) = -6(2)^{-4} = -\frac{3}{8} \Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{x}{4}$, $P_2(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$, $P_3(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$

10. $f(x) = (1-x)^{1/2}$, $f'(x) = -\frac{1}{2}(1-x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1-x)^{-3/2}$, $f'''(x) = -\frac{3}{8}(1-x)^{-5/2}$; $f(0) = (1)^{1/2} = 1$, $f'(0) = -\frac{1}{2}(1)^{-1/2} = -\frac{1}{2}$, $f''(0) = -\frac{1}{4}(1)^{-3/2} = -\frac{1}{4}$, $f'''(0) = -\frac{3}{8}(1)^{-5/2} = -\frac{3}{8} \Rightarrow P_0(x) = 1$, $P_1(x) = 1 - \frac{1}{2}x$, $P_2(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2$, $P_3(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

11. $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x} \Rightarrow \ldots f^{(k)}(x) = (-1)^k e^{-x}$; $f(0) = e^{-(0)} = 1$, $f'(0) = -1$, $f''(0) = 1$, $f'''(0) = -1, \ldots, f^{(k)}(0) = (-1)^k \Rightarrow e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n$

16. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{x}{2} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^5}{2^5 \cdot 5!} + \ldots$

19. $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x^2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

22. $f(x) = \frac{x^2}{x+1} \Rightarrow f'(x) = \frac{2x+x^2}{(x+1)^2}$; $f''(x) = \frac{2}{(x+1)^3}$; $f'''(x) = \frac{-6}{(x+1)^4} \Rightarrow f^{(n)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}}$; $f(0) = 0$, $f'(0) = 0$, $f''(0) = 2$, $f'''(0) = -6$, $f^{(n)}(0) = (-1)^n n!$ if $n \geq 2 \Rightarrow x^2 - x^3 + x^4 - x^5 + \ldots = \sum_{n=2}^{\infty} (-1)^n x^n$

27. $f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3}, f''(x) = 3!\,x^{-4}, f'''(x) = -4!\,x^{-5} \Rightarrow f^{(n)}(x) = (-1)^n(n+1)!\,x^{-n-2}$;
$f(1) = 1, f'(1) = -2, f''(1) = 3!, f'''(1) = -4!, f^{(n)}(1) = (-1)^n(n+1)! \Rightarrow \frac{1}{x^2}$
$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots = \sum_{n=0}^{\infty} (-1)^n(n+1)(x-1)^n$

28. $f(x) = \frac{1}{(1-x)^3} \Rightarrow f'(x) = 3(1-x)^{-4}, f''(x) = 12(1-x)^{-5}, f'''(x) = 60\,(1-x)^{-6} \Rightarrow f^{(n)}(x) = \frac{(n+2)!}{2}(1-x)^{-n-3}$;
$f(0) = 1, f'(0) = 3, f''(0) = 12, f'''(0) = 60, \dots, f^{(n)}(0) = \frac{(n+2)!}{2} \Rightarrow \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \dots$
$= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$

29. $f(x) = e^x \Rightarrow f'(x) = e^x, f''(x) = e^x \Rightarrow f^{(n)}(x) = e^x; f(2) = e^2, f'(2) = e^2, \dots f^{(n)}(2) = e^2$
$\Rightarrow e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^3}{3!}(x-2)^3 + \dots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

30. $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2, f''(x) = 2^x(\ln 2)^2, f'''(x) = 2^x(\ln 2)^3 \Rightarrow f^{(n)}(x) = 2^x(\ln 2)^n; f(1) = 2, f'(1) = 2\ln 2,$
$f''(1) = 2(\ln 2)^2, f'''(1) = 2(\ln 2)^3, \dots, f^{(n)}(1) = 2(\ln 2)^n$
$\Rightarrow 2^x = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \dots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n(x-1)^n}{n!}$

