MTH 201. Homework III Solution.

Section 10.1:

35. $\lim_{n \to \infty} (1 + (-1)^n)$ does not exist $\Rightarrow$ diverges

36. $\lim_{n \to \infty} (-1)^n \left(1 - \frac{1}{n}\right)$ does not exist $\Rightarrow$ diverges

40. $\lim_{n \to \infty} \left(-\frac{1}{2}\right)^n = \lim_{n \to \infty} \frac{(-1)^n}{2^n} = 0 \Rightarrow$ converges

41. $\lim_{n \to \infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n \to \infty} \frac{2n}{n+1}} = \sqrt{\lim_{n \to \infty} \left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2} \Rightarrow$ converges

43. $\lim_{n \to \infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin\left(\lim_{n \to \infty} \left(\frac{\pi}{2} + \frac{1}{n}\right)\right) = \sin \frac{\pi}{2} = 1 \Rightarrow$ converges

44. $\lim_{n \to \infty} n\pi \cos(n\pi) = \lim_{n \to \infty} (n\pi)(-1)^n$ does not exist $\Rightarrow$ diverges

45. $\lim_{n \to \infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

47. $\lim_{n \to \infty} \frac{n}{2^n} = \lim_{n \to \infty} \frac{1}{2^n \ln 2} = 0 \Rightarrow$ converges (using l'Hôpital's rule)

50. $\lim_{n \to \infty} \frac{\ln n}{\ln 2n} = \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{2}{2n}\right)} = 1 \Rightarrow$ converges

51. $\lim_{n \to \infty} 8^{1/n} = 1 \Rightarrow$ converges (Theorem 5, #3)

52. $\lim_{n \to \infty} (0.03)^{1/n} = 1 \Rightarrow$ converges (Theorem 5, #3)

53. $\lim_{n \to \infty} \left(1 + \frac{7}{n}\right)^n = e^7 \Rightarrow$ converges (Theorem 5, #5)

55. $\lim_{n \to \infty} \sqrt[n]{10n} = \lim_{n \to \infty} 10^{1/n} \cdot n^{1/n} = 1 \cdot 1 = 1 \Rightarrow$ converges (Theorem 5, #3 and #2)

56. $\lim_{n \to \infty} \sqrt[n]{n^2} = \lim_{n \to \infty} \left(\sqrt[n]{n}\right)^2 = 1^2 = 1 \Rightarrow$ converges (Theorem 5, #2)

57. $\lim_{n \to \infty} \left(\frac{3}{n}\right)^{1/n} = \frac{\lim_{n\to\infty} 3^{1/n}}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1 \Rightarrow$ converges (Theorem 5, #3 and #2)

59. $\lim_{n \to \infty} \frac{\ln n}{n^{1/n}} = \frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n^{1/n}} = \frac{\infty}{1} = \infty \Rightarrow$ diverges (Theorem 5, #2)

63. $\lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} \frac{1\cdot 2\cdot 3\cdots(n-1)(n)}{n\cdot n\cdot n\cdots n\cdot n} \le \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0$ and $\frac{n!}{n^n} \ge 0 \Rightarrow \lim_{n \to \infty} \frac{n!}{n^n} = 0 \Rightarrow$ converges

67. $\lim_{n \to \infty} \left(\frac{1}{n}\right)^{1/(\ln n)} = \lim_{n \to \infty} \exp\left(\frac{1}{\ln n} \ln\left(\frac{1}{n}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln 1 - \ln n}{\ln n}\right) = e^{-1} \Rightarrow$ converges

72. $\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} \exp\left(n \ln\left(1-\frac{1}{n^2}\right)\right) = \lim_{n\to\infty} \exp\left(\frac{\ln\left(1-\frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)}\right) = \lim_{n\to\infty} \exp\left[\frac{\left(\frac{2}{n^3}\right)/\left(1-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}\right]$
$= \lim_{n\to\infty} \exp\left(\frac{-2n}{n^2-1}\right) = e^0 = 1 \Rightarrow$ converges

78. $\lim_{n\to\infty} n\left(1-\cos\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\left(1-\cos\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\left[\sin\left(\frac{1}{n}\right)\right]\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0 \Rightarrow$ converges

81. $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} \Rightarrow$ converges

82. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \tan^{-1} n = 0 \cdot \frac{\pi}{2} = 0 \Rightarrow$ converges

