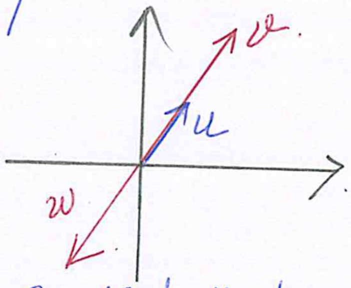
section 2.3

Linear independence

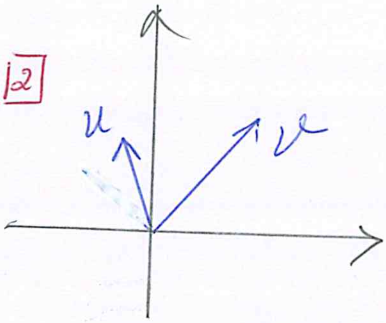
Motivation

1



notice that $v = ku$ $k>0$
& $w = ku$ $k<0$
$\Rightarrow$ v is a linear Combination of u as well as w.
$\Rightarrow$ u & v are linearly dependent
& u & w are liniarly dependent

2



u & v are linearly independent
$v \neq ku$.

definition 2.3.1 (Linear dependence & independence)

let $S = \{v_1, v_2, \dots, v_m\}$ be a set of vectors in $\mathbb{R}^n$.

1 S is said to be linearly independent iff:
for any linear Combination of vectors in S which is equal to zero. $k_1 v_1 + \dots + k_m v_m = 0$ then $k_1 = k_2 = \dots = k_m = 0$ (ie $k_i = 0$ $\forall i = 1, m$)

(ie) S linearly independent $\Longleftrightarrow$ the equation $k_1 v_1 + \dots + k_m v_m = 0$ has only the trivial solution

2 S is said to be linearly dependent iff:
there exists a linear Combination of the vectors in S which is equal to zero $k_1 v_1 + k_2 v_2 + \dots + k_m v_m = 0$ where at least one of the scalars is not equal to zero (ie $\exists$ at least one k_i / $k_i \neq 0$)

(ie) S linearly dependent $\Longleftrightarrow$ the equation $k_1 v_1 + \dots + k_m v_m = 0$ has a nontrival solution (ie has ∞ many sol)

examples

$\boxed{1}$ $S = \{e_1, e_2, \ldots, e_n\}$ is a linearly indep set of vectors in $\mathbb{R}^n$ $\left(e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \to i\text{th component}\right)$

$\boxed{2}$ determine whether S is linearly dependent or independent in $\underbrace{\mathbb{R}^4}_{(a)}$ & $\underbrace{\mathbb{R}^3}_{(b)}$

$\boxed{a}$ $S = \{v_1, v_2, v_3\}$ where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ & $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$

$\boxed{b}$ $S = \{v_1, v_2, v_3, v_4\}$ where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ & $v_4 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Answer

$\boxed{a}$ let $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ ($0_{\mathbb{R}^4}$ since we are in $\mathbb{R}^4$)

$$\Rightarrow k_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} + k_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} k_1 + k_3 \\ k_2 + k_3 \\ k_1 + k_2 + k_3 \\ 2k_1 + 2k_2 + 3k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} k_1 + k_3 = 0 \\ k_2 + k_3 = 0 \\ k_1 + k_2 + k_3 = 0 \\ 2k_1 + 2k_2 + 3k_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} k_1 + k_3 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{cases} \Rightarrow k_1 = k_2 = k_3 = 0 \text{ is the only solute}$$

$\Rightarrow$ S is a linearly independent set of vectors in $\mathbb{R}^4$

b let $k_1 v_1 + k_2 v_2 + k_3 v_3 + k_4 v_4 = 0$ (ie $0_{\mathbb{R}^3}$ since we are in $\mathbb{R}^3$)

$$\Rightarrow k_1\begin{pmatrix}1\\0\\2\end{pmatrix} + k_2\begin{pmatrix}-1\\1\\2\end{pmatrix} + k_3\begin{pmatrix}-2\\3\\1\end{pmatrix} + k_4\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}k_1 - k_2 - 2k_3 + 2k_4\\ k_2 + 3k_3 + k_4\\ 2k_1 + 2k_2 + k_3 + k_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases} k_1 - k_2 - 2k_3 + 2k_4 = 0\\ k_2 + 3k_3 + k_4 = 0\\ 2k_1 - 2k_2 + k_3 + k_4 = 0\end{cases}$$