31. $f(x) = \cos\left(2x + \frac{\pi}{2}\right), f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right), f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right), f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right),$
$f^{(4)}(x) = 2^4\cos\left(2x + \frac{\pi}{2}\right),\ f^{(5)}(x) = -2^5\sin\left(2x + \frac{\pi}{2}\right), \dots; f\left(\frac{\pi}{4}\right) = -1, f'\left(\frac{\pi}{4}\right) = 0,\ f''\left(\frac{\pi}{4}\right) = 4, f'''\left(\frac{\pi}{4}\right) = 0, f^{(4)}\left(\frac{\pi}{4}\right) = 2^4,$
$f^{(5)}\left(\frac{\pi}{4}\right) = 0, \dots, f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n} \Rightarrow \cos\left(2x + \frac{\pi}{2}\right) = -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \dots$
$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n}$

32. $f(x) = \sqrt{x+1}, f'(x) = \frac{1}{2}(x+1)^{-1/2}, f''(x) = -\frac{1}{4}(x+1)^{-3/2}, f'''(x) = \frac{3}{8}(x+1)^{-5/2}, f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2}, \dots;$
$f(0) = 1, f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}, f'''(0) = \frac{3}{8}, f^{(4)}(0) = -\frac{15}{16}, \dots \Rightarrow \sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$

34. The Maclaurin series generated by e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges on $(-\infty, \infty)$. The Maclaurin series generated by $f(x) = (1 - x + x^2)e^x$ is given by $(1 - x + x^2)\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{1}{2}x^2 + \frac{2}{3}x^3 \dots$. which converges on $(-\infty, \infty)$.

35. The Maclaurin series generated by $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ which converges on $(-\infty, \infty)$ and the Maclaurin series generated by $\ln(1+x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n$ which converges on $(-1, 1)$. Thus the Maclaurin series genereated by $f(x) = \sin x \cdot \ln(1+x)$ is given by $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n\right) = x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 - \dots$. which converges on the intersection of $(-\infty, \infty)$ and $(-1, 1)$, so the interval of convergence is $(-1, 1)$.

36. The Maclaurin series generated by $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ which converges on $(-\infty, \infty)$. The Maclaurin series genereated by $f(x) = x\sin^2 x$ is given by $x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)^2 = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)$
$= x^3 - \frac{1}{3}x^5 + \frac{2}{45}x^7 + \dots$ which converges on $(-\infty, \infty)$.

41. $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$; $f(0) = 0, f'(0) = 0, f''(0) = -1 \Rightarrow L(x) = 0$ and $Q(x) = -\frac{x^2}{2}$

42. $f(x) = e^{\sin x} \Rightarrow f'(x) = (\cos x)e^{\sin x}$ and $f''(x) = (-\sin x)e^{\sin x} + (\cos x)^2 e^{\sin x}$; $f(0) = 1, f'(0) = 1, f''(0) = 1$
$\Rightarrow L(x) = 1 + x$ and $Q(x) = 1 + x + \frac{x^2}{2}$

45. $f(x)=\sin x \Rightarrow f'(x)=\cos x$ and $f''(x)=-\sin x$; $f(0)=0, f'(0)=1, f''(0)=0 \Rightarrow L(x)=x$ and $Q(x)=x$

46. $f(x)=\tan x \Rightarrow f'(x)=\sec^2 x$ and $f''(x)=2\sec^2 x \tan x$; $f(0)=0, f'(0)=1, f''=0 \Rightarrow L(x)=x$ and $Q(x)=x$

Section 10.9:

1. $e^x = 1+x+\frac{x^2}{2!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow e^{-5x}=1+(-5x)+\frac{(-5x)^2}{2!}+\ldots=1-5x+\frac{5^2x^2}{2!}-\frac{5^3x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n5^nx^n}{n!}$

4. $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!} \Rightarrow \sin\frac{\pi x}{2}=\frac{\pi x}{2}-\frac{\left(\frac{\pi x}{2}\right)^3}{3!}+\frac{\left(\frac{\pi x}{2}\right)^5}{5!}-\frac{\left(\frac{\pi x}{2}\right)^7}{7!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}x^{2n+1}}{2^{2n+1}(2n+1)!}$

