MATH 201. Homework IV Solution.

Section 10.4:

3. Compare with $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \le 1$. Both series have nonnegative terms for $n \ge 2$. For $n \ge 2$, we have $\sqrt{n} - 1 \le \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}-1} \ge \frac{1}{\sqrt{n}}$. Then by Comparison Test, $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges.

4. Compare with $\sum\limits_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \le 1$. Both series have nonnegative terms for $n \ge 2$. For $n \ge 2$, we have $n^2 - n \le n^2 \Rightarrow \frac{1}{n^2-n} \ge \frac{1}{n^2} \Rightarrow \frac{n}{n^2-n} \ge \frac{n}{n^2} = \frac{1}{n} \Rightarrow \frac{n+2}{n^2-n} \ge \frac{n}{n^2-n} \ge \frac{1}{n}$. Thus $\sum\limits_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges.

5. Compare with $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$. Both series have nonnegative terms for $n \ge 1$. For $n \ge 1$, we have $0 \le \cos^2 n \le 1 \Rightarrow \frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$. Then by Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ converges.

6. Compare with $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$, which is a convergent geometric series, since $|r| = \left|\frac{1}{3}\right| < 1$. Both series have nonnegative terms for $n \ge 1$. For $n \ge 1$, we have $n \cdot 3^n \ge 3^n \Rightarrow \frac{1}{n \cdot 3^n} \le \frac{1}{3^n}$. Then by Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$ converges.

7. Compare with $\sum\limits_{n=1}^{\infty} \frac{\sqrt{5}}{n^{3/2}}$. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, since $p = \frac{3}{2} > 1$, and the series $\sum\limits_{n=1}^{\infty} \frac{\sqrt{5}}{n^{3/2}}$ $= \sqrt{5} \sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by Theorem 8 part 3. Both series have nonnegative terms for $n \ge 1$. For $n \ge 1$, we have
$n^3 \le n^4 \Rightarrow 4n^3 \le 4n^4 \Rightarrow n^4 + 4n^3 \le n^4 + 4n^4 = 5n^4 \Rightarrow n^4 + 4n^3 \le 5n^4 + 20 = 5(n^4 + 4) \Rightarrow \frac{n^4+4n^3}{n^4+4} \le 5$.
$\Rightarrow \frac{n^3(n+4)}{n^4+4} \le 5 \Rightarrow \frac{n+4}{n^4+4} \le \frac{5}{n^3} \Rightarrow \sqrt{\frac{n+4}{n^4+4}} \le \sqrt{\frac{5}{n^3}} = \frac{\sqrt{5}}{n^{3/2}}$ Then by Comparison Test, $\sum\limits_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$ converges.

9. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n}$ $= \lim_{n\to\infty} \frac{\frac{n-2}{n^3-n^2+3}}{1/n^2} = \lim_{n\to\infty} \frac{n^3-2n^2}{n^3-n^2+3} = \lim_{n\to\infty} \frac{3n^2-4n}{3n^2-2n} = \lim_{n\to\infty} \frac{6n-4}{6n-2} = \lim_{n\to\infty} \frac{6}{6} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$ converges.

10. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n}$ $= \lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+2}}}{1/\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}} = \sqrt{\lim_{n\to\infty} \frac{n^2+n}{n^2+2}} = \sqrt{\lim_{n\to\infty} \frac{2n+1}{2n}} = \sqrt{\lim_{n\to\infty} \frac{2}{2}} = \sqrt{1} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$ diverges.

11. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have positive terms for $n \geq 2$. $\lim_{n\to\infty} \frac{a_n}{b_n}$ $= \lim_{n\to\infty} \frac{\frac{n(n+1)}{(n^2+1)(n-1)}}{1/n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n^2+n-1} = \lim_{n\to\infty} \frac{3n^2+2n}{3n^2-2n+1} = \lim_{n\to\infty} \frac{6n+2}{6n-2} = \lim_{n\to\infty} \frac{6}{6} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$ diverges.

12. Compare with $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent geometric series, since $|r| = \left|\frac{1}{2}\right| < 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2^n}{3+4^n}}{1/2^n} = \lim_{n\to\infty} \frac{4^n}{3+4^n} = \lim_{n\to\infty} \frac{4^n \ln 4}{4^n \ln 4} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ converges.

13. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n}$ $= \lim_{n\to\infty} \frac{\frac{5^n}{\sqrt{n}\cdot 4^n}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{5^n}{4^n} = \lim_{n\to\infty} \left(\frac{5}{4}\right)^n = \infty$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\cdot 4^n}$ diverges.

