CHAPTER 7 TRANSCENDENTAL FUNCTIONS

7.1 INVERSE FUNCTIONS AND THEIR DERIVATIVES

1. Yes one-to-one, the graph passes the horizontal line test.

2. Not one-to-one, the graph fails the horizontal line test.

3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.

4. Not one-to-one, the graph fails the horizontal line test.

5. Yes one-to-one, the graph passes the horizontal line test

6. Yes one-to-one, the graph passes the horizontal line test

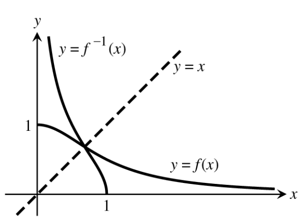
7. Not one-to-one since the horizontal line $y = 3$ intersects the graph an infinite number of times.

8. Yes one-to-one, the graph passes the horizontal line test

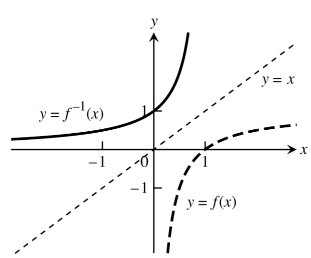
9. Yes one-to-one, the graph passes the horizontal line test

10. Not one-to-one since (for example) the horizontal line $y = 1$ intersects the graph twice.

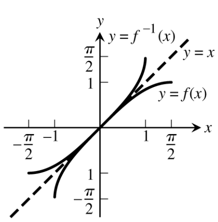
11. Domain: $0 < x \le 1$, Range: $0 \le y$



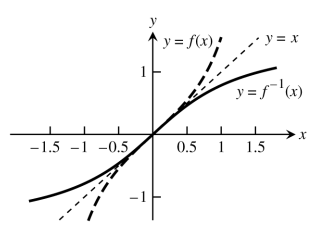
12. Domain: $x < 1$, Range: $y > 0$



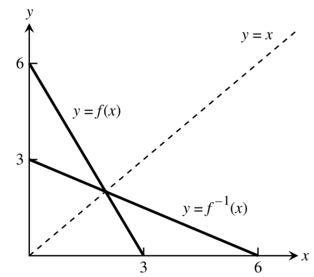
13. Domain: $-1 \le x \le 1$, Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



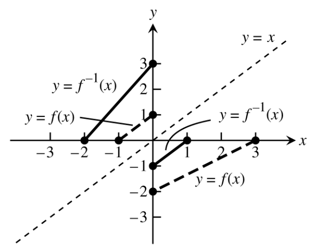
14. Domain: $-\infty < x < \infty$, Range: $-\frac{\pi}{2} < y \le \frac{\pi}{2}$



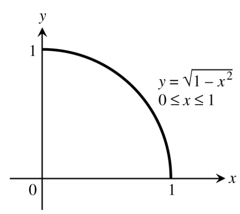
15. Domain: $0 \le x \le 6$, Range: $0 \le y \le 3$



16. Domain: $-2 \le x \le 1$, Range: $-1 \le y < 3$

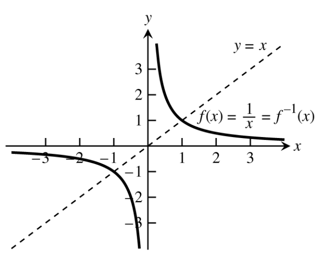


17. The graph is symmetric about $y = x$.



(b) $y = \sqrt{1 - x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \sqrt{1 - y^2} \Rightarrow y = \sqrt{1 - x^2} = f^{-1}(x)$

18. The graph is symmetric about $y = x$.



$y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x} = f^{-1}(x)$

19. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y - 1}$
Step 2: $y = \sqrt{x - 1} = f^{-1}(x)$

20. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$, since $x \le 0$.
Step 2: $y = -\sqrt{x} = f^{-1}(x)$

21. Step 1: $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y + 1)^{1/3}$
Step 2: $y = \sqrt[3]{x + 1} = f^{-1}(x)$

22. Step 1: $y = x^2 - 2x + 1 \Rightarrow y = (x - 1)^2 \Rightarrow \sqrt{y} = x - 1$, since $x \ge 1 \Rightarrow x = 1 + \sqrt{y}$
Step 2: $y = 1 + \sqrt{x} = f^{-1}(x)$

23. Step 1: $y = (x + 1)^2 \Rightarrow \sqrt{y} = x + 1$, since $x \ge -1 \Rightarrow x = \sqrt{y} - 1$
Step 2: $y = \sqrt{x} - 1 = f^{-1}(x)$

24. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$
Step 2: $y = x^{3/2} = f^{-1}(x)$

25. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$
Step 2: $y = \sqrt[5]{x} = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \left(x^{1/5}\right)^5 = x$ and $f^{-1}(f(x)) = \left(x^5\right)^{1/5} = x$

26. Step 1: $y = x^4 \Rightarrow x = y^{1/4}$
Step 2: $y = \sqrt[4]{x} = f^{-1}(x)$;
Domain of f^{-1}: $x \ge 0$, Range of f^{-1}: $y \ge 0$;
$f(f^{-1}(x)) = \left(x^{1/4}\right)^4 = x$ and $f^{-1}(f(x)) = \left(x^4\right)^{1/4} = x$

27. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y-1)^{1/3}$
Step 2: $y = \sqrt[3]{x-1} = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \left((x-1)^{1/3}\right)^3 + 1 = (x-1) + 1 = x$ and $f^{-1}(f(x)) = \left((x^3+1)-1\right)^{1/3} = \left(x^3\right)^{1/3} = x$

28. Step 1: $y = \frac{1}{2}x - \frac{7}{2} \Rightarrow \frac{1}{2}x = y + \frac{7}{2} \Rightarrow x = 2y + 7$
Step 2: $y = 2x + 7 = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \frac{1}{2}(2x+7) - \frac{7}{2} = \left(x + \frac{7}{2}\right) - \frac{7}{2} = x$ and $f^{-1}(f(x)) = 2\left(\frac{1}{2}x - \frac{7}{2}\right) + 7 = (x-7) + 7 = x$