7. $\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \Rightarrow \ln(1+x^2)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(x^2\right)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}=x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\ldots$

11. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow xe^x=x\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}=x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\frac{x^5}{4!}+\ldots$

12. $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!} \Rightarrow x^2\sin x=x^2\left(\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}\right)=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+3}}{(2n+1)!}=x^3-\frac{x^5}{3!}+\frac{x^7}{5!}-\frac{x^9}{7!}+\ldots$

13. $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!} \Rightarrow \frac{x^2}{2}-1+\cos x=\frac{x^2}{2}-1+\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!}=\frac{x^2}{2}-1+1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\ldots$
$=\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\ldots=\sum_{n=2}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!}$

14. $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!} \Rightarrow \sin x-x+\frac{x^3}{3!}=\left(\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}\right)-x+\frac{x^3}{3!}$
$=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\ldots\right)-x+\frac{x^3}{3!}=\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\ldots=\sum_{n=2}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}$

15. $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!} \Rightarrow x\cos\pi x=x\sum_{n=0}^{\infty}\frac{(-1)^n(\pi x)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}x^{2n+1}}{(2n)!}=x-\frac{\pi^2x^3}{2!}+\frac{\pi^4x^5}{4!}-\frac{\pi^6x^7}{6!}+\ldots$

16. $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!} \Rightarrow x^2\cos\left(x^2\right)=x^2\sum_{n=0}^{\infty}\frac{(-1)^n\left(x^2\right)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^nx^{4n+2}}{(2n)!}=x^2-\frac{x^6}{2!}+\frac{x^{10}}{4!}-\frac{x^{14}}{6!}+\ldots$

17. $\cos^2 x=\frac{1}{2}+\frac{\cos 2x}{2}=\frac{1}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}=\frac{1}{2}+\frac{1}{2}\left[1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\frac{(2x)^8}{8!}-\ldots\right]$
$=1-\frac{(2x)^2}{2\cdot 2!}+\frac{(2x)^4}{2\cdot 4!}-\frac{(2x)^6}{2\cdot 6!}+\frac{(2x)^8}{2\cdot 8!}-\ldots=1+\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{2\cdot(2n)!}=1+\sum_{n=1}^{\infty}\frac{(-1)^n\,2^{2n-1}\,x^{2n}}{(2n)!}$

18. $\sin^2 x=\left(\frac{1-\cos 2x}{2}\right)=\frac{1}{2}-\frac{1}{2}\cos 2x=\frac{1}{2}-\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\ldots\right)=\frac{(2x)^2}{2\cdot 2!}-\frac{(2x)^4}{2\cdot 4!}+\frac{(2x)^6}{2\cdot 6!}-\ldots$
$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2x)^{2n}}{2\cdot(2n)!}=\sum_{n=1}^{\infty}\frac{(-1)^n\,2^{2n-1}\,x^{2n}}{(2n)!}$

19. $\frac{x^2}{1-2x} = x^2\left(\frac{1}{1-2x}\right) = x^2 \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^{n+2} = x^2 + 2x^3 + 2^2x^4 + 2^3x^5 + \dots$

20. $x\ln(1+2x) = x\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n x^{n+1}}{n} = 2x^2 - \frac{2^2x^3}{2} + \frac{2^3x^4}{4} - \frac{2^4x^5}{5} + \dots$

21. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$

22. $\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}(1 + 2x + 3x^2 + \dots) = 2 + 6x + 12x^2 + \dots = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$
$= \sum_{n=0}^{\infty} (n+2)(n+1)x^n$

23. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \Rightarrow x\tan^{-1}x^2 = x\left(x^2 - \frac{1}{3}(x^2)^3 + \frac{1}{5}(x^2)^5 - \frac{1}{7}(x^2)^7 + \dots\right)$
$= x^3 - \frac{1}{3}x^7 + \frac{1}{5}x^{11} - \frac{1}{7}x^{15} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-1}}{2n-1}$

24. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \Rightarrow \sin x \cdot \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots\right)$
$= x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \frac{64x^7}{7!} + \dots = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!}$

25. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ and $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \Rightarrow e^x + \frac{1}{1+x}$
$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + (1 - x + x^2 - x^3 + \dots) = 2 + \frac{3}{2}x^2 - \frac{5}{6}x^3 + \frac{25}{24}x^4 + \dots = \sum_{n=0}^{\infty}\left(\frac{1}{n!} + (-1)^n\right)x^n$

26. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \Rightarrow \cos x - \sin x$
$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \dots$
$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!} - \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$

27. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \Rightarrow \frac{x}{3}\ln(1+x^2) = \frac{x}{3}\left(x^2 - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 - \frac{1}{4}(x^2)^4 + \dots\right)$
$= \frac{1}{3}x^3 - \frac{1}{6}x^5 + \frac{1}{9}x^7 - \frac{1}{12}x^9 + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n} x^{2n+1}$

28. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ and $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \Rightarrow \ln(1+x) - \ln(1-x)$
$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots = \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$

35. Since $n = 3$, then $f^{(4)}(x) = \sin x$, $|f^{(4)}(x)| \le M$ on $[0, 0.1] \Rightarrow |\sin x| \le 1$ on $[0, 0.1] \Rightarrow M = 1$. Then $|R_3(0.1)| \le 1\frac{|0.1-0|^4}{4!}$
$= 4.2 \times 10^{-6} \Rightarrow$ error $\le 4.2 \times 10^{-6}$

37. By the Alternating Series Estimation Theorem, the error is less than $\frac{|x|^5}{5!} \Rightarrow |x|^5 < (5!)(5 \times 10^{-4}) \Rightarrow |x|^5 < 600 \times 10^{-4}$
$\Rightarrow |x| < \sqrt[5]{6 \times 10^{-2}} \approx 0.56968$

39. If $\sin x = x$ and $|x| < 10^{-3}$, then the error is less than $\frac{(10^{-3})^3}{3!} \approx 1.67 \times 10^{-10}$, by Alternating Series Estimation Theorem; The Alternating Series Estimation Theorem says $R_2(x)$ has the same sign as $-\frac{x^3}{3!}$. Moreover, $x < \sin x$
$\Rightarrow 0 < \sin x - x = R_2(x) \Rightarrow x < 0 \Rightarrow -10^{-3} < x < 0$.

41. $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{3^{(0.1)}(0.1)^3}{3!} < 1.87 \times 10^{-4}$, where c is between 0 and x

42. $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{(0.1)^3}{3!} = 1.67 \times 10^{-4}$, where c is between 0 and x

48. $f(x) = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f^{(3)}(x) = 6(1-x)^{-4}$
$\Rightarrow f^{(4)}(x) = 24(1-x)^{-5}$; therefore $\frac{1}{1-x} \approx 1 + x + x^2 + x^3$. $|x| < 0.1 \Rightarrow \frac{10}{11} < \frac{1}{1-x} < \frac{10}{9} \Rightarrow \left|\frac{1}{(1-x)^5}\right| < \left(\frac{10}{9}\right)^5$
$\Rightarrow \left|\frac{x^4}{(1-x)^5}\right| < x^4\left(\frac{10}{9}\right)^5 \Rightarrow$ the error $e_3 \le \left|\frac{\max f^{(4)}(x)\, x^4}{4!}\right| < (0.1)^4\left(\frac{10}{9}\right)^5 = 0.00016935 < 0.00017$, since $\left|\frac{f^{(4)}(x)}{4!}\right| = \left|\frac{1}{(1-x)^5}\right|$.