14. Compare with $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$, which is a convergent geometric series, since $|r| = \left|\frac{2}{5}\right| < 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{(2/5)^n} = \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} \ln\left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} n \ln\left(\frac{10n+15}{10n+8}\right)$ $= \exp \lim_{n\to\infty} \frac{\ln\left(\frac{10n+15}{10n+8}\right)}{1/n} = \exp \lim_{n\to\infty} \frac{\frac{10}{10n+15} - \frac{10}{10n+8}}{-1/n^2} = \exp \lim_{n\to\infty} \frac{70n^2}{(10n+15)(10n+8)} = \exp \lim_{n\to\infty} \frac{70n^2}{100n^2+230n+120}$ $= \exp \lim_{n\to\infty} \frac{140n}{200n+230} = \exp \lim_{n\to\infty} \frac{140}{200} = e^{7/10} > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$ converges.

15. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have positive terms for $n \geq 2$. $\lim_{n\to\infty} \frac{a_n}{b_n}$ $= \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{1/n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$. Then by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

16. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n}$ $= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{1/n^2} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\left(-\frac{2}{n^3}\right)}{\left(-\frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$ converges.

18. diverges by the Direct Comparison Test since $n + n + n > n + \sqrt{n} + 0 \Rightarrow \frac{3}{n+\sqrt{n}} > \frac{1}{n}$, which is the nth term of the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ or use Limit Comparison Test with $b_n = \frac{1}{n}$

19. converges by the Direct Comparison Test; $\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$, which is the nth term of a convergent geometric series

21. diverges since $\lim_{n \to \infty} \frac{2n}{3n-1} = \frac{2}{3} \neq 0$

22. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series:

$$\lim_{n \to \infty} \frac{\left(\frac{n+1}{n^2\sqrt{n}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n \to \infty} \left(\frac{n+1}{n}\right) = 1$$

23. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the nth term of a convergent p-series:

$$\lim_{n \to \infty} \frac{\left(\frac{10n+1}{n(n+1)(n+2)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \frac{10n^2+n}{n^2+3n+2} = \lim_{n \to \infty} \frac{20n+1}{2n+3} = \lim_{n \to \infty} \frac{20}{2} = 10$$

28. converges by the Limit Comparison Test (part 2) when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series:

$$\lim_{n \to \infty} \frac{\left[\frac{(\ln n)^2}{n^3}\right]}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \frac{(\ln n)^2}{n} = \lim_{n \to \infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = 2 \lim_{n \to \infty} \frac{\ln n}{n} = 0$$

29. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n \to \infty} \frac{\left[\frac{1}{\sqrt{n}\ln n}\right]}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{\sqrt{n}}{\ln n} = \lim_{n \to \infty} \frac{\left(\frac{1}{2\sqrt{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{\sqrt{n}}{2} = \infty$$

32. diverges by the Integral Test: $\int_2^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \int_{\ln 3}^{\infty} u\,du = \lim_{b \to \infty} \left[\frac{1}{2}u^2\right]_{\ln 3}^{b} = \lim_{b \to \infty} \frac{1}{2}\left(b^2 - \ln^2 3\right) = \infty$

34. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series: $n^2+1 > n^2$ $\Rightarrow n^2+1 > \sqrt{n}n^{3/2} \Rightarrow \frac{n^2+1}{\sqrt{n}} > n^{3/2} \Rightarrow \frac{\sqrt{n}}{n^2+1} < \frac{1}{n^{3/2}}$ or use Limit Comparison Test with $\frac{1}{n^{3/2}}$.

38. diverges; $\lim_{n \to \infty} \left(\frac{3^{n-1}+1}{3^n}\right) = \lim_{n \to \infty} \left(\frac{1}{3} + \frac{1}{3^n}\right) = \frac{1}{3} \neq 0$

40. converges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$, which is a convergent geometric series with $|r| = \frac{1}{5} < 1$,

$$\lim_{n \to \infty} \frac{\left(\frac{2^n+3^n}{3^n+4^n}\right)}{(3/4)^n} = \lim_{n \to \infty} \frac{8^n+12^n}{9^n+12^n} = \lim_{n \to \infty} \frac{\left(\frac{8}{12}\right)^n+1}{\left(\frac{9}{12}\right)^n+1} = \frac{1}{1} = 1 > 0.$$