29. Step 1: $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \frac{1}{\sqrt{y}}$
Step 2: $y = \frac{1}{\sqrt{x}} = f^{-1}(x)$
Domain of f^{-1}: $x > 0$, Range of f^{-1}: $y > 0$;
$f(f^{-1}(x)) = \frac{1}{\left(\frac{1}{\sqrt{x}}\right)^2} = \frac{1}{\left(\frac{1}{x}\right)} = x$ and $f^{-1}(f(x)) = \frac{1}{\sqrt{\frac{1}{x^2}}} = \frac{1}{\left(\frac{1}{x}\right)} = x$ since $x > 0$

30. Step 1: $y = \frac{1}{x^3} \Rightarrow x^3 = \frac{1}{y} \Rightarrow x = \frac{1}{y^{1/3}}$
Step 2: $y = \frac{1}{x^{1/3}} = \sqrt[3]{\frac{1}{x}} = f^{-1}(x)$;
Domain of f^{-1}: $x \ne 0$, Range of f^{-1}: $y \ne 0$;
$f(f^{-1}(x)) = \frac{1}{\left(x^{-1/3}\right)^3} = \frac{1}{x^{-1}} = x$ and $f^{-1}(f(x)) = \left(\frac{1}{x^3}\right)^{-1/3} = \left(\frac{1}{x}\right)^{-1} = x$

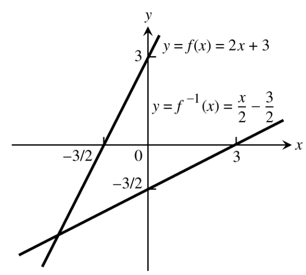
31. Step 1: $y = \frac{x+3}{x-2} \Rightarrow y(x-2) = x + 3 \Rightarrow xy - 2y = x + 3 \Rightarrow xy - x = 2y + 3 \Rightarrow x = \frac{2y+3}{y-1}$
Step 2: $y = \frac{2x+3}{x-1} = f^{-1}(x)$;
Domain of f^{-1}: $x \ne 1$, Range of f^{-1}: $y \ne 2$;
$f(f^{-1}(x)) = \frac{\left(\frac{2x+3}{x-1}\right)+3}{\left(\frac{2x+3}{x-1}\right)-2} = \frac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)} = \frac{5x}{5} = x$ and $f^{-1}(f(x)) = \frac{2\left(\frac{x+3}{x-2}\right)+3}{\left(\frac{x+3}{x-2}\right)-1} = \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)} = \frac{5x}{5} = x$

32. Step 1: $y = \frac{\sqrt{x}}{\sqrt{x}-3} \Rightarrow y\left(\sqrt{x} - 3\right) = \sqrt{x} \Rightarrow y\sqrt{x} - 3y = \sqrt{x} \Rightarrow y\sqrt{x} - \sqrt{x} = 3y \Rightarrow x = \left(\frac{3y}{y-1}\right)^2$
Step 2: $y = \left(\frac{3x}{x-1}\right)^2 = f^{-1}(x)$;
Domain of f^{-1}: $(-\infty, 0] \cup (1, \infty)$, Range of f^{-1}: $[0, 9) \cup (9, \infty)$;
$f(f^{-1}(x)) = \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3}$; If $x > 1$ or $x \le 0 \Rightarrow \frac{3x}{x-1} \ge 0 \Rightarrow \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-3} = \frac{3x}{3x-3(x-1)} = \frac{3x}{3} = x$ and
$f^{-1}(f(x)) = \left(\frac{3\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)}{\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)-1}\right)^2 = \frac{9x}{\left(\sqrt{x}-\left(\sqrt{x}-3\right)\right)^2} = \frac{9x}{9} = x$

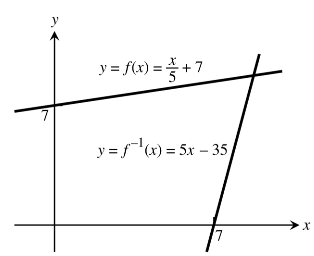
33. Step 1: $y = x^2 - 2x,\ x \le 1 \Rightarrow y + 1 = (x-1)^2,\ x \le 1 \Rightarrow -\sqrt{y+1} = x - 1,\ x \le 1 \Rightarrow x = 1 - \sqrt{y+1}$
Step 2: $y = 1 - \sqrt{x+1} = f^{-1}(x)$;
Domain of f^{-1}: $[-1, \infty)$, Range of f^{-1}: $(-\infty, 1]$;
$f\left(f^{-1}(x)\right) = \left(1 - \sqrt{x+1}\right)^2 - 2\left(1 - \sqrt{x+1}\right) = 1 - 2\sqrt{x+1} + x + 1 - 2 + 2\sqrt{x+1} = x$ and
$f^{-1}(f(x)) = 1 - \sqrt{(x^2 - 2x) + 1},\ x \le 1 = 1 - \sqrt{(x-1)^2},\ x \le 1 = 1 - |x - 1| = 1 - (1 - x) = x$

34. Step 1: $y = (2x^3 + 1)^{1/5} \Rightarrow y^5 = 2x^3 + 1 \Rightarrow y^5 - 1 = 2x^3 \Rightarrow \frac{y^5 - 1}{2} = x^3 \Rightarrow x = \sqrt[3]{\frac{y^5 - 1}{2}}$
Step 2: $y = \sqrt[3]{\frac{x^5 - 1}{2}} = f^{-1}(x)$;
Domain of f^{-1}: $(-\infty, \infty)$, Range of f^{-1}: $(-\infty, \infty)$;
$f\left(f^{-1}(x)\right) = \left(2\left(\sqrt[3]{\frac{x^5-1}{2}}\right)^3 + 1\right)^{1/5} = \left(2\left(\frac{x^5-1}{2}\right) + 1\right)^{1/5} = \left((x^5 - 1) + 1\right)^{1/5} = \left(x^5\right)^{1/5} = x$ and
$f^{-1}(f(x)) = \sqrt[3]{\frac{\left[(2x^3+1)^{1/5}\right]^5 - 1}{2}} = \sqrt[3]{\frac{(2x^3+1) - 1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = x$