Section 10.10:

11. $(1+x)^4 = 1 + 4x + \frac{(4)(3)x^2}{2!} + \frac{(4)(3)(2)x^3}{3!} + \frac{(4)(3)(2)x^4}{4!} = 1 + 4x + 6x^2 + 4x^3 + x^4$

12. $(1+x^2)^3 = 1 + 3x^2 + \frac{(3)(2)(x^2)^2}{2!} + \frac{(3)(2)(1)(x^2)^3}{3!} = 1 + 3x^2 + 3x^4 + x^6$

13. $(1-2x)^3 = 1 + 3(-2x) + \frac{(3)(2)(-2x)^2}{2!} + \frac{(3)(2)(1)(-2x)^3}{3!} = 1 - 6x + 12x^2 - 8x^3$

14. $\left(1-\frac{x}{2}\right)^4 = 1 + 4\left(-\frac{x}{2}\right) + \frac{(4)(3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(4)(3)(2)\left(-\frac{x}{2}\right)^3}{3!} + \frac{(4)(3)(2)(1)\left(-\frac{x}{2}\right)^4}{4!} = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

15. $\int_0^{0.2} \sin x^2\, dx = \int_0^{0.2}\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \ldots\right]_0^{0.2} \approx \left[\frac{x^3}{3}\right]_0^{0.2} \approx 0.00267$ with error
$|E| \le \frac{(.2)^7}{7\cdot 3!} \approx 0.0000003$

16. $\int_0^{0.2} \frac{e^{-x}-1}{x}\, dx = \int_0^{0.2} \frac{1}{x}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots - 1\right) dx = \int_0^{0.2}\left(-1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{24} - \ldots\right) dx$
$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \ldots\right]_0^{0.2} \approx -0.19044$ with error $|E| \le \frac{(0.2)^4}{96} \approx 0.00002$

17. $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\, dx = \int_0^{0.1}\left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \ldots\right) dx = \left[x - \frac{x^5}{10} + \ldots\right]_0^{0.1} \approx [x]_0^{0.1} \approx 0.1$ with error
$|E| \le \frac{(0.1)^5}{10} = 0.000001$

18. $\int_0^{0.25} \sqrt[3]{1+x^2}\, dx = \int_0^{0.25}\left(1 + \frac{x^2}{3} - \frac{x^4}{9} + \ldots\right) dx = \left[x + \frac{x^3}{9} - \frac{x^5}{45} + \ldots\right]_0^{0.25} \approx \left[x + \frac{x^3}{9}\right]_0^{0.25} \approx 0.25174$ with error
$|E| \le \frac{(0.25)^5}{45} \approx 0.0000217$

19. $\int_0^{0.1} \frac{\sin x}{x}\, dx = \int_0^{0.1}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right) dx = \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \ldots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!}\right]_0^{0.1}$
≈ 0.0999444611, $|E| \le \frac{(0.1)^7}{7\cdot 7!} \approx 2.8 \times 10^{-12}$

20. $\int_0^{0.1} \exp(-x^2)\, dx = \int_0^{0.1}\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \ldots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.1}$
≈ 0.0996676643, $|E| \le \frac{(0.1)^9}{216} \approx 4.6 \times 10^{-12}$

21. $(1+x^4)^{1/2} = (1)^{1/2} + \frac{\left(\frac{1}{2}\right)}{1}(1)^{-1/2}(x^4) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(1)^{-3/2}(x^4)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(1)^{-5/2}(x^4)^3$
$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(1)^{-7/2}(x^4)^4 + \ldots = 1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \ldots$
$\Rightarrow \int_0^{0.1}\left(1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \ldots\right) dx \approx \left[x + \frac{x^5}{10}\right]_0^{0.1} \approx 0.100001$, $|E| \le \frac{(0.1)^9}{72} \approx 1.39 \times 10^{-11}$