42. diverges by the definition of an infinite series: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \left[\ln n - \ln(n+1)\right]$, $s_k = (\ln 1 - \ln 2) + (\ln 2 - \ln 3)$ $+ \ldots + (\ln(k-1) - \ln k) + (\ln k - \ln(k+1)) = -\ln(k+1) \Rightarrow \lim_{k \to \infty} s_k = -\infty$

43. converges by Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ which converges since $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n}\right]$, and $s_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{k-2} - \frac{1}{k-1}\right) + \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 - \frac{1}{k} \Rightarrow \lim_{k \to \infty} s_k = 1$; for $n \geq 2$, $(n-2)! \geq 1$ $\Rightarrow n(n-1)(n-2)! \geq n(n-1) \Rightarrow n! \geq n(n-1) \Rightarrow \frac{1}{n!} \leq \frac{1}{n(n-1)}$

45. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n \to \infty} \frac{\left(\sin \frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

47. converges by the Direct Comparison Test: $\frac{\tan^{-1} n}{n^{1.1}} < \frac{\frac{\pi}{2}}{n^{1.1}}$ and $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.1}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is the product of a convergent p-series and a nonzero constant

50. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{\tanh n}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \tanh n = \lim_{n \to \infty} \frac{e^n - e^{-n}}{e^n + e^{-n}}$

$$= \lim_{n \to \infty} \frac{1 - e^{-2n}}{1 + e^{-2n}} = 1$$

53. $\frac{1}{1+2+3+\ldots+n} = \frac{1}{\left(\frac{n(n+1)}{2}\right)} = \frac{2}{n(n+1)}$. The series converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\left(\frac{2}{n(n+1)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{2n^2}{n^2+n} = \lim_{n\to\infty} \frac{4n}{2n+1} = \lim_{n\to\infty} \frac{4}{2} = 2.$$

60. Since $a_n > 0$ and $\lim_{n\to\infty}(n^2 \cdot a_n) = 0$, compare $\sum a_n$ with $\sum \frac{1}{n^2}$, which is a convergent p-series; $\lim_{n\to\infty} \frac{a_n}{1/n^2}$ $= \lim_{n\to\infty}(n^2 \cdot a_n) = 0 \Rightarrow \sum a_n$ converges by Limit Comparison Test

61. Let $-\infty < q < \infty$ and $p > 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a convergent p-series. If $q \neq 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$ where $1 < r < p$, then $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}}$, and $p - r > 0$. If $q < 0 \Rightarrow -q > 0$ and $\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}}$ $= \lim_{n\to\infty} \frac{1}{(\ln n)^{-q} n^{p-r}} = 0$. If $q > 0$, $\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{q(\ln n)^{q-1}\left(\frac{1}{n}\right)}{(p-r)n^{p-r-1}} = \lim_{n\to\infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}}$. If $q - 1 \leq 0 \Rightarrow 1 - q \geq 0$ and $\lim_{n\to\infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}} = \lim_{n\to\infty} \frac{q}{(p-r)n^{p-r}(\ln n)^{1-q}} = 0$, otherwise, we apply L'Hopital's Rule again. $\lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}\left(\frac{1}{n}\right)}{(p-r)^2 n^{p-r-1}}$ $= \lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}}$. If $q - 2 \leq 0 \Rightarrow 2 - q \geq 0$ and $\lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}} = \lim_{n\to\infty} \frac{q(q-1)}{(p-r)^2 n^{p-r}(\ln n)^{2-q}} = 0$; otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $q - k \leq 0 \Rightarrow k - q \geq 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n\to\infty} \frac{q(q-1)\cdots(q-k+1)(\ln n)^{q-k}}{(p-r)^k n^{p-r}} = \lim_{n\to\infty} \frac{q(q-1)\cdots(q-k+1)}{(p-r)^k n^{p-r}(\ln n)^{k-q}} = 0$. Since the limit is 0 in every case, by Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^p}$ converges.

