

Section 8.7:

$$17. \int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}; \left[\begin{array}{l} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{array} \right] \rightarrow \int_0^{\infty} \frac{2du}{u^2+1} = \lim_{b \rightarrow \infty} \int_0^b \frac{2du}{u^2+1} = \lim_{b \rightarrow \infty} [2 \tan^{-1} u]_0^b \\ = \lim_{b \rightarrow \infty} (2 \tan^{-1} b - 2 \tan^{-1} 0) = 2 \left(\frac{\pi}{2} \right) - 2(0) = \pi$$

$$25. \int_0^1 x \ln x \, dx = \lim_{b \rightarrow 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_b^1 = \left(\frac{1}{2} \ln 1 - \frac{1}{4} \right) - \lim_{b \rightarrow 0^+} \left(\frac{b^2}{2} \ln b - \frac{b^2}{4} \right) = -\frac{1}{4} - \lim_{b \rightarrow 0^+} \frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0 \\ = -\frac{1}{4} - \lim_{b \rightarrow 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim_{b \rightarrow 0^+} \left(\frac{b^2}{4} \right) = -\frac{1}{4} + 0 = -\frac{1}{4}$$

$$29. \int_1^2 \frac{ds}{s\sqrt{s^2-1}} = \lim_{b \rightarrow 1^+} [\sec^{-1} s]_b^2 = \sec^{-1} 2 - \lim_{b \rightarrow 1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

$$39. \int_0^{\ln 2} x^{-2} e^{-1/x} \, dx; \left[\frac{1}{x} = y \right] \rightarrow \int_{\infty}^{1/\ln 2} \frac{y^2 e^{-y} dy}{-y^2} = \int_{1/\ln 2}^{\infty} e^{-y} \, dy = \lim_{b \rightarrow \infty} [-e^{-y}]_{1/\ln 2}^b = \lim_{b \rightarrow \infty} [-e^{-b}] - [-e^{-1/\ln 2}] \\ = 0 + e^{-1/\ln 2} = e^{-1/\ln 2}, \text{ so the integral converges.}$$

$$43. \int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2} \text{ and } \int_0^1 \frac{dx}{1-x^2} = \lim_{b \rightarrow 1^-} \left[\frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| \right]_0^b = \lim_{b \rightarrow 1^-} \left[\frac{1}{2} \ln \left| \frac{1+b}{1-b} \right| \right] - 0 = \infty, \text{ which} \\ \text{diverges} \Rightarrow \int_0^2 \frac{dx}{1-x^2} \text{ diverges as well.}$$

$$49. \int_2^{\infty} \frac{dv}{\sqrt{v-1}}; \lim_{v \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim_{v \rightarrow \infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim_{v \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1 \text{ and } \int_2^{\infty} \frac{dv}{\sqrt{v}} = \lim_{b \rightarrow \infty} [2\sqrt{v}]_2^b = \infty, \\ \text{which diverges} \Rightarrow \int_2^{\infty} \frac{dv}{\sqrt{v-1}} \text{ diverges by the Limit Comparison Test.}$$

$$57. \int_4^{\infty} \frac{2 \, dt}{t^{3/2}-1}; \lim_{t \rightarrow \infty} \frac{t^{3/2}}{t^{3/2}-1} = 1 \text{ and } \int_4^{\infty} \frac{2 \, dt}{t^{3/2}} = \lim_{b \rightarrow \infty} [-4t^{-1/2}]_4^b = \lim_{b \rightarrow \infty} \left(\frac{-4}{\sqrt{b}} + 2 \right) = 2 \Rightarrow \int_4^{\infty} \frac{2 \, dt}{t^{3/2}} \text{ converges} \\ \Rightarrow \int_4^{\infty} \frac{2 \, dt}{t^{3/2}-1} \text{ converges by the Limit Comparison Test.}$$

$$59. \int_1^{\infty} \frac{e^x}{x} \, dx; 0 < \frac{1}{x} < \frac{e^x}{x} \text{ for } x > 1 \text{ and } \int_1^{\infty} \frac{dx}{x} \text{ diverges} \Rightarrow \int_1^{\infty} \frac{e^x}{x} \, dx \text{ diverges by the Direct Comparison Test.}$$

$$61. \int_1^{\infty} \frac{dx}{\sqrt{e^x-x}}; \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{e^x}}{\sqrt{e^x-x}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1; \int_1^{\infty} \frac{dx}{\sqrt{e^x}} = \int_1^{\infty} e^{-x/2} \, dx \\ = \lim_{b \rightarrow \infty} [-2e^{-x/2}]_1^b = \lim_{b \rightarrow \infty} (-2e^{-b/2} + 2e^{-1/2}) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^{\infty} e^{-x/2} \, dx \text{ converges} \Rightarrow \int_1^{\infty} \frac{dx}{\sqrt{e^x-x}} \text{ converges} \\ \text{by the Limit Comparison Test.}$$