22. $\int_0^1 \left(\frac{1-\cos x}{x^2}\right) dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \frac{x^8}{10!} - \ldots\right) dx \approx \left[\frac{x}{2} - \frac{x^3}{3\cdot 4!} + \frac{x^5}{5\cdot 6!} - \frac{x^7}{7\cdot 8!} + \frac{x^9}{9\cdot 10!}\right]_0^1$
≈ 0.4863853764, $|E| \le \frac{1}{11\cdot 12!} \approx 1.9 \times 10^{-10}$

25. $F(x)=\int_0^x \left(t^2-\frac{t^6}{3!}+\frac{t^{10}}{5!}-\frac{t^{14}}{7!}+\ldots\right)dt=\left[\frac{t^3}{3}-\frac{t^7}{7\cdot 3!}+\frac{t^{11}}{11\cdot 5!}-\frac{t^{15}}{15\cdot 7!}+\ldots\right]_0^x \approx \frac{x^3}{3}-\frac{x^7}{7\cdot 3!}+\frac{x^{11}}{11\cdot 5!}$
$\Rightarrow\ |\text{error}|<\frac{1}{15\cdot 7!}\approx 0.000013$

26. $F(x)=\int_0^x \left(t^2-t^4+\frac{t^6}{2!}-\frac{t^8}{3!}+\frac{t^{10}}{4!}-\frac{t^{12}}{5!}+\ldots\right)dt=\left[\frac{t^3}{3}-\frac{t^5}{5}+\frac{t^7}{7\cdot 2!}-\frac{t^9}{9\cdot 3!}+\frac{t^{11}}{11\cdot 4!}-\frac{t^{13}}{13\cdot 5!}+\ldots\right]_0^x$
$\approx \frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7\cdot 2!}-\frac{x^9}{9\cdot 3!}+\frac{x^{11}}{11\cdot 4!}\ \Rightarrow\ |\text{error}|<\frac{1}{13\cdot 5!}\approx 0.00064$

27. (a) $F(x)=\int_0^x \left(t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\ldots\right)dt=\left[\frac{t^2}{2}-\frac{t^4}{12}+\frac{t^6}{30}-\ldots\right]_0^x \approx \frac{x^2}{2}-\frac{x^4}{12}\ \Rightarrow\ |\text{error}|<\frac{(0.5)^6}{30}\approx .00052$
(b) $|\text{error}|<\frac{1}{33\cdot 34}\approx .00089$ when $F(x)\approx \frac{x^2}{2}-\frac{x^4}{3\cdot 4}+\frac{x^6}{5\cdot 6}-\frac{x^8}{7\cdot 8}+\ldots+(-1)^{15}\frac{x^{32}}{31\cdot 32}$

29. $\frac{1}{x^2}\left(e^x-(1+x)\right)=\frac{1}{x^2}\left(\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots\right)-1-x\right)=\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\ldots\ \Rightarrow\ \lim_{x\to 0}\frac{e^x-(1+x)}{x^2}$
$=\lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\ldots\right)=\frac{1}{2}$

30. $\frac{1}{x}\left(e^x-e^{-x}\right)=\frac{1}{x}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\ldots\right)\right]=\frac{1}{x}\left(2x+\frac{2x^3}{3!}+\frac{2x^5}{5!}+\frac{2x^7}{7!}+\ldots\right)$
$=2+\frac{2x^2}{3!}+\frac{2x^4}{5!}+\frac{2x^6}{7!}+\ldots\ \Rightarrow\ \lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to\infty}\left(2+\frac{2x^2}{3!}+\frac{2x^4}{5!}+\frac{2x^6}{7!}+\ldots\right)=2$

31. $\frac{1}{t^4}\left(1-\cos t-\frac{t^2}{2}\right)=\frac{1}{t^4}\left[1-\frac{t^2}{2}-\left(1-\frac{t^2}{2}+\frac{t^4}{4!}-\frac{t^6}{6!}+\ldots\right)\right]=-\frac{1}{4!}+\frac{t^2}{6!}-\frac{t^4}{8!}+\ldots\ \Rightarrow\ \lim_{t\to 0}\frac{1-\cos t-\left(\frac{t^2}{2}\right)}{t^4}$
$=\lim_{t\to 0}\left(-\frac{1}{4!}+\frac{t^2}{6!}-\frac{t^4}{8!}+\ldots\right)=-\frac{1}{24}$