62. Let $-\infty < q < \infty$ and $p \leq 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. If $q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. Then $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^p} = \lim_{n\to\infty} (\ln n)^q = \infty$. If $q < 0 \Rightarrow -q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$, where $0 < p < r \leq 1$. $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{n^{r-p}}{(\ln n)^{-q}}$ since $r - p > 0$. Apply L'Hopital's to obtain $\lim_{n\to\infty} \frac{(r-p)n^{r-p-1}}{(-q)(\ln n)^{-q-1}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)n^{r-p}}{(-q)(\ln n)^{-q-1}}$. If $-q - 1 \leq 0 \Rightarrow q + 1 \geq 0$ and $\lim_{n\to\infty} \frac{(r-p)n^{r-p}(\ln n)^{q+1}}{(-q)} = \infty$, otherwise, we apply L'Hopital's Rule again to obtain $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p-1}}{(-q)(-q-1)(\ln n)^{-q-2}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}}$. If $-q - 2 \leq 0 \Rightarrow q + 2 \geq 0$ and $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}(\ln n)^{q+2}}{(-q)(-q-1)} = \infty$, otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $-q - k \leq 0 \Rightarrow q + k \geq 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n\to\infty} \frac{(r-p)^k n^{r-p}}{(-q)(-q-1)\cdots(-q-k+1)(\ln n)^{-q-k}} = \lim_{n\to\infty} \frac{(r-p)^k n^{r-p}(\ln n)^{q+k}}{(-q)(-q-1)\cdots(-q-k+1)} = \infty$.

Since the limit is ∞ if $q > 0$ or if $q < 0$ and $p < 1$, by Limit comparison test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges. Finally if $q < 0$ and $p = 1$ then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series. For $n \geq 3$, $\ln n \geq 1$ $\Rightarrow (\ln n)^q \geq 1 \Rightarrow \frac{(\ln n)^q}{n} \geq \frac{1}{n}$. Thus $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$ diverges by Comparison Test. Thus, if $-\infty < q < \infty$ and $p \leq 1$, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges.

Section 10.5:

1. $\frac{2^n}{n!} > 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right) = \lim\limits_{n\to\infty}\left(\frac{2^n\cdot 2}{(n+1)\cdot n!}\cdot\frac{n!}{2^n}\right) = \lim\limits_{n\to\infty}\left(\frac{2}{n+1}\right) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{2^n}{n!}$ converges

2. $\frac{n+2}{3^n} > 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{(n+1)+2}{3^{n+1}}}{\frac{n+2}{3^n}}\right) = \lim\limits_{n\to\infty}\left(\frac{n+3}{3^n\cdot 3}\cdot\frac{3^n}{n+2}\right) = \lim\limits_{n\to\infty}\left(\frac{n+3}{3n+6}\right) = \lim\limits_{n\to\infty}\left(\frac{1}{3}\right) = \frac{1}{3} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{n+2}{3^n}$ conver

3. $\frac{(n-1)!}{(n+1)^2} > 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{((n+1)-1)!}{((n+1)+1)^2}}{\frac{(n-1)!}{(n+1)^2}}\right) = \lim\limits_{n\to\infty}\left(\frac{n\cdot(n-1)!}{(n+2)^2}\cdot\frac{(n+1)^2}{(n-1)!}\right) = \lim\limits_{n\to\infty}\left(\frac{n^3+2n^2+n}{n^2+4n+4}\right) = \lim\limits_{n\to\infty}\left(\frac{3n^2+4n+1}{2n+4}\right)$
$= \lim\limits_{n\to\infty}\left(\frac{6n+4}{2}\right) = \infty > 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{(n-1)!}{(n+1)^2}$ diverges

4. $\frac{2^{n+1}}{n\cdot 3^{n-1}} > 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{2^{(n+1)+1}}{(n+1)\cdot 3^{(n+1)-1}}}{\frac{2^{n+1}}{n\cdot 3^{n-1}}}\right) = \lim\limits_{n\to\infty}\left(\frac{2^{n+1}\cdot 2}{(n+1)\cdot 3^{n-1}\cdot 3}\cdot\frac{n\cdot 3^{n-1}}{2^{n+1}}\right) = \lim\limits_{n\to\infty}\left(\frac{2n}{3n+3}\right) = \lim\limits_{n\to\infty}\left(\frac{2}{3}\right) = \frac{2}{3} < 1$
$\Rightarrow \sum\limits_{n=1}^{\infty}\frac{2^{n+1}}{n\cdot 3^{n-1}}$ converges

5. $\frac{n^4}{4^n} > 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{(n+1)^4}{4^{n+1}}}{\frac{n^4}{4^n}}\right) = \lim\limits_{n\to\infty}\left(\frac{(n+1)^4}{4^n\cdot 4}\cdot\frac{4^n}{n^4}\right) = \lim\limits_{n\to\infty}\left(\frac{n^4+4n^3+6n^2+4n+1}{4n^4}\right)$
$= \lim\limits_{n\to\infty}\left(\frac{1}{4}+\frac{1}{n}+\frac{3}{2n^2}+\frac{1}{n^3}+\frac{1}{4n^4}\right) = \frac{1}{4} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{n^4}{4^n}$ converges

