Chapter II

Linear Combinations & Linear Independence

section 2.1

Vectors in $\mathbb{R}^n$

In this section we shall extend many familiar ideas beyond 3 space by working analytically rather than geometrically since our Geometric visualization doesn't extend beyond 3 space

$\mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} / x_1, x_2 \text{ are real numbers} \right\}$ (Euclidean 2-space)

set of all vectors with two real entries

$\mathbb{R}^3 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} / x_1, x_2, x_3 \text{ are real numbers} \right\}$ (Euclidean 3-space)

set of all vectors with three real entries.

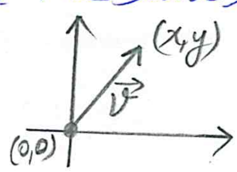
definition 2.1.1

The Euclidean n-space denoted by $\mathbb{R}^n$ is the set of all vectors with n entries (The entries of a vector are called Components of the vector) $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ; \underbrace{x_i \text{ a real number}}_{x_i \in \mathbb{R}} \ \forall i = 1, \ldots, n \right\}$

Geometric representation of a vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$)

The vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is the directed line segment from the origin $(0,0)$ (initial pt) to the pt (x,y) (terminal pt)

The length "L" of the vector is the length of the line segment joining $(0,0)$ & (x,y) : $L = \sqrt{x^2 + y^2}$



NB

1. When the initial pt of a vector is the origin we say that the vector is in standard position
2. a vector may have more than one representation.

example: the directed line segment between $(0,0)$ & $(1,2)$ & between $(2,2)$ & $(3,4)$ and between $(0,1)$ & $(1,3)$ etc --- are many representations of the same vector.

definition 21.2

let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$ & $k \in \mathbb{R}$

1. The equality $u = v$ is defined by $u_i = v_i \ \forall i = 1, \dots, n$
   (ie $u_1 = v_1, u_2 = v_2, \dots, u_n = v_n$)

2. The sum $u+v$ is defined by $u+v = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix}$

3. The scalar multiplication of k and u is defined by:
   $$k.u = \begin{pmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{pmatrix}$$

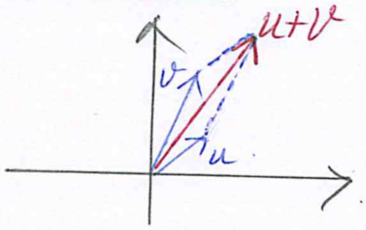
4. There exists the zero vector of $\mathbb{R}^n$ denoted by O or $O_{\mathbb{R}^n}$ which is a vector in $\mathbb{R}^n$ ie of the form $O_{\mathbb{R}^n} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ satisfying
   $u + O_{\mathbb{R}^n} = u = O_{\mathbb{R}^n} + u$
   find $O_{\mathbb{R}^n}$
   take $u + O_{\mathbb{R}^n} = u \Rightarrow \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \Rightarrow \begin{pmatrix} u_1+\alpha_1 \\ \vdots \\ u_n+\alpha_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$
   $\Rightarrow u_i + \alpha_i = u_i \ \forall i = 1 \to n$
   $\Rightarrow \alpha_i = 0 \ \forall i = 1 \to n \Rightarrow O_{\mathbb{R}^n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
   (under the standard operation + on $\mathbb{R}^n$, $O_{\mathbb{R}^n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$)

5. for any $u \in \mathbb{R}^n$, there exists the additive inverse of u (or symetric of u) denoted by $(-u)$ which is a vector in $\mathbb{R}^n$. ie of the form $(-u) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$ satisfying $u + (-u) = O_{\mathbb{R}^n} = (-u) + u$.
   find $(-u)$ given u.
   take $u + (-u) = O_{\mathbb{R}^n} \Rightarrow \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1+\beta_1 \\ \vdots \\ u_n+\beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow u_i + \beta_i = 0 \ \forall i = 1 \to n.$
   $\Rightarrow \beta_i = -u_i \ \forall i = 1 \to n$
   $\Rightarrow (-u) = \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \end{pmatrix}$ (under the std operation + on $\mathbb{R}^n$ additive inverse of u, $(-u) = \begin{pmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{pmatrix}$)

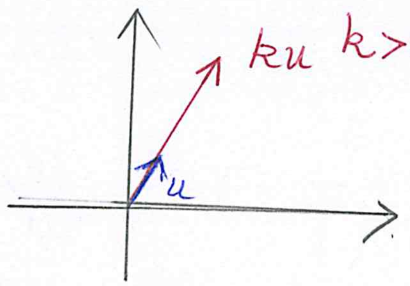
$\boxed{6}$ The difference $u - v = u + (-v) = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_n - v_n \end{pmatrix}$