32. $\frac{1}{\theta^5}\left(-\theta+\frac{\theta^3}{6}+\sin\theta\right)=\frac{1}{\theta^5}\left(-\theta+\frac{\theta^3}{6}+\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right)=\frac{1}{5!}-\frac{\theta^2}{7!}+\frac{\theta^4}{9!}-\ldots\ \Rightarrow\ \lim_{\theta\to 0}\frac{\sin\theta-\theta+\left(\frac{\theta^3}{6}\right)}{\theta^5}$
$=\lim_{\theta\to 0}\left(\frac{1}{5!}-\frac{\theta^2}{7!}+\frac{\theta^4}{9!}-\ldots\right)=\frac{1}{120}$

33. $\frac{1}{y^3}\left(y-\tan^{-1}y\right)=\frac{1}{y^3}\left[y-\left(y-\frac{y^3}{3}+\frac{y^5}{5}-\ldots\right)\right]=\frac{1}{3}-\frac{y^2}{5}+\frac{y^4}{7}-\ldots\ \Rightarrow\ \lim_{y\to 0}\frac{y-\tan^{-1}y}{y^3}=\lim_{y\to 0}\left(\frac{1}{3}-\frac{y^2}{5}+\frac{y^4}{7}-\ldots\right)$
$=\frac{1}{3}$

34. $\frac{\tan^{-1}y-\sin y}{y^3\cos y}=\frac{\left(y-\frac{y^3}{3}+\frac{y^5}{5}-\ldots\right)-\left(y-\frac{y^3}{3!}+\frac{y^5}{5!}-\ldots\right)}{y^3\cos y}=\frac{\left(-\frac{y^3}{6}+\frac{23y^5}{5!}-\ldots\right)}{y^3\cos y}=\frac{\left(-\frac{1}{6}+\frac{23y^2}{5!}-\ldots\right)}{\cos y}$
$\Rightarrow\ \lim_{y\to 0}\frac{\tan^{-1}y-\sin y}{y^3\cos y}=\lim_{y\to 0}\frac{\left(-\frac{1}{6}+\frac{23y^2}{5!}-\ldots\right)}{\cos y}=-\frac{1}{6}$

35. $x^2\left(-1+e^{-1/x^2}\right)=x^2\left(-1+1-\frac{1}{x^2}+\frac{1}{2x^4}-\frac{1}{6x^6}+\ldots\right)=-1+\frac{1}{2x^2}-\frac{1}{6x^4}+\ldots\ \Rightarrow\ \lim_{x\to\infty}x^2\left(e^{-1/x^2}-1\right)$
$=\lim_{x\to\infty}\left(-1+\frac{1}{2x^2}-\frac{1}{6x^4}+\ldots\right)=-1$

36. $(x+1)\sin\left(\frac{1}{x+1}\right) = (x+1)\left(\frac{1}{x+1} - \frac{1}{3!(x+1)^3} + \frac{1}{5!(x+1)^5} - \ldots\right) = 1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots$

$\Rightarrow \lim\limits_{x\to\infty} (x+1)\sin\left(\frac{1}{x+1}\right) = \lim\limits_{x\to\infty} \left(1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots\right) = 1$

37. $\frac{\ln(1+x^2)}{1-\cos x} = \frac{\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \ldots\right)}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)} = \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} \Rightarrow \lim\limits_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim\limits_{x\to 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} = 2! = 2$

38. $\frac{x^2-4}{\ln(x-1)} = \frac{(x-2)(x+2)}{\left[(x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \ldots\right]} = \frac{x+2}{\left[1 - \frac{x-2}{2} + \frac{(x-2)^2}{3} - \ldots\right]} \Rightarrow \lim\limits_{x\to 2} \frac{x^2-4}{\ln(x-1)}$