(don't need to solve we can deduce directly that it has ∞ many solutions)

we obtain a homogenous system (always consistent) with # eq < # unknowns

$\Rightarrow$ the system has ∞ many solutions

$\Rightarrow k_1 = k_2 = k_3 = k_4 = 0$ is not the only solution of the system

$\Rightarrow$ S is linearly dependent in $\mathbb{R}^3$.

Theorem 2.3.1

let $S = \{v_1, \dots, v_r\}$ be a set of r vectors in $\mathbb{R}^n$. If $\underline{\underline{r > n}}$ then the set S is $\underline{\underline{\text{linearly dependent}}}$

proof

let $k_1 v_1 + k_2 v_2 + \dots + k_r v_r = 0$ where $v_1, \dots, v_r$ vectors in $\mathbb{R}^n$

ie $v_1 = \begin{pmatrix}v_{11}\\v_{12}\\ \vdots\\ v_{1n}\end{pmatrix} \cdots v_r = \begin{pmatrix}v_{r1}\\v_{r2}\\ \vdots \\ v_{rn}\end{pmatrix}$

$$\Rightarrow k_1 v_1 + k_2 v_2 + \dots + k_r v_r = 0$$

$$\Rightarrow \begin{cases} k_1 v_{11} + k_2 v_{21} + \dots + k_r v_{r1} = 0\\ k_1 v_{12} + k_2 v_{22} + \dots + k_r v_{r2} = 0\\ \vdots\\ k_1 v_{1n} + k_2 v_{2n} + \dots + k_r v_{rn} = 0\end{cases} \Leftrightarrow \left(\begin{array}{cccc|c} v_{11} & v_{21} & \dots & v_{r1} & 0\\ v_{12} & v_{22} & \dots & v_{r2} & 0\\ \vdots & & & & \vdots\\ v_{1n} & v_{2n} & \dots & v_{rn} & 0\end{array}\right)$$

we obtain a homogenous system with n equations & r unknowns where $\underline{\underline{n < r}}$ (ie # eq < # unknowns)

$\Rightarrow$ the system has ∞ many solutions

$\Rightarrow$ S is linearly dependent.

Theorem 2.3.2

Any set S of vectors in $\mathbb{R}^n$ containing the zero vector is linearly dependent (This is also true for general vector spaces V ch3)

proof

let $S = \{0, v_1, v_2, \dots, v_k\}$ be a set of vectors in $\mathbb{R}^n$ containing the zero vector

notice that we can find (there exists) a lin combination of vectors in S which is equal to zero where at least one of the scalars is $\neq 0$

$$\underset{\neq 0}{1} \cdot 0_{\mathbb{R}^n} + 0v_1 + 0v_2 + \dots + 0v_k = 0 \quad (\text{ie } 0_{\mathbb{R}^n})$$

Theorem 2.3.3

A set of nonzero vectors in $\mathbb{R}^n$ is linearly dependent iff at least one of the vectors can be written as a linear combination of the other vectors in the set.

proof

let $S = \{v_1, \dots, v_r\}$ be a linearly dependent set

$\Rightarrow \exists\ k_1v_1 + \dots + k_rv_r = 0$ where at least one of the scalars $\neq 0$

suppose $k_i \neq 0$ (since $k_i \neq 0$ then $\frac{1}{k_i}$ exists)

multiply both sides by $\frac{1}{k_i}$

$$\Rightarrow \frac{1}{k_i}\left(k_1v_1 + \dots + k_{i-1}v_{i-1} + k_iv_i + k_{i+1}v_{i+1} + \dots + k_rv_r\right) = \frac{1}{k_i}(0)$$

$$\Rightarrow \frac{k_1}{k_i}v_1 + \dots + \frac{k_{i-1}}{k_i}v_{i-1} + \underbrace{\frac{k_i}{k_i}}_{1}v_i + \frac{k_{i+1}}{k_i}v_{i+1} + \dots + \frac{k_r}{k_i}v_r = 0$$

$$\Rightarrow \frac{k_1}{k_i}v_1 + \dots + \frac{k_{i-1}}{k_i}v_{i-1} + v_i + \frac{k_{i+1}}{k_i}v_{i+1} + \dots + \frac{k_r}{k_i}v_r = 0$$