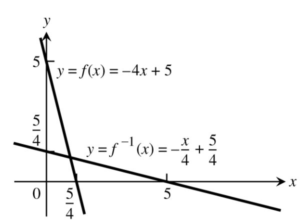
35. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$
$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$
(c) $\left.\frac{df}{dx}\right|_{x=-1} = 2,\ \left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{2}$

(b) 

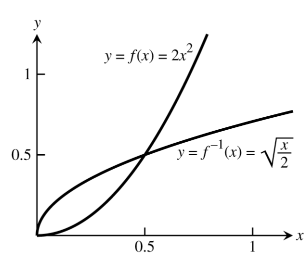
36. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$
$\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$
(c) $\left.\frac{df}{dx}\right|_{x=-1} = \frac{1}{5},\ \left.\frac{df^{-1}}{dx}\right|_{x=34/5} = 5$

(b) 

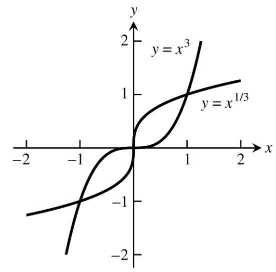
37. (a) $y = 5 - 4x \Rightarrow 4x = 5 - y$
$\Rightarrow x = \frac{5}{4} - \frac{y}{4} \Rightarrow f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$
(c) $\left.\frac{df}{dx}\right|_{x=1/2} = -4,\ \left.\frac{df^{-1}}{dx}\right|_{x=3} = -\frac{1}{4}$

(b) 

38. (a) $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y$
$\Rightarrow x = \frac{1}{\sqrt{2}}\sqrt{y} \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$
(c) $\left.\frac{df}{dx}\right|_{x=5} = 4x\big|_{x=5} = 20,$
$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \left.\frac{1}{2\sqrt{2}}x^{-1/2}\right|_{x=50} = \frac{1}{20}$

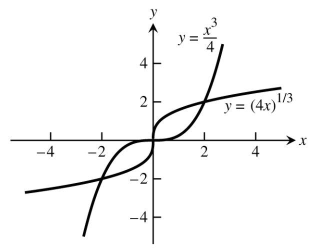
(b) 

39. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$, $g(f(x)) = \sqrt[3]{x^3} = x$
(b)



(c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3$, $f'(-1) = 3$;
$g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(1) = \frac{1}{3}$, $g'(-1) = \frac{1}{3}$
(d) The line $y = 0$ is tangent to $f(x) = x^3$ at $(0, 0)$;
the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at $(0, 0)$

40. (a) $h(k(x)) = \frac{1}{4}\left((4x)^{1/3}\right)^3 = x$,
$k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$
(b)



(c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3$, $h'(-2) = 3$;
$k'(x) = \frac{4}{3}(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3}$, $k'(-2) = \frac{1}{3}$
(d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at $(0, 0)$;
the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at $(0, 0)$

41. $\frac{df}{dx} = 3x^2 - 6x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(3)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=3} = \frac{1}{9}$

42. $\frac{df}{dx} = 2x - 4 \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=5} = \frac{1}{6}$

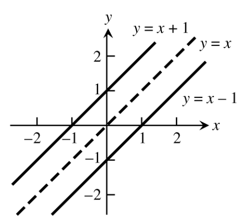
43. $\left.\frac{df^{-1}}{dx}\right|_{x=4} = \left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=2} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

44. $\left.\frac{dg^{-1}}{dx}\right|_{x=0} = \left.\frac{dg^{-1}}{dx}\right|_{x=f(0)} = \left.\frac{1}{\frac{dg}{dx}}\right|_{x=0} = \frac{1}{2}$

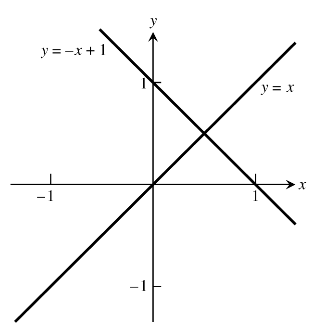
45. (a) $y = mx \Rightarrow x = \frac{1}{m}y \Rightarrow f^{-1}(x) = \frac{1}{m}x$
(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

46. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

47. (a) $y = x + 1 \Rightarrow x = y - 1 \Rightarrow f^{-1}(x) = x - 1$
(b) $y = x + b \Rightarrow x = y - b \Rightarrow f^{-1}(x) = x - b$
(c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.



48. (a) $y = -x + 1 \Rightarrow x = -y + 1 \Rightarrow f^{-1}(x) = 1 - x$;
the lines intersect at a right angle
(b) $y = -x + b \Rightarrow x = -y + b \Rightarrow f^{-1}(x) = b - x$;
the lines intersect at a right angle
(c) Such a function is its own inverse.



49. Let $x_1 \neq x_2$ be two numbers in the domain of an increasing function f. Then, either $x_1 < x_2$ or $x_1 > x_2$ which implies $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, since f(x) is increasing. In either case, $f(x_1) \neq f(x_2)$ and f is one-to-one. Similar arguments hold if f is decreasing.