89. $\lim_{n\to\infty} \frac{1}{n}\int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$ converges (Theorem 5, #1)

Section 10.2:

8. $\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$, the sum of this geometric series is $\frac{\left(\frac{1}{16}\right)}{1-\left(\frac{1}{4}\right)} = \frac{1}{12}$

9. $\frac{7}{4}+\frac{7}{16}+\frac{7}{64}+\ldots$, the sum of this geometric series is $\frac{\left(\frac{7}{4}\right)}{1-\left(\frac{1}{4}\right)} = \frac{7}{3}$

10. $5-\frac{5}{4}+\frac{5}{16}-\frac{5}{64}+\ldots$, the sum of this geometric series is $\frac{5}{1-\left(-\frac{1}{4}\right)} = 4$

14. $2+\frac{4}{5}+\frac{8}{25}+\frac{16}{125}+\ldots = 2\left(1+\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+\ldots\right)$; the sum of this geometric series is $2\left(\frac{1}{1-\left(\frac{2}{5}\right)}\right) = \frac{10}{3}$

29. $\lim_{n\to\infty} \frac{1}{n+4} = 0 \Rightarrow$ test inconclusive

33. $\lim_{n\to\infty} \ln\frac{1}{n} = -\infty \neq 0 \Rightarrow$ diverges

34. $\lim_{n\to\infty} \cos n\pi =$ does not exist $\Rightarrow$ diverges

35. $s_k = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{k-1}-\frac{1}{k}\right)+\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1-\frac{1}{k+1} \Rightarrow \lim_{k\to\infty} s_k$
$= \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right) = 1$, series converges to 1

36. $s_k = \left(\frac{3}{1}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{3}{9}\right)+\left(\frac{3}{9}-\frac{3}{16}\right)+\ldots+\left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right)+\left(\frac{3}{k^2}-\frac{3}{(k+1)^2}\right) = 3-\frac{3}{(k+1)^2} \Rightarrow \lim_{k\to\infty} s_k$
$= \lim_{k\to\infty}\left(3-\frac{3}{(k+1)^2}\right) = 3$, series converges to 3

37. $s_k = \left(\ln\sqrt{2} - \ln\sqrt{1}\right) + \left(\ln\sqrt{3} - \ln\sqrt{2}\right) + \left(\ln\sqrt{4} - \ln\sqrt{3}\right) + \ldots + \left(\ln\sqrt{k} - \ln\sqrt{k-1}\right) + \left(\ln\sqrt{k+1} - \ln\sqrt{k}\right)$
$= \ln\sqrt{k+1} - \ln\sqrt{1} = \ln\sqrt{k+1} \Rightarrow \lim\limits_{k \to \infty} s_k = \lim\limits_{k \to \infty} \ln\sqrt{k+1} = \infty$; series diverges

39. $s_k = \left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{1}{3}\right)\right) + \left(\cos^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(\frac{1}{4}\right)\right) + \left(\cos^{-1}\left(\frac{1}{4}\right) - \cos^{-1}\left(\frac{1}{5}\right)\right) + \ldots$
$+ \left(\cos^{-1}\left(\frac{1}{k}\right) - \cos^{-1}\left(\frac{1}{k+1}\right)\right) + \left(\cos^{-1}\left(\frac{1}{k+1}\right) - \cos^{-1}\left(\frac{1}{k+2}\right)\right) = \frac{\pi}{3} - \cos^{-1}\left(\frac{1}{k+2}\right)$
$\Rightarrow \lim\limits_{k \to \infty} s_k = \lim\limits_{k \to \infty} \left[\frac{\pi}{3} - \cos^{-1}\left(\frac{1}{k+2}\right)\right] = \frac{\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$, series converges to $\frac{\pi}{6}$

44. $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2} \Rightarrow s_k = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) + \ldots + \left[\frac{1}{(k-1)^2} - \frac{1}{k^2}\right] + \left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right]$
$\Rightarrow \lim\limits_{k \to \infty} s_k = \lim\limits_{k \to \infty} \left[1 - \frac{1}{(k+1)^2}\right] = 1$

47. $s_k = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4}\right) + \ldots + \left(\frac{1}{\ln(k+1)} - \frac{1}{\ln k}\right) + \left(\frac{1}{\ln(k+2)} - \frac{1}{\ln(k+1)}\right)$
$= -\frac{1}{\ln 2} + \frac{1}{\ln(k+2)} \Rightarrow \lim\limits_{k \to \infty} s_k = -\frac{1}{\ln 2}$