6. $\frac{3^{n+2}}{\ln n} > 0$ for all $n \geq 2$; $\lim\limits_{n\to\infty}\left(\frac{\frac{3^{(n+1)+2}}{\ln(n+1)}}{\frac{3^{n+2}}{\ln n}}\right) = \lim\limits_{n\to\infty}\left(\frac{3^{n+2}\cdot 3}{\ln(n+1)}\cdot\frac{\ln n}{3^{n+2}}\right) = \lim\limits_{n\to\infty}\left(\frac{3\ln n}{\ln(n+1)}\right) = \lim\limits_{n\to\infty}\left(\frac{\frac{3}{n}}{\frac{1}{n+1}}\right) = \lim\limits_{n\to\infty}\left(\frac{3n+3}{n}\right)$
$= \lim\limits_{n\to\infty}\left(\frac{3}{1}\right) = 3 > 1 \Rightarrow \sum\limits_{n=2}^{\infty}\frac{3^{n+2}}{\ln n}$ diverges

10. $\frac{4^n}{(3n)^n} \geq 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{4^n}{(3n)^n}} = \lim\limits_{n\to\infty}\left(\frac{4}{3n}\right) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{4^n}{(3n)^n}$ converges

11. $\left(\frac{4n+3}{3n-5}\right)^n \geq 0$ for all $n \geq 2$; $\lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^n} = \lim\limits_{n\to\infty}\left(\frac{4n+3}{3n-5}\right) = \lim\limits_{n\to\infty}\left(\frac{4}{3}\right) = \frac{4}{3} > 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left(\frac{4n+3}{3n-5}\right)^n$ diverges

12. $\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1} \geq 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}} = \lim\limits_{n\to\infty}\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+1/n} = \ln(e^2) = 2 > 1$
$\Rightarrow \sum\limits_{n=1}^{\infty}\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}$ diverges

13. $\frac{8}{\left(3+\frac{1}{n}\right)^{2n}} \geq 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{8}{\left(3+\frac{1}{n}\right)^{2n}}} = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{8}}{\left(3+\frac{1}{n}\right)^2}\right) = \frac{1}{9} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{8}{\left(3+\frac{1}{n}\right)^{2n}}$ converges

14. $\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n \geq 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n} = \lim\limits_{n\to\infty}\sin\left(\frac{1}{\sqrt{n}}\right) = \sin(0) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n$ converges

15. $\left(1-\frac{1}{n}\right)^{n^2} \geq 0$ for all $n \geq 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$ converges

19. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)!}{e^{n+1}}\right)}{\left(\frac{n!}{e^n}\right)} = \lim_{n\to\infty} \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \lim_{n\to\infty} \frac{n+1}{e} = \infty$

20. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^n}\right)} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n}{10} = \infty$

21. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(n+1)^{10}}{10^{n+1}}\right)}{\left(\frac{n^{10}}{10^n}\right)} = \lim_{n\to\infty} \frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10}\left(\frac{1}{10}\right) = \frac{1}{10} < 1$

22. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{-2}{n}\right)^n = e^{-2} \neq 0$

23. converges by the Direct Comparison Test: $\frac{2+(-1)^n}{(1.25)^n} = \left(\frac{4}{5}\right)^n [2+(-1)^n] \le \left(\frac{4}{5}\right)^n (3)$ which is the n^{th} term of a convergent geometric series

24. converges; a geometric series with $|r| = \left|-\frac{2}{3}\right| < 1$

25. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1-\frac{3}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{-3}{n}\right)^n = e^{-3} \approx 0.05 \neq 0$

26. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1-\frac{1}{3n}\right)^n = \lim_{n\to\infty} \left(1+\frac{\left(-\frac{1}{3}\right)}{n}\right)^n = e^{-1/3} \approx 0.72 \neq 0$

27. converges by the Direct Comparison Test: $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \ge 2$, the n^{th} term of a convergent p-series.

28. converges by the nth-Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{((\ln n)^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 < 1$

29. diverges by the Direct Comparison Test: $\frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} > \frac{1}{2}\left(\frac{1}{n}\right)$ for $n > 2$ or by the Limit Comparison Test (part 1) with $\frac{1}{n}$.