50. f(x) is increasing since $x_2 > x_1 \Rightarrow \frac{1}{3}x_2 + \frac{5}{6} > \frac{1}{3}x_1 + \frac{5}{6}$; $\frac{df}{dx} = \frac{1}{3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

51. f(x) is increasing since $x_2 > x_1 \Rightarrow 27x_2^3 > 27x_1^3$; $y = 27x^3 \Rightarrow x = \frac{1}{3}y^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{3}x^{1/3}$; $\frac{df}{dx} = 81x^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{81x^2}\Big|_{\frac{1}{3}x^{1/3}} = \frac{1}{9x^{2/3}} = \frac{1}{9}x^{-2/3}$

52. f(x) is decreasing since $x_2 > x_1 \Rightarrow 1 - 8x_2^3 < 1 - 8x_1^3$; $y = 1 - 8x^3 \Rightarrow x = \frac{1}{2}(1-y)^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{2}(1-x)^{1/3}$; $\frac{df}{dx} = -24x^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-24x^2}\Big|_{\frac{1}{2}(1-x)^{1/3}} = \frac{-1}{6(1-x)^{2/3}} = -\frac{1}{6}(1-x)^{-2/3}$

53. f(x) is decreasing since $x_2 > x_1 \Rightarrow (1-x_2)^3 < (1-x_1)^3$; $y = (1-x)^3 \Rightarrow x = 1 - y^{1/3} \Rightarrow f^{-1}(x) = 1 - x^{1/3}$; $\frac{df}{dx} = -3(1-x)^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-3(1-x)^2}\Big|_{1-x^{1/3}} = \frac{-1}{3x^{2/3}} = -\frac{1}{3}x^{-2/3}$

54. f(x) is increasing since $x_2 > x_1 \Rightarrow x_2^{5/3} > x_1^{5/3}$; $y = x^{5/3} \Rightarrow x = y^{3/5} \Rightarrow f^{-1}(x) = x^{3/5}$; $\frac{df}{dx} = \frac{5}{3}x^{2/3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\frac{5}{3}x^{2/3}}\Big|_{x^{3/5}} = \frac{3}{5x^{2/5}} = \frac{3}{5}x^{-2/5}$

55. The function g(x) is also one-to-one. The reasoning: f(x) is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $-f(x_1) \neq -f(x_2)$ and therefore $g(x_1) \neq g(x_2)$. Therefore g(x) is one-to-one as well.

56. The function h(x) is also one-to-one. The reasoning: f(x) is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $\frac{1}{f(x_1)} \neq \frac{1}{f(x_2)}$, and therefore $h(x_1) \neq h(x_2)$.

57. The composite is one-to-one also. The reasoning: If $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$ because g is one-to-one. Since $g(x_1) \neq g(x_2)$, we also have $f(g(x_1)) \neq f(g(x_2))$ because f is one-to-one. Thus, $f \circ g$ is one-to-one because $x_1 \neq x_2 \Rightarrow f(g(x_1)) \neq f(g(x_2))$.

58. Yes, g must be one-to-one. If g were not one-to-one, there would exist numbers $x_1 \neq x_2$ in the domain of g with $g(x_1) = g(x_2)$. For these numbers we would also have $f(g(x_1)) = f(g(x_2))$, contradicting the assumption that $f \circ g$ is one-to-one.

59. $(g \circ f)(x) = x \Rightarrow g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

60. $W(a) = \int_{f(a)}^{f(a)} \pi\left[\left(f^{-1}(y)\right)^2 - a^2\right] dy = 0 = \int_a^a 2\pi x[f(a) - f(x)]\,dx = S(a)$; $W'(t) = \pi\left[\left(f^{-1}(f(t))\right)^2 - a^2\right] f'(t)$ $= \pi\left(t^2 - a^2\right) f'(t)$; also $S(t) = 2\pi f(t)\int_a^t x\,dx - 2\pi\int_a^t xf(x)\,dx = \left[\pi f(t)t^2 - \pi f(t)a^2\right] - 2\pi\int_a^t xf(x)\,dx \Rightarrow S'(t)$ $= \pi t^2 f'(t) + 2\pi t f(t) - \pi a^2 f'(t) - 2\pi t f(t) = \pi\left(t^2 - a^2\right) f'(t) \Rightarrow W'(t) = S'(t)$. Therefore, $W(t) = S(t)$ for all $t \in [a, b]$.

61-68. Example CAS commands:

Maple:

```
with( plots );#63
f := x -> sqrt(3*x-2);
domain := 2/3 .. 4;
x0 := 3;
Df := D(f);                        # (a)
```

```
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f '(x)"],
      title="#61(a) (Section 7.1)" );
q1 := solve( y=f(x), x );          # (b)
g := unapply( q1, y );
m1 := Df(x0);                      # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                    # (d)
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain);     # (e)
p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#63(e) (Section 7.1)" );
```

Mathematica: (assigned function and values for a, b, and x0 may vary)

If a function requires the odd root of a negative number, begin by loading the RealOnly package that allows Mathematica to do this. See section 2.5 for details.

```
<<Miscellaneous `RealOnly`
Clear[x, y]
{a,b} = {−2, 1}; x0 = 1/2 ;
f[x_] = (3x + 2) / (2x − 11)
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[y == f[x], x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x-x0)
gtan[y_] = x0 + 1/ f'[x0] (y − y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]},{x, a, b},
Epilog → Line[{{x0, y0},{y0, x0}}], PlotRange → {{a,b},{a,b}}, AspectRatio → Automatic]
```

69-70. Example CAS commands:

Maple:

```
with( plots );
eq := cos(y) = x^(1/5);
domain := 0 .. 1;
x0 := 1/2;
f := unapply( solve( eq, y ), x );   # (a)
Df := D(f);
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f '(x)"],
      title="#70(a) (Section 7.1)" );
q1 := solve( eq, x );              # (b)
g := unapply( q1, y );
m1 := Df(x0);                      # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                    # (d)
```

```
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain);        # (e)
p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#70(e) (Section 7.1)" );
```

Mathematica: (assigned function and values for a, b, and x0 may vary)

For problems 69 and 70, the code is just slightly altered. At times, different "parts" of solutions need to be used, as in the definitions of f[x] and g[y]

```
Clear[x, y]
{a,b} = {0, 1}; x0 = 1/2 ;
eqn = Cos[y] == x^(1/5)
soly = Solve[eqn, y]
f[x_] = y /. soly[[2]]
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[eqn, x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x − x0)
gtan[y_] = x0 + 1/ f'[x0] (y − y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]},{x, a, b},
Epilog → Line[{{x0, y0},{y0, x0}}], PlotRange → {{a, b}, {a, b}}, AspectRatio → Automatic]
```