48. $s_k = [\tan^{-1}(1) - \tan^{-1}(2)] + [\tan^{-1}(2) - \tan^{-1}(3)] + \ldots + [\tan^{-1}(k-1) - \tan^{-1}(k)]$
$+ [\tan^{-1}(k) - \tan^{-1}(k+1)] = \tan^{-1}(1) - \tan^{-1}(k+1) \Rightarrow \lim\limits_{k \to \infty} s_k = \tan^{-1}(1) - \frac{\pi}{2} = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

49. convergent geometric series with sum $\frac{1}{1 - \left(\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \sqrt{2}$

50. divergent geometric series with $|r| = \sqrt{2} > 1$

51. convergent geometric series with sum $\frac{\left(\frac{3}{2}\right)}{1 - \left(-\frac{1}{2}\right)} = 1$

52. $\lim\limits_{n \to \infty} (-1)^{n+1} n \neq 0 \Rightarrow$ diverges

53. $\lim\limits_{n \to \infty} \cos(n\pi) = \lim\limits_{n \to \infty} (-1)^n \neq 0 \Rightarrow$ diverges

54. $\cos(n\pi) = (-1)^n \Rightarrow$ convergent geometric series with sum $\frac{1}{1 - \left(-\frac{1}{5}\right)} = \frac{5}{6}$

55. convergent geometric series with sum $\frac{1}{1 - \left(\frac{1}{e^2}\right)} = \frac{e^2}{e^2 - 1}$

56. $\lim\limits_{n \to \infty} \ln \frac{1}{3^n} = -\infty \neq 0 \Rightarrow$ diverges

57. convergent geometric series with sum $\frac{2}{1 - \left(\frac{1}{10}\right)} - 2 = \frac{20}{9} - \frac{18}{9} = \frac{2}{9}$

58. convergent geometric series with sum $\frac{1}{1 - \left(\frac{1}{x}\right)} = \frac{x}{x-1}$

59. difference of two geometric series with sum $\frac{1}{1 - \left(\frac{2}{3}\right)} - \frac{1}{1 - \left(\frac{1}{3}\right)} = 3 - \frac{3}{2} = \frac{3}{2}$

60. $\lim\limits_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \lim\limits_{n \to \infty} \left(1 + \frac{-1}{n}\right)^n = e^{-1} \neq 0 \Rightarrow$ diverges

61. $\lim\limits_{n \to \infty} \frac{n!}{1000^n} = \infty \neq 0 \Rightarrow$ diverges

62. $\lim\limits_{n \to \infty} \frac{n^n}{n!} = \lim\limits_{n \to \infty} \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} > \lim\limits_{n \to \infty} n = \infty \Rightarrow$ diverges

63. $\sum\limits_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum\limits_{n=1}^{\infty} \frac{2^n}{4^n} + \sum\limits_{n=1}^{\infty} \frac{3^n}{4^n} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum\limits_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$; both $= \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ and $\sum\limits_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ are geometric series, and both converge since $r = \frac{1}{2} \Rightarrow \left|\frac{1}{2}\right| < 1$ and $r = \frac{3}{4} \Rightarrow \left|\frac{3}{4}\right| < 1$, respectivley $\Rightarrow \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$ and $\sum\limits_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}}{1 - \frac{3}{4}} = 3 \Rightarrow$
$\sum\limits_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = 1 + 3$ 4 by Theorem 8, part (1)

64. $\lim\limits_{n\to\infty} \frac{2^n+4^n}{3^n+4^n} = \lim\limits_{n\to\infty} \frac{\frac{2^n}{4^n}+1}{\frac{3^n}{4^n}+1} = \lim\limits_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n+1}{\left(\frac{3}{4}\right)^n+1} = \frac{1}{1} = 1 \neq 0 \Rightarrow$ diverges by n^{th} term test for divergence

65. $\sum\limits_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum\limits_{n=1}^{\infty} [\ln(n) - \ln(n+1)] \Rightarrow s_k = [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \ldots$
$+ [\ln(k-1) - \ln(k)] + [\ln(k) - \ln(k+1)] = -\ln(k+1) \Rightarrow \lim\limits_{k\to\infty} s_k = -\infty, \Rightarrow$ diverges

66. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \ln\left(\frac{n}{2n+1}\right) = \ln\left(\frac{1}{2}\right) \neq 0 \Rightarrow$ diverges

67. convergent geometric series with sum $\frac{1}{1-\left(\frac{e}{\pi}\right)} = \frac{\pi}{\pi - e}$

68. divergent geometric series with $|r| = \frac{e^\pi}{\pi^e} \approx \frac{23.141}{22.459} > 1$

69. $\sum\limits_{n=0}^{\infty} (-1)^n x^n = \sum\limits_{n=0}^{\infty} (-x)^n$; $a = 1, r = -x$; converges to $\frac{1}{1-(-x)} = \frac{1}{1+x}$ for $|x| < 1$

71. $a = 3, r = \frac{x-1}{2}$; converges to $\frac{3}{1-\left(\frac{x-1}{2}\right)} = \frac{6}{3-x}$ for $-1 < \frac{x-1}{2} < 1$ or $-1 < x < 3$

Section 10.3:

7. $f(x) = \frac{x}{x^2+4}$ is positive and continuous for $x \geq 1$, $f'(x) = \frac{4-x^2}{(x^2+4)^2} < 0$ for $x > 2$, thus f is decreasing for $x \geq 3$;
$\int_3^\infty \frac{x}{x^2+4}\,dx = \lim\limits_{b\to\infty} \int_3^b \frac{x}{x^2+4}\,dx = \lim\limits_{b\to\infty} \left[\frac{1}{2}\ln(x^2+4)\right]_3^b = \lim\limits_{b\to\infty} \left(\frac{1}{2}\ln(b^2+4) - \frac{1}{2}\ln(13)\right) = \infty \Rightarrow \int_3^\infty \frac{x}{x^2+4}\,dx$
diverges $\Rightarrow \sum\limits_{n=3}^{\infty} \frac{n}{n^2+4}$ diverges $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{n}{n^2+4} = \frac{1}{5} + \frac{2}{8} + \sum\limits_{n=3}^{\infty} \frac{n}{n^2+4}$ diverges

8. $f(x) = \frac{\ln x^2}{x}$ is positive and continuous for $x \geq 2$, $f'(x) = \frac{2-\ln x^2}{x^2} < 0$ for $x > e$, thus f is decreasing for $x \geq 3$;
$\int_3^\infty \frac{\ln x^2}{x}\,dx = \lim\limits_{b\to\infty} \int_3^b \frac{\ln x^2}{x}\,dx = \lim\limits_{b\to\infty} \left[2(\ln x)\right]_3^b = \lim\limits_{b\to\infty} (2(\ln b) - 2(\ln 3)) = \infty \Rightarrow \int_3^\infty \frac{\ln x^2}{x}\,dx$
diverges $\Rightarrow \sum\limits_{n=3}^{\infty} \frac{\ln n^2}{n}$ diverges $\Rightarrow \sum\limits_{n=2}^{\infty} \frac{\ln n^2}{n} = \frac{\ln 4}{2} + \sum\limits_{n=3}^{\infty} \frac{\ln n^2}{n}$ diverges

9. $f(x) = \frac{x^2}{e^{x/3}}$ is positive and continuous for $x \geq 1$, $f'(x) = \frac{-x(x-6)}{3e^{x/3}} < 0$ for $x > 6$, thus f is decreasing for $x \geq 7$;
$\int_7^\infty \frac{x^2}{e^{x/3}}\,dx = \lim\limits_{b\to\infty} \int_7^b \frac{x^2}{e^{x/3}}\,dx = \lim\limits_{b\to\infty} \left[-\frac{3x^2}{e^{x/3}} - \frac{18x}{e^{x/3}} - \frac{54}{e^{x/3}}\right]_7^b = \lim\limits_{b\to\infty} \left(\frac{-3b^2-18b-54}{e^{b/3}} + \frac{327}{e^{7/3}}\right) =$
$= \lim\limits_{b\to\infty} \left(\frac{3(-6b-18)}{e^{b/3}}\right) + \frac{327}{e^{7/3}} = \lim\limits_{b\to\infty} \left(\frac{-54}{e^{b/3}}\right) + \frac{327}{e^{7/3}} = \frac{327}{e^{7/3}} \Rightarrow \int_7^\infty \frac{x^2}{e^{x/3}}\,dx$ converges $\Rightarrow \sum\limits_{n=7}^{\infty} \frac{n^2}{e^{n/3}}$ converges
$\Rightarrow \sum\limits_{n=1}^{\infty} \frac{n^2}{e^{n/3}} = \frac{1}{e^{1/3}} + \frac{4}{e^{2/3}} + \frac{9}{e^1} + \frac{16}{e^{4/3}} + \frac{25}{e^{5/3}} + \frac{36}{e^2} + \sum\limits_{n=7}^{\infty} \frac{n^2}{e^{n/3}}$ converges