(we can find v_i by adding to both sides the additive inverses of all the other $\frac{k_j}{k_i}v_j$ $\forall\, j = 1, \dots, i-1, i+1, \dots, r$ since all vectors in $\mathbb{R}^n$ have additive inverses)

$$\Rightarrow v_i = -\frac{k_1}{k_i}v_1 - \dots - \frac{k_{i-1}}{k_i}v_{i-1} - \frac{k_{i+1}}{k_i}v_{i+1} - \dots - \frac{k_r}{k_i}v_r$$

$\Rightarrow v_i$ is a lin combination of the other vectors in S.

example

$v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ & $v_3 = \begin{pmatrix}2\\-1\\3\end{pmatrix}$ are linearly dependent since $v_3 = 3v_1 - v_2$

N.B

If a set of vectors is linearly dependent it's not necessarily that every vector in the set can be written as linear Combinatio of the others but there exists at least one vector that can be written as linear Combination of the others

example

$S = \{ \underbrace{\begin{pmatrix}-1\\0\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{v_3} \}$ notice that S is linearly dependent since S is a set of 3 vectors in $\mathbb{R}^2$ $(3>2)$

look at the vector v_2, v_2 is not a linear Combination of the others

(S dependent $\Rightarrow$ there exists at least one vector in S lin comb of the others)

$v_1 = 0v_2 - v_3$ & $v_3 = -v_1 + 0v_2$.

Theorem 2.3.4

$\boxed{1}$ any subset of a linearly independent set of vectors is linearly indep

$\boxed{2}$ any set Containing a linearly dependent set is linearly dependent

proof

let $T = \{v_1, \ldots, v_r\}$ be a subset of a linearly indep set $S = \{v_1, \ldots, v_r, \ldots, v_n\}$.

show T independent

let $k_1 v_1 + \cdots + k_r v_r = 0$ show that $k_1 = k_2 = \cdots = k_r = 0$??

but $k_1 v_1 + \cdots + k_r v_r = 0 \Longrightarrow k_1 v_1 + \cdots + k_r v_r + 0v_{r+1} + \cdots + 0v_n = 0$ (*)

but since $S = \{v_1, \ldots, v_r, \ldots, v_n\}$ is linearly indep

$\Rightarrow$ the equation (*) has only the trival solut $k_1 = k_2 = \cdots = k_r = 0$

$\Rightarrow T$ is linearly indep

2. let $S=\{v_1 \to v_r, v_{r+1} \to v_n\}$ & $T=\{v_1 \to v_r\}$ linearly dependent subset of S

since T dependent $\Rightarrow$ There exists $c_1v_1 + \cdots + c_rv_r = 0$ where at least one $c_i \neq 0$ (*)

show that S is linearly dep (ie must find a linear Combination of vectors in S which is $=0$ where at least one of the scalars $\neq 0$)

(*) $\Rightarrow c_1v_1 + c_2v_2 + \cdots + c_rv_r = 0$ where at least one $c_i \neq 0$

$\Rightarrow c_1v_1 + c_2v_2 + \cdots + c_rv_r + 0v_{r+1} + \cdots + 0v_n = 0$ where at least one $c_i \neq 0$

$\Rightarrow$ There exists a lin Comb of the vectors in S which is equal to zero where at least one of the scalars $\neq 0$

$\Rightarrow S$ is dependent

Theorem 2.3.5

let S be a linearly independent set of vectors. suppose that the vector v is a linear Combination of the vectors in S Then v has a unique representation with respect to the vectors in S. (where $S=\{v_1, v_2, \ldots, v_r\}$)

proof

Suppose v has 2 representations as linear Combination of the vectors in S.