7.2 NATURAL LOGARITHMS

1. (a) $\ln 0.75 = \ln \frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2 \ln 2$
   (b) $\ln \frac{4}{9} = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2 \ln 2 - 2 \ln 3$
   (c) $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$
   (d) $\ln \sqrt[3]{9} = \frac{1}{3} \ln 9 = \frac{1}{3} \ln 3^2 = \frac{2}{3} \ln 3$
   (e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2} \ln 2$
   (f) $\ln \sqrt{13.5} = \frac{1}{2} \ln 13.5 = \frac{1}{2} \ln \frac{27}{2} = \frac{1}{2} (\ln 3^3 - \ln 2) = \frac{1}{2} (3 \ln 3 - \ln 2)$

2. (a) $\ln \frac{1}{125} = \ln 1 - 3 \ln 5 = -3 \ln 5$
   (b) $\ln 9.8 = \ln \frac{49}{5} = \ln 7^2 - \ln 5 = 2 \ln 7 - \ln 5$
   (c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2} \ln 7$
   (d) $\ln 1225 = \ln 35^2 = 2 \ln 35 = 2 \ln 5 + 2 \ln 7$
   (e) $\ln 0.056 = \ln \frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3 \ln 5$
   (f) $\frac{\ln 35 + \ln \frac{1}{7}}{\ln 25} = \frac{\ln 5 + \ln 7 - \ln 7}{2 \ln 5} = \frac{1}{2}$

3. (a) $\ln \sin \theta - \ln \left(\frac{\sin \theta}{5}\right) = \ln \left(\frac{\sin \theta}{\left(\frac{\sin \theta}{5}\right)}\right) = \ln 5$
   (b) $\ln (3x^2 - 9x) + \ln \left(\frac{1}{3x}\right) = \ln \left(\frac{3x^2 - 9x}{3x}\right) = \ln (x - 3)$
   (c) $\frac{1}{2} \ln (4t^4) - \ln 2 = \ln \sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln \left(\frac{2t^2}{2}\right) = \ln (t^2)$

4. (a) $\ln \sec \theta + \ln \cos \theta = \ln [(\sec \theta)(\cos \theta)] = \ln 1 = 0$
   (b) $\ln (8x + 4) - \ln 2^2 = \ln (8x + 4) - \ln 4 = \ln \left(\frac{8x + 4}{4}\right) = \ln (2x + 1)$
   (c) $3 \ln \sqrt[3]{t^2 - 1} - \ln (t + 1) = 3 \ln (t^2 - 1)^{1/3} - \ln (t + 1) = 3 \left(\frac{1}{3}\right) \ln (t^2 - 1) - \ln (t + 1) = \ln \left(\frac{(t + 1)(t - 1)}{(t + 1)}\right)$
   $= \ln (t - 1)$

5. $y = \ln 3x \;\Rightarrow\; y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$

6. $y = \ln kx \;\Rightarrow\; y' = \left(\frac{1}{kx}\right)(k) = \frac{1}{x}$

7. $y = \ln\left(t^2\right) \;\Rightarrow\; \frac{dy}{dt} = \left(\frac{1}{t^2}\right)(2t) = \frac{2}{t}$

8. $y = \ln\left(t^{3/2}\right) \;\Rightarrow\; \frac{dy}{dt} = \left(\frac{1}{t^{3/2}}\right)\left(\frac{3}{2}\,t^{1/2}\right) = \frac{3}{2t}$

9. $y = \ln\frac{3}{x} = \ln 3x^{-1} \;\Rightarrow\; \frac{dy}{dx} = \left(\frac{1}{3x^{-1}}\right)\left(-3x^{-2}\right) = -\frac{1}{x}$

10. $y = \ln\frac{10}{x} = \ln 10x^{-1} \;\Rightarrow\; \frac{dy}{dx} = \left(\frac{1}{10x^{-1}}\right)\left(-10x^{-2}\right) = -\frac{1}{x}$

11. $y = \ln(\theta+1) \;\Rightarrow\; \frac{dy}{d\theta} = \left(\frac{1}{\theta+1}\right)(1) = \frac{1}{\theta+1}$

12. $y = \ln(2\theta+2) \;\Rightarrow\; \frac{dy}{d\theta} = \left(\frac{1}{2\theta+2}\right)(2) = \frac{1}{\theta+1}$

13. $y = \ln x^3 \;\Rightarrow\; \frac{dy}{dx} = \left(\frac{1}{x^3}\right)\left(3x^2\right) = \frac{3}{x}$

14. $y = (\ln x)^3 \;\Rightarrow\; \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$

15. $y = t(\ln t)^2 \;\Rightarrow\; \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t)\cdot\frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t\ln t}{t} = (\ln t)^2 + 2\ln t$

16. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \;\Rightarrow\; \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2}\,t(\ln t)^{-1/2}\cdot\frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t}$
$= (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

17. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16} \;\Rightarrow\; \frac{dy}{dx} = x^3\ln x + \frac{x^4}{4}\cdot\frac{1}{x} - \frac{4x^3}{16} = x^3\ln x$

18. $y = \left(x^2\ln x\right)^4 \Rightarrow\; \frac{dy}{dx} = 4\left(x^2\ln x\right)^3\left(x^2\cdot\frac{1}{x} + 2x\ln x\right) = 4x^6(\ln x)^3(x + 2x\ln x) = 4x^7(\ln x)^3 + 8x^7(\ln x)^4$

19. $y = \frac{\ln t}{t} \;\Rightarrow\; \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1-\ln t}{t^2}$

20. $y = \frac{1+\ln t}{t} \;\Rightarrow\; \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1+\ln t)(1)}{t^2} = \frac{1-1-\ln t}{t^2} = -\frac{\ln t}{t^2}$

21. $y = \frac{\ln x}{1+\ln x} \;\Rightarrow\; y' = \frac{(1+\ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1+\ln x)^2} = \frac{1}{x(1+\ln x)^2}$

22. $y = \frac{x\ln x}{1+\ln x} \;\Rightarrow\; y' = \frac{(1+\ln x)\left(\ln x + x\cdot\frac{1}{x}\right) - (x\ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \frac{(1+\ln x)^2 - \ln x}{(1+\ln x)^2} = 1 - \frac{\ln x}{(1+\ln x)^2}$