12. converges; a geometric series with $r = \frac{1}{e} < 1$

13. diverges; by the nth-Term Test for Divergence, $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

15. diverges; $\sum\limits_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series ($p = \frac{1}{2}$)

20. diverges by the Integral Test: $\int_2^\infty \frac{\ln x}{\sqrt{x}}\,dx$; $\begin{bmatrix} t = \ln x \\ dt = \frac{dx}{x} \\ dx = e^t\,dt \end{bmatrix} \rightarrow \int_{\ln 2}^\infty te^{t/2}\,dt = \lim\limits_{b\to\infty} \left[2te^{t/2} - 4e^{t/2}\right]_{\ln 2}^b$
$= \lim\limits_{b\to\infty} \left[2e^{b/2}(b-2) - 2e^{(\ln 2)/2}(\ln 2 - 2)\right] = \infty$

26. diverges by the Integral Test: $\int_1^n \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$; $\left[\begin{array}{c} u = \sqrt{x}+1 \\ du = \frac{dx}{\sqrt{x}} \end{array}\right] \to \int_2^{\sqrt{n}+1} \frac{du}{u} = \ln\left(\sqrt{n}+1\right) - \ln 2 \to \infty$ as $n \to \infty$

28. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$

33. diverges by the nth-Term Test for divergence; $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0$

34. diverges by the nth-Term Test for divergence; $\lim_{n\to\infty} n \tan\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{1}{n^2}\right)\sec^2\left(\frac{1}{n}\right)}{\left(-\frac{1}{n^2}\right)}$

$= \lim_{n\to\infty} \sec^2\left(\frac{1}{n}\right) = \sec^2 0 = 1 \neq 0$

35. converges by the Integral Test: $\int_1^\infty \frac{e^x}{1+e^{2x}}\,dx$; $\left[\begin{array}{c} u = e^x \\ du = e^x\,dx \end{array}\right] \to \int_e^\infty \frac{1}{1+u^2}\,du = \lim_{n\to\infty} \left[\tan^{-1} u\right]_e^b$

$= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} e\right) = \frac{\pi}{2} - \tan^{-1} e \approx 0.35$

36. converges by the Integral Test: $\int_1^\infty \frac{2}{1+e^x}\,dx$; $\left[\begin{array}{c} u = e^x \\ du = e^x\,dx \\ dx = \frac{1}{u}\,du \end{array}\right] \to \int_e^\infty \frac{2}{u(1+u)}\,du = \int_e^\infty \left(\frac{2}{u} - \frac{2}{u+1}\right) du$

38. diverges by the Integral Test: $\int_1^\infty \frac{x}{x^2+1}\,dx$; $\left[\begin{array}{c} u = x^2+1 \\ du = 2x\,dx \end{array}\right] \to \frac{1}{2}\int_2^\infty \frac{du}{4} = \lim_{b\to\infty} \left[\frac{1}{2}\ln u\right]_2^b = \lim_{b\to\infty} \frac{1}{2}(\ln b - \ln 2) = \infty$

57. (a) From Fig. 10.11(a) in the text with $f(x) = \frac{1}{x}$ and $a_k = \frac{1}{k}$, we have $\int_1^{n+1} \frac{1}{x}\,dx \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$
$\le 1 + \int_1^n f(x)\,dx \Rightarrow \ln(n+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \le 1 + \ln n \Rightarrow 0 \le \ln(n+1) - \ln n$
$\le \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n \le 1$. Therefore the sequence $\left\{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n\right\}$ is bounded above by 1 and below by 0.

(b) From the graph in Fig. 10.11(b) with $f(x) = \frac{1}{x}$, $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln n$
$\Rightarrow 0 > \frac{1}{n+1} - [\ln(n+1) - \ln n] = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right)$.
If we define $a_n = 1 + \frac{1}{2} = \frac{1}{3} + \frac{1}{n} - \ln n$, then $0 > a_{n+1} - a_n \Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is a decreasing sequence of nonnegative terms.