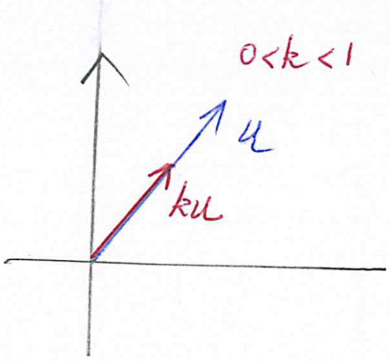
Geometric representation of the algebraic definition addition, subtraction & scalar multiplication (in $\mathbb{R}^2$)

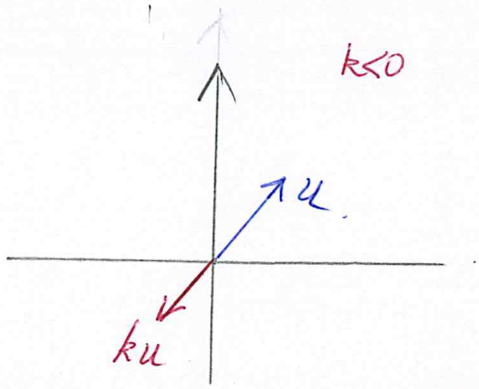
addition



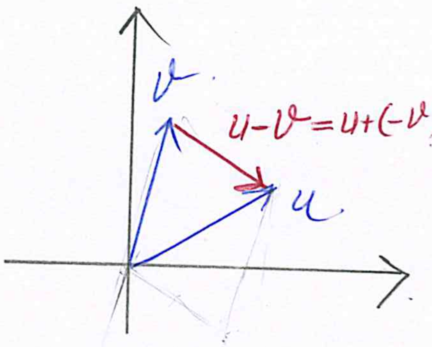
scalar multiplication







difference



Theorem 2.1.2.

Let u, v & w be vectors in $\mathbb{R}^n$ & let k, m scalars in $\mathbb{R}$. Then

1. $u+v \in \mathbb{R}^n$ ($\mathbb{R}^n$ closed under $+$)
2. $ku \in \mathbb{R}^n$ ($\mathbb{R}^n$ closed under $\cdot$)
3. $u+v = v+u$ ($+$ commutative)
4. $(u+v)+w = u+(v+w)$
5. there exists the zero vector of $\mathbb{R}^n$ denoted by $0_{\mathbb{R}^n}$ (vector in $\mathbb{R}^n$ of the form $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$) satisfying $u+0 = u = 0+u$
6. for any $u \in \mathbb{R}^n$, there exists the additive inverse of u (or symetric of u) denoted by $\underbrace{(-u)}_{\text{vector in } \mathbb{R}^n \text{ ie of the form } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}$ satisfying $u+(-u) = (-u)+u = 0_{\mathbb{R}^n}$
7. $k(u+v) = ku+kv$
8. $(k+m)u = ku+mu$
9. $(km)u = k(mu)$
10. $1.u = u$

Some useful properties

1. $\underset{\text{scalar}}{0}.u = 0.\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \overset{\text{vector}}{0_{\mathbb{R}^n}} \Rightarrow \underline{\underline{\overset{\text{scalar}}{0}.u = \overset{\text{vector}}{0}}}$
2. $k.\underset{\text{vector ie } 0_{\mathbb{R}^n}}{0} = k.\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0_{\mathbb{R}^n} \Rightarrow \underline{\underline{k.\underset{\text{vector}}{0} = \underset{\text{vector}}{0}}}$ (ie $k0_{\mathbb{R}^n} = 0_{\mathbb{R}^n}$)
3. $(-1).u = -\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{pmatrix} = (-u) \Rightarrow \underline{\underline{(-1).u = u}}$ (additive inverse)
4. if $ku = 0$ then $k = 0$ or $u = 0_{\mathbb{R}^n}$.
   (since $ku = 0 \Leftrightarrow k\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} ku_1 \\ \vdots \\ ku_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow ku_i = 0 \ \forall i = 1 \to n$
   $\Rightarrow k = 0$ or $u_i = 0 \ \forall i = 1 \to n$
   $\Rightarrow k = 0$ or $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0_{\mathbb{R}^n}$