23. $y = \ln(\ln x) \;\Rightarrow\; y' = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{1}{x\ln x}$

24. $y = \ln(\ln(\ln x)) \;\Rightarrow\; y' = \frac{1}{\ln(\ln x)}\cdot\frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)}\cdot\frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x) = \frac{1}{x(\ln x)\ln(\ln x)}$

25. $y = \theta[\sin(\ln\theta) + \cos(\ln\theta)] \;\Rightarrow\; \frac{dy}{d\theta} = [\sin(\ln\theta) + \cos(\ln\theta)] + \theta\left[\cos(\ln\theta)\cdot\frac{1}{\theta} - \sin(\ln\theta)\cdot\frac{1}{\theta}\right]$
$= \sin(\ln\theta) + \cos(\ln\theta) + \cos(\ln\theta) - \sin(\ln\theta) = 2\cos(\ln\theta)$

26. $y = \ln(\sec\theta + \tan\theta) \;\Rightarrow\; \frac{dy}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} = \frac{\sec\theta(\tan\theta + \sec\theta)}{\tan\theta + \sec\theta} = \sec\theta$

27. $y = \ln\frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2}\ln(x+1) \;\Rightarrow\; y' = -\frac{1}{x} - \frac{1}{2}\left(\frac{1}{x+1}\right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$

28. $y = \frac{1}{2}\ln\frac{1+x}{1-x} = \frac{1}{2}[\ln(1+x) - \ln(1-x)] \;\Rightarrow\; y' = \frac{1}{2}\left[\frac{1}{1+x} - \left(\frac{1}{1-x}\right)(-1)\right] = \frac{1}{2}\left[\frac{1-x+1+x}{(1+x)(1-x)}\right] = \frac{1}{1-x^2}$

29. $y = \frac{1+\ln t}{1-\ln t} \Rightarrow \frac{dy}{dt} = \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$

30. $y = \sqrt{\ln \sqrt{t}} = \left(\ln t^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{d}{dt}\left(\ln t^{1/2}\right) = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{d}{dt}\left(t^{1/2}\right)$
$= \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{1}{2} t^{-1/2} = \frac{1}{4t\sqrt{\ln \sqrt{t}}}$

31. $y = \ln(\sec(\ln \theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\sec(\ln \theta)) = \frac{\sec(\ln \theta)\tan(\ln \theta)}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\ln \theta) = \frac{\tan(\ln \theta)}{\theta}$

32. $y = \ln \frac{\sqrt{\sin \theta \cos \theta}}{1 + 2\ln \theta} = \frac{1}{2}(\ln \sin \theta + \ln \cos \theta) - \ln(1 + 2\ln \theta) \Rightarrow \frac{dy}{d\theta} = \frac{1}{2}\left(\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}\right) - \frac{\frac{2}{\theta}}{1 + 2\ln \theta}$
$= \frac{1}{2}\left[\cot \theta - \tan \theta - \frac{4}{\theta(1 + 2\ln \theta)}\right]$

33. $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right) = 5\ln(x^2+1) - \frac{1}{2}\ln(1-x) \Rightarrow y' = \frac{5 \cdot 2x}{x^2+1} - \frac{1}{2}\left(\frac{1}{1-x}\right)(-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$

34. $y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \frac{1}{2}[5\ln(x+1) - 20\ln(x+2)] \Rightarrow y' = \frac{1}{2}\left(\frac{5}{x+1} - \frac{20}{x+2}\right) = \frac{5}{2}\left[\frac{(x+2) - 4(x+1)}{(x+1)(x+2)}\right]$
$= -\frac{5}{2}\left[\frac{3x+2}{(x+1)(x+2)}\right]$

35. $y = \int_{x^2/2}^{x^2} \ln \sqrt{t}\, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt{x^2}\right) \cdot \frac{d}{dx}(x^2) - \left(\ln \sqrt{\frac{x^2}{2}}\right) \cdot \frac{d}{dx}\left(\frac{x^2}{2}\right) = 2x\ln|x| - x\ln\frac{|x|}{\sqrt{2}}$

36. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\, dt \Rightarrow \frac{dy}{dx} = \left(\ln \sqrt[3]{x}\right) \cdot \frac{d}{dx}\left(\sqrt[3]{x}\right) - \left(\ln \sqrt{x}\right) \cdot \frac{d}{dx}\left(\sqrt{x}\right) = \left(\ln \sqrt[3]{x}\right)\left(\frac{1}{3}x^{-2/3}\right) - \left(\ln \sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)$
$= \frac{\ln \sqrt[3]{x}}{3\sqrt[3]{x^2}} - \frac{\ln \sqrt{x}}{2\sqrt{x}}$

37. $\int_{-3}^{-2} \frac{1}{x}\, dx = [\ln|x|]_{-3}^{-2} = \ln 2 - \ln 3 = \ln \frac{2}{3}$

38. $\int_{-1}^{0} \frac{3}{3x-2}\, dx = [\ln|3x-2|]_{-1}^{0} = \ln 2 - \ln 5 = \ln \frac{2}{5}$

39. $\int \frac{2y}{y^2-25}\, dy = \ln|y^2 - 25| + C$

40. $\int \frac{8r}{4r^2-5}\, dr = \ln|4r^2 - 5| + C$

41. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\, dt = [\ln|2-\cos t|]_0^{\pi} = \ln 3 - \ln 1 = \ln 3$; or let $u = 2 - \cos t \Rightarrow du = \sin t\, dt$ with $t = 0$
$\Rightarrow u = 1$ and $t = \pi \Rightarrow u = 3 \Rightarrow \int_0^{\pi} \frac{\sin t}{2-\cos t}\, dt = \int_1^3 \frac{1}{u}\, du = [\ln|u|]_1^3 = \ln 3 - \ln 1 = \ln 3$