65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \rightarrow \int_0^{\ln 2} \frac{dt}{t^p} = \lim_{b \rightarrow 0^+} \left[\frac{1}{-p+1} t^{1-p} \right]_b^{\ln 2} = \lim_{b \rightarrow 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p} (\ln 2)^{1-p}$
 \Rightarrow the integral converges for $p < 1$ and diverges for $p \geq 1$
- (b) $\int_2^\infty \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \rightarrow \int_{\ln 2}^\infty \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 65(a): it converges for $p > 1$ and diverges for $p \leq 1$

Practice Exercises:

$$59. \int_0^\infty x^2 e^{-x} dx = \lim_{b \rightarrow \infty} [-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}]_0^b = \lim_{b \rightarrow \infty} (-b^2 e^{-b} - 2b e^{-b} - 2e^{-b}) - (-2) = 0 + 2 = 2$$

$$61. \int_{-\infty}^\infty \frac{dx}{4x^2+9} = 2 \int_0^\infty \frac{dx}{4x^2+9} = \frac{1}{2} \int_0^\infty \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{2}{3} \tan^{-1} \left(\frac{2x}{3} \right) \right]_0^b = \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{2}{3} \tan^{-1} \left(\frac{2b}{3} \right) \right] - \frac{1}{3} \tan^{-1}(0)$$

$$= \frac{1}{2} \left(\frac{2}{3} \cdot \frac{\pi}{2} \right) - 0 = \frac{\pi}{6}$$

$$65. \int_1^\infty \frac{\ln z}{z} dz = \int_1^e \frac{\ln z}{z} dz + \int_e^\infty \frac{\ln z}{z} dz = \left[\frac{(\ln z)^2}{2} \right]_1^e + \lim_{b \rightarrow \infty} \left[\frac{(\ln z)^2}{2} \right]_e^b = \left(\frac{1^2}{2} - 0 \right) + \lim_{b \rightarrow \infty} \left[\frac{(\ln b)^2}{2} - \frac{1}{2} \right] = \infty$$

\Rightarrow diverges

$$67. \int_{-\infty}^\infty \frac{2 dx}{e^x + e^{-x}} = 2 \int_0^\infty \frac{2 dx}{e^x + e^{-x}} < \int_0^\infty \frac{4 dx}{e^x} \text{ converges} \Rightarrow \int_{-\infty}^\infty \frac{2 dx}{e^x + e^{-x}} \text{ converges}$$

1. Is it true that the improper integral $\int_a^\infty f(x)$ converges, whenever $f(x) \rightarrow 0$. Prove or give a counterexample.

This is NOT true. For example consider

$$\int_1^\infty \frac{1}{x} dx.$$

We know from the p-integral test that this integral diverges (case $p = 1$). Yet, $\lim_{x \rightarrow \infty} 1/x = 0$.

2. Evaluate the following integrals:

Both integrals are improper of Type I. In both cases, we can explicitly compute the answer.

- $\int_e^\infty \frac{1}{x(\ln x + 3)^2} dx$

We need to compute

$$\lim_{b \rightarrow \infty} \int_e^b \frac{1}{x(\ln x + 3)^2} dx.$$

We first evaluate the integral by substitution by letting $u = \ln x$, $\frac{du}{dx} = 1/x$, to get (up to a constant)

$$\int \frac{du}{(u + 3)^2} = -(u + 3)^{-2+1} = -(\ln x + 3)^{-1}$$

We now compute the limit:

$$\lim_{b \rightarrow \infty} [-(\ln x + 3)^{-1}]_e^b = 0 + 1/4 = 1/4$$

- $\int_0^\infty \frac{\sqrt{\tan^{-1}(x)}}{1+x^2} dx$

We need to compute

$$\lim_{b \rightarrow \infty} \int_0^b \frac{\sqrt{\tan^{-1}(x)}}{1+x^2} dx.$$

We evaluate this integral by substitution. Let $u = \tan^{-1} x$, then $du = \frac{1}{1+x^2} dx$, obtaining (up to a constant):

$$\int \frac{\sqrt{\tan^{-1}(x)}}{1+x^2} dx = \int u^{1/2} du = (2/3)u^{3/2} = (2/3)(\tan^{-1} x)^{3/2}.$$