30. converges by the nth-Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \left(\left(\frac{1}{n}-\frac{1}{n^2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right) = 0 < 1$

31. diverges by the Direct Comparison Test: $\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 3$

32. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1)}{2^{n+1}} \cdot \frac{2^n}{n\ln(n)} = \frac{1}{2} < 1$

33. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} = 0 < 1$

38. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$
$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

39. converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$

40. converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^{n/2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt{\ln n}} = \frac{\lim_{n\to\infty} \sqrt[n]{n}}{\lim_{n\to\infty} \sqrt{\ln n}} = 0 < 1$

$\left(\lim_{n\to\infty} \sqrt[n]{n} = 1\right)$

41. converges by the Direct Comparison Test: $\frac{n!\,\ln n}{n(n+2)!} = \frac{\ln n}{n(n+1)(n+2)} < \frac{n}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$ which is the nth-term of a convergent p-series

42. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3\,2^{n+1}} \cdot \frac{n^3 2^n}{3^n} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$

43. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2}{[2(n+1)]!} \cdot \frac{(2n)!}{[n!]^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$

44. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+5)(2^{n+1}+3)}{3^{n+1}+2} \cdot \frac{3^n+2}{(2n+3)(2^n+3)} = \lim_{n\to\infty} \left[\frac{2n+5}{2n+3} \cdot \frac{2\cdot 6^n + 4\cdot 2^n + 3\cdot 3^n + 6}{3\cdot 6^n + 9\cdot 3^n + 2\cdot 2^n + 6}\right]$

$= \lim_{n\to\infty} \left[\frac{2n+5}{2n+3}\right] \cdot \lim_{n\to\infty} \left[\frac{2\cdot 6^n + 4\cdot 2^n + 3\cdot 3^n + 6}{3\cdot 6^n + 9\cdot 3^n + 2\cdot 2^n + 6}\right] = 1 \cdot \frac{2}{3} = \frac{2}{3} < 1$

45. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\sin n}{n}\right) a_n}{a_n} = 0 < 1$

60. diverges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n\to\infty} \frac{n}{4} = \infty > 1$

61. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1\cdot 3\cdot \cdots \cdot (2n-1)(2n+1)}{4^{n+1}2^{n+1}(n+1)!} \cdot \frac{4^n\,2^n\,n!}{1\cdot 3\cdot \cdots \cdot (2n-1)} = \lim_{n\to\infty} \frac{2n+1}{(4\cdot 2)(n+1)} = \frac{1}{4} < 1$

62. converges by the Ratio Test: $a_n = \frac{1\cdot 3\cdots(2n-1)}{(2\cdot 4\cdots 2n)\,(3^n+1)} = \frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}{(2\cdot 4\cdots 2n)^2\,(3^n+1)} = \frac{(2n)!}{(2^n n!)^2\,(3^n+1)}$

$\Rightarrow \lim_{n\to\infty} \frac{(2n+2)!}{[2^{n+1}(n+1)!]^2\,(3^{n+1}+1)} \cdot \frac{(2^n n!)^2\,(3^n+1)}{(2n)!} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)\,(3^n+1)}{2^2(n+1)^2\,(3^{n+1}+1)}$

$= \lim_{n\to\infty} \left(\frac{4n^2+6n+2}{4n^2+8n+4}\right) \frac{(1+3^{-n})}{(3+3^{-n})} = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1$

63. Ratio: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1^p = 1 \Rightarrow$ no conclusion

Root: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

64. Ratio: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(\ln(n+1))^p} \cdot \frac{(\ln n)^p}{1} = \left[\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right]^p = \left[\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right]^p = \left(\lim_{n\to\infty} \frac{n+1}{n}\right)^p$

$= (1)^p = 1 \Rightarrow$ no conclusion

Root: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^p}} = \frac{1}{\left(\lim_{n\to\infty} (\ln n)^{1/n}\right)^p}$; let $f(n) = (\ln n)^{1/n}$, then $\ln f(n) = \frac{\ln(\ln n)}{n}$

$\Rightarrow \lim_{n\to\infty} \ln f(n) = \lim_{n\to\infty} \frac{\ln(\ln n)}{n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n \ln n}\right)}{1} = \lim_{n\to\infty} \frac{1}{n \ln n} = 0 \Rightarrow \lim_{n\to\infty} (\ln n)^{1/n}$

$= \lim_{n\to\infty} e^{\ln f(n)} = e^0 = 1$; therefore $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\left(\lim_{n\to\infty} (\ln n)^{1/n}\right)^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

65. $a_n \le \frac{n}{2^n}$ for every n and the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by the Ratio Test since $\lim_{n\to\infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test