42. $\int_0^{\pi/3} \frac{4\sin \theta}{1-4\cos \theta}\, d\theta = [\ln|1-4\cos \theta|]_0^{\pi/3} = \ln|1-2| = -\ln 3 = \ln \frac{1}{3}$; or let $u = 1 - 4\cos \theta \Rightarrow du = 4\sin \theta\, d\theta$
with $\theta = 0 \Rightarrow u = -3$ and $\theta = \frac{\pi}{3} \Rightarrow u = -1 \Rightarrow \int_0^{\pi/3} \frac{4\sin \theta}{1-4\cos \theta}\, d\theta = \int_{-3}^{-1} \frac{1}{u}\, du = [\ln|u|]_{-3}^{-1} = -\ln 3 = \ln \frac{1}{3}$

43. Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$; $x = 1 \Rightarrow u = 0$ and $x = 2 \Rightarrow u = \ln 2$;
$\int_1^2 \frac{2\ln x}{x}\, dx = \int_0^{\ln 2} 2u\, du = [u^2]_0^{\ln 2} = (\ln 2)^2$

44. Let $u = \ln x \Rightarrow du = \frac{1}{x}\, dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;
$\int_2^4 \frac{dx}{x\ln x} = \int_{\ln 2}^{\ln 4} \frac{1}{u}\, du = [\ln u]_{\ln 2}^{\ln 4} = \ln(\ln 4) - \ln(\ln 2) = \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{\ln 2^2}{\ln 2}\right) = \ln\left(\frac{2\ln 2}{\ln 2}\right) = \ln 2$

45. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 4 \Rightarrow u = \ln 4$;
$\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4} u^{-2}\,du = \left[-\frac{1}{u}\right]_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} + \frac{1}{\ln 2} = -\frac{1}{\ln 2^2} + \frac{1}{\ln 2} = -\frac{1}{2\ln 2} + \frac{1}{\ln 2} = \frac{1}{2\ln 2} = \frac{1}{\ln 4}$

46. Let $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$; $x = 2 \Rightarrow u = \ln 2$ and $x = 16 \Rightarrow u = \ln 16$;
$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2}\int_{\ln 2}^{\ln 16} u^{-1/2}\,du = \left[u^{1/2}\right]_{\ln 2}^{\ln 16} = \sqrt{\ln 16} - \sqrt{\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$

47. Let $u = 6 + 3\tan t \Rightarrow du = 3\sec^2 t\,dt$;
$\int \frac{3\sec^2 t}{6 + 3\tan t}\,dt = \int \frac{du}{u} = \ln|u| + C = \ln|6 + 3\tan t| + C$

48. Let $u = 2 + \sec y \Rightarrow du = \sec y \tan y\,dy$;
$\int \frac{\sec y \tan y}{2 + \sec y}\,dy = \int \frac{du}{u} = \ln|u| + C = \ln|2 + \sec y| + C$

49. Let $u = \cos\frac{x}{2} \Rightarrow du = -\frac{1}{2}\sin\frac{x}{2}\,dx \Rightarrow -2\,du = \sin\frac{x}{2}\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{2} \Rightarrow u = \frac{1}{\sqrt{2}}$;
$\int_0^{\pi/2} \tan\frac{x}{2}\,dx = \int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = \left[-2\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} = 2\ln\sqrt{2} = \ln 2$

50. Let $u = \sin t \Rightarrow du = \cos t\,dt$; $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $t = \frac{\pi}{2} \Rightarrow u = 1$;
$\int_{\pi/4}^{\pi/2} \cot t\,dt = \int_{\pi/4}^{\pi/2} \frac{\cos t}{\sin t}\,dt = \int_{1/\sqrt{2}}^{1} \frac{du}{u} = \left[\ln|u|\right]_{1/\sqrt{2}}^{1} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$

51. Let $u = \sin\frac{\theta}{3} \Rightarrow du = \frac{1}{3}\cos\frac{\theta}{3}\,d\theta \Rightarrow 6\,du = 2\cos\frac{\theta}{3}\,d\theta$; $\theta = \frac{\pi}{2} \Rightarrow u = \frac{1}{2}$ and $\theta = \pi \Rightarrow u = \frac{\sqrt{3}}{2}$;
$\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta = \int_{\pi/2}^{\pi} \frac{2\cos\frac{\theta}{3}}{\sin\frac{\theta}{3}}\,d\theta = 6\int_{1/2}^{\sqrt{3}/2} \frac{du}{u} = 6\left[\ln|u|\right]_{1/2}^{\sqrt{3}/2} = 6\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right) = 6\ln\sqrt{3} = \ln 27$

52. Let $u = \cos 3x \Rightarrow du = -3\sin 3x\,dx \Rightarrow -2\,du = 6\sin 3x\,dx$; $x = 0 \Rightarrow u = 1$ and $x = \frac{\pi}{12} \Rightarrow u = \frac{1}{\sqrt{2}}$;
$\int_0^{\pi/12} 6\tan 3x\,dx = \int_0^{\pi/12} \frac{6\sin 3x}{\cos 3x}\,dx = -2\int_1^{1/\sqrt{2}} \frac{du}{u} = -2\left[\ln|u|\right]_1^{1/\sqrt{2}} = -2\ln\frac{1}{\sqrt{2}} - \ln 1 = 2\ln\sqrt{2} = \ln 2$

53. $\int \frac{dx}{2\sqrt{x} + 2x} = \int \frac{dx}{2\sqrt{x}\left(1 + \sqrt{x}\right)}$; let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$; $\int \frac{dx}{2\sqrt{x}\left(1 + \sqrt{x}\right)} = \int \frac{du}{u} = \ln|u| + C$
$= \ln\left|1 + \sqrt{x}\right| + C = \ln\left(1 + \sqrt{x}\right) + C$

54. Let $u = \sec x + \tan x \Rightarrow du = \left(\sec x \tan x + \sec^2 x\right)dx = (\sec x)(\tan x + \sec x)\,dx \Rightarrow \sec x\,dx = \frac{du}{u}$;
$\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}} = \int \frac{du}{u\sqrt{\ln u}} = \int (\ln u)^{-1/2} \cdot \frac{1}{u}\,du = 2(\ln u)^{1/2} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$