$= \lim\limits_{x\to 2} \frac{x+2}{\left[1 - \frac{x-2}{2} + \frac{(x-2)^2}{3} - \ldots\right]} = 4$

39. $\sin 3x^2 = 3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} - \ldots$ and $1 - \cos 2x = 2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \ldots \Rightarrow \lim\limits_{x\to 0} \frac{\sin 3x^2}{1-\cos 2x}$

$= \lim\limits_{x\to 0} \frac{3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} - \ldots}{2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \ldots} = \lim\limits_{x\to 0} \frac{3 - \frac{9}{2}x^4 + \frac{81}{40}x^8 - \ldots}{2 - \frac{2}{3}x^2 + \frac{4}{45}x^4 - \ldots} = \frac{3}{2}$

40. $\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots$ and $x\sin x^2 = x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \frac{1}{5040}x^{15} + \ldots \Rightarrow \lim\limits_{x\to 0} \frac{\ln(1+x^3)}{x\sin x^2}$

$= \lim\limits_{x\to 0} \frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots}{x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \frac{1}{5040}x^{15} + \ldots} = \lim\limits_{x\to 0} \frac{1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \ldots}{1 - \frac{1}{6}x^4 + \frac{1}{120}x^8 - \frac{1}{5040}x^{12} + \ldots} = 1$

53. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

59. (a) $(1-x^2)^{-1/2} \approx 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \Rightarrow \sin^{-1} x \approx x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$; Using the Ratio Test:

$\lim\limits_{n\to\infty} \left|\frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)x^{2n+3}}{2\cdot4\cdot6\cdots(2n)(2n+2)(2n+3)} \cdot \frac{2\cdot4\cdot6\cdots(2n)(2n+1)}{1\cdot3\cdot5\cdots(2n-1)x^{2n+1}}\right| < 1 \Rightarrow x^2 \lim\limits_{n\to\infty} \left|\frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}\right| < 1$

$\Rightarrow |x| < 1 \Rightarrow$ the radius of convergence is 1. See Exercise 69.

(b) $\frac{d}{dx}(\cos^{-1} x) = -(1-x^2)^{-1/2} \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \approx \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}\right) \approx \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$

66. $[\tan^{-1} t]_x^{\infty} = \frac{\pi}{2} - \tan^{-1} x = \int_x^{\infty} \frac{dt}{1+t^2} = \int_x^{\infty} \left[\frac{\left(\frac{1}{t^2}\right)}{1+\left(\frac{1}{t^2}\right)}\right] dt = \int_x^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \ldots\right) dt$

$= \int_x^{\infty} \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \ldots\right) dt = \lim\limits_{b\to\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_x^b = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \ldots$

$\Rightarrow \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,\ x > 1$; $[\tan^{-1} t]_{-\infty}^{x} = \tan^{-1} x + \frac{\pi}{2} = \int_{-\infty}^{x} \frac{dt}{1+t^2}$

$= \lim\limits_{b\to-\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_b^x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots \Rightarrow \tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,$

$x < -1$

67. (a) $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1 + i(0) = -1$

(b) $e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)(1+i)$

(c) $e^{-i\pi/2} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i(-1) = -i$

68. $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$;

$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$;

$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta \Rightarrow \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

72. $\frac{d}{dx}\left(e^{(a+ib)}\right) = \frac{d}{dx}[e^{ax}(\cos bx + i\sin bx)] = ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + bi\cos bx)$

$= ae^{ax}(\cos bx + i\sin bx) + bie^{ax}(\cos bx + i\sin bx) = ae^{(a+ib)x} + ibe^{(a+ib)x} = (a+ib)e^{(a+ib)x}$

73. (a) $e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$
$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}$

(b) $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = (\cos\theta - i\sin\theta)\left(\frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta}\right) = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{e^{i\theta}}$