We can now take the limit

$$\lim_{b \rightarrow \infty} [(2/3)(\tan^{-1} x)^{3/2}]_0^b = (2/3) \lim_{b \rightarrow \infty} [(\tan^{-1} b)^{3/2} - (\tan^{-1} 0)^{3/2}] = (2/3) [(\frac{\pi}{2})^{3/2} - 0]$$

3. Determine whether the following integrals diverge or converge:

- $\int_1^{\infty} \frac{x}{(x^3+2)^2} dx$

We first observe that this integral is roughly like $\sim \int \frac{x}{x^6} dx = \int \frac{1}{x^5} dx$. This suggests to us that we can use the limit comparison test (LCT) with $g(x) = \frac{1}{x^5}$ (remember that $f(x) = \frac{x}{(x^3+2)^2}$).

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^6}{(x^3+2)^2} = 1, \quad \text{using l'Hopital's rule.}$$

So the corresponding integrals behave the same. But we know by the p-integral test that $\int_1^{\infty} \frac{1}{x^5} dx$ converges. So the original integral converges.

- $\int_{-\infty}^{-3} \frac{1}{x^3} dx$

We first observe that since the integrand is an odd function, we can write

$$\int_{-\infty}^{-3} \frac{1}{x^3} dx = - \int_3^{\infty} \frac{1}{x^3} dx.$$

We know that $\int_3^{\infty} \frac{1}{x^3} dx$ converges being a p - integral with $p = 3$. So the original integral converges.

- $\int_{-\infty}^{\infty} \frac{x}{e^x+e^{-x}} dx$

Since the integrand is odd, it is enough to analyze $\int_0^{\infty} \frac{x}{e^x+e^{-x}} dx$.

We can do LCT with $g(x) = \frac{x}{e^x}$, and conclude that the integrals behave alike. So let us now look at $\int_0^{\infty} \frac{x}{e^x} dx$. One way would be to do DCT with $\int_0^{\infty} e^{-x/2} dx$ and conclude it is convergent so $\int_0^{\infty} \frac{x}{e^x+e^{-x}} dx$ converges to a finite limit $L > 0$. This means that $\int_{-\infty}^0 \frac{x}{e^x+e^{-x}} dx$ converges to $-L$, which means that the original integral goes to 0.

- $\int_2^{\infty} \frac{1}{\sqrt{x} \ln x} dx$

We first note that for all $x > 0$, $\ln x < \sqrt{x}$. Using this fact, we see that

$$\int_2^{\infty} \frac{1}{\sqrt{x} \ln x} dx \geq \int_2^{\infty} \frac{1}{\sqrt{x} \sqrt{x}} dx = \int_2^{\infty} \frac{1}{x} dx.$$

So the original integral diverges by the direct comparison test and the fact that $\int_2^{\infty} \frac{1}{x} dx$ is a divergent p - integral ($p = 1$).

- $\int_1^{\infty} \frac{x-e^x}{x^2} dx$.

We first observe that since the numerator is dominated by the exponential, we

suspect this integral to diverge. We can prove so by using the limit comparison test and comparing the integral with $\int \frac{e^x}{x^2}$. But before doing that, we need to insure that the integrand is positive (to be able to use the limit comparison test). We first rewrite

$$\int_1^{\infty} \frac{x - e^x}{x^2} dx = - \int_1^{\infty} \frac{e^x - x}{x^2} dx.$$

and work with

$$\int_1^{\infty} \frac{e^x - x}{x^2} dx.$$

Now let $g(x) = \frac{e^x}{x^2}$ and $f(x) = \frac{e^x - x}{x^2}$. We compute

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{e^x - x}{e^x} = 1.$$

So the integrals of f and g behave alike. Then we see that

$$\int_1^{\infty} \frac{e^x}{x^2} dx$$

diverges because the integrand tends to infinity as $x \rightarrow \infty$. So we conclude that the original integral diverges.

- This is an integral of mixed type, so we split it as

$$\int_0^1 \frac{1}{x^2} dx + \int_1^{\infty} \frac{1}{x^2} dx.$$

The second is a convergent p-integral. As for the first one, we integrate it explicitly and evaluate the limit

$$\lim_{b \rightarrow 0} \int_b^1 x^{-2} dx = \lim_{b \rightarrow 0} -x^{-1} \Big|_b^1 = \infty.$$

We conclude that the original integral diverges.