$\Rightarrow v = c_1v_1 + \cdots + c_rv_r$ & $v = k_1v_1 + \cdots + k_rv_r$ (*)

take $v - v$ (ie $v + (-v)$)

$v - v = 0$ & $v - v = (c_1v_1 + \cdots + c_rv_r) - (k_1v_1 + \cdots + k_rv_r)$ by (*)

$\Rightarrow (c_1v_1 - k_1v_1) + \cdots + (c_rv_r - k_rv_r) = 0$

$\Rightarrow (c_1 - k_1)v_1 + \cdots + (c_r - k_r)v_r = 0$

we obtain a linear Combination of the vectors in S which is equal to 0

but since S is linearly independent $\Rightarrow c_i - k_i = 0 \ \forall i = 1 \to r$

$\Rightarrow c_i = k_i = 0 \ \forall i = 1 \to r$

$\Rightarrow c_1 = k_1, c_2 = k_2, \ldots, c_r = k_r$

$\Rightarrow v$ has a unique representation with respect to the vectors in S

Theorem 2.3.6

Let $Ax=b$ be a Consistent $m\times n$ linear system

$Ax=b$ has a unique solution $\overset{\text{iff}}{\Longleftrightarrow}$ the column vectors of A are linearly independent

proof

$\Leftarrow$ done by previous Theorem

since $Ax=B$ is Consistent $\Leftrightarrow$ b can be written as linear Combination of the column vectors of A.

$\Rightarrow b=x_1A_1+\dots+x_nA_n$ for $x_1,x_2,\dots,x_n\in\mathbb{R}$.

but since $A_1,\dots,A_n$ the col vectors of A are lin indep

$\Rightarrow b$ has a unique representation with respect to the col vectors $A_1,\dots,A_n$

$\Rightarrow Ax=b$ has a unique solution.

$\Rightarrow$ Suppose $Ax=b$ has a unique solut $x=\begin{pmatrix}x_1\\x_2\\ \vdots\\x_n\end{pmatrix}$ & show that the column vectors of A are linearly indep

take a linear Combinat of the column vectors of A which is equal to zero. $k_1A_1+\dots+k_nA_n=0$ & show that the only solut is the trival sol

i.e must show that the system $A\cdot\underbrace{\begin{pmatrix}k_1\\ \vdots\\k_n\end{pmatrix}}_{k}=0$ has only trival sol.

now given $Ax=b$ Consistent with solution $y=\begin{pmatrix}y_1\\y_2\\ \vdots\\y_n\end{pmatrix}\in\mathbb{R}^n\Rightarrow A\cdot y=b$

but $\underset{\text{can be written in the form}}{b=}Ay+0=A\begin{pmatrix}y_1\\ \vdots\\y_n\end{pmatrix}+A\begin{pmatrix}k_1\\ \vdots\\k_n\end{pmatrix}=A\begin{pmatrix}y_1+k_1\\ \vdots\\y_n+k_n\end{pmatrix}=A(y+k)$

$\Rightarrow y+k=\begin{pmatrix}y_1+k_1\\ \vdots\\y_n+k_n\end{pmatrix}$ is a solution of $Ax=b$ but since $Ax=b$ has unique solution

$\Rightarrow y=y+k\Rightarrow\begin{pmatrix}y_1\\ \vdots\\y_n\end{pmatrix}=\begin{pmatrix}y_1+k_1\\ \vdots\\y_n+k_n\end{pmatrix}\Rightarrow y_i+k_i=y_i\ \forall i=1,\dots,n$

$\Rightarrow k_i=0\ \forall i=1,\dots,n$

i.e $k_1=k_2=\dots=k_n=0$

Therefore $k_1A_1+\dots+k_nA_n=0$ has only trival solut

$\Rightarrow$ the col vectors of A: $\{A_1,\dots,A_n\}$ are linearly indep.

Theorem 2.3.7

let A be a square matrix TFAE:

(a) A is invertible

(b) the system $AX=b$ has a unique solutn for any $n\times 1$ matrix b ($b \in \mathbb{R}^n$)

(c) $AX=0$ has only the trivial solutn

(d) the matrix A is row equivalent to I (rref of A is I)

(e) $\det(A) \neq 0$

(f) the column vectors of A are linearly independent