55. $y = \sqrt{x(x+1)} = (x(x+1))^{1/2} \Rightarrow \ln y = \frac{1}{2}\ln(x(x+1)) \Rightarrow 2\ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1}$
$\Rightarrow y' = \left(\frac{1}{2}\right)\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{\sqrt{x(x+1)}\,(2x+1)}{2x(x+1)} = \frac{2x+1}{2\sqrt{x(x+1)}}$

56. $y = \sqrt{(x^2+1)(x-1)^2} \Rightarrow \ln y = \frac{1}{2}\left[\ln(x^2+1) + 2\ln(x-1)\right] \Rightarrow \frac{y'}{y} = \frac{1}{2}\left(\frac{2x}{x^2+1} + \frac{2}{x-1}\right)$
$\Rightarrow y' = \sqrt{(x^2+1)(x-1)^2}\left(\frac{x}{x^2+1} + \frac{1}{x-1}\right) = \sqrt{(x^2+1)(x-1)^2}\left[\frac{x^2 - x + x^2 + 1}{(x^2+1)(x-1)}\right] = \frac{(2x^2 - x + 1)\,|x-1|}{\sqrt{x^2+1}\,(x-1)}$

57. $y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2}\left[\ln t - \ln(t+1)\right] \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+1}\right)$
$\Rightarrow \frac{dy}{dt} = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left[\frac{1}{t(t+1)}\right] = \frac{1}{2\sqrt{t}\,(t+1)^{3/2}}$

58. $y = \sqrt{\frac{1}{t(t+1)}} = [t(t+1)]^{-1/2} \Rightarrow \ln y = \frac{1}{2}[\ln t + \ln(t+1)] \Rightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)$
$\Rightarrow \frac{dy}{dt} = -\frac{1}{2}\sqrt{\frac{1}{t(t+1)}}\left[\frac{2t+1}{t(t+1)}\right] = -\frac{2t+1}{2(t^2+t)^{3/2}}$

59. $y = \sqrt{\theta+3}\,(\sin\theta) = (\theta+3)^{1/2}\sin\theta \Rightarrow \ln y = \frac{1}{2}\ln(\theta+3) + \ln(\sin\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{2(\theta+3)} + \frac{\cos\theta}{\sin\theta}$
$\Rightarrow \frac{dy}{d\theta} = \sqrt{\theta+3}\,(\sin\theta)\left[\frac{1}{2(\theta+3)} + \cot\theta\right]$

60. $y = (\tan\theta)\sqrt{2\theta+1} = (\tan\theta)(2\theta+1)^{1/2} \Rightarrow \ln y = \ln(\tan\theta) + \frac{1}{2}\ln(2\theta+1) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{\sec^2\theta}{\tan\theta} + \left(\frac{1}{2}\right)\left(\frac{2}{2\theta+1}\right)$
$\Rightarrow \frac{dy}{d\theta} = (\tan\theta)\sqrt{2\theta+1}\left(\frac{\sec^2\theta}{\tan\theta} + \frac{1}{2\theta+1}\right) = (\sec^2\theta)\sqrt{2\theta+1} + \frac{\tan\theta}{\sqrt{2\theta+1}}$

61. $y = t(t+1)(t+2) \Rightarrow \ln y = \ln t + \ln(t+1) + \ln(t+2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}$
$\Rightarrow \frac{dy}{dt} = t(t+1)(t+2)\left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right) = t(t+1)(t+2)\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right] = 3t^2+6t+2$

62. $y = \frac{1}{t(t+1)(t+2)} \Rightarrow \ln y = \ln 1 - \ln t - \ln(t+1) - \ln(t+2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}$
$\Rightarrow \frac{dy}{dt} = \frac{1}{t(t+1)(t+2)}\left[-\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}\right] = \frac{-1}{t(t+1)(t+2)}\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right]$
$= -\frac{3t^2+6t+2}{(t^3+3t^2+2t)^2}$

63. $y = \frac{\theta+5}{\theta\cos\theta} \Rightarrow \ln y = \ln(\theta+5) - \ln\theta - \ln(\cos\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta} \Rightarrow \frac{dy}{d\theta} = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right)$

64. $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}} \Rightarrow \ln y = \ln\theta + \ln(\sin\theta) - \frac{1}{2}\ln(\sec\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \left[\frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{(\sec\theta)(\tan\theta)}{2\sec\theta}\right]$
$\Rightarrow \frac{dy}{d\theta} = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}\left(\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right)$

65. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1) \Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$
$\Rightarrow y' = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$

66. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \Rightarrow \ln y = \frac{1}{2}[10\ln(x+1) - 5\ln(2x+1)] \Rightarrow \frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$
$\Rightarrow y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)$

67. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}} \Rightarrow \ln y = \frac{1}{3}[\ln x + \ln(x-2) - \ln(x^2+1)] \Rightarrow \frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$
$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$

68. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \Rightarrow \ln y = \frac{1}{3}[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)]$
$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)$

69. (a) $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\frac{\sin x}{\cos x} = -\tan x = 0 \Rightarrow x = 0$; $f'(x) > 0$ for $-\frac{\pi}{4} \le x < 0$ and $f'(x) < 0$ for $0 < x \le \frac{\pi}{3} \Rightarrow$ there is a relative maximum at $x = 0$ with $f(0) = \ln(\cos 0) = \ln 1 = 0$; $f\left(-\frac{\pi}{4}\right) = \ln\left(\cos\left(-\frac{\pi}{4}\right)\right)$ $= \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ and $f\left(\frac{\pi}{3}\right) = \ln\left(\cos\left(\frac{\pi}{3}\right)\right) = \ln\frac{1}{2} = -\ln 2$. Therefore, the absolute minimum occurs at $x = \frac{\pi}{3}$ with $f\left(\frac{\pi}{3}\right) = -\ln 2$ and the absolute maximum occurs at $x = 0$ with $f(0) = 0$.

(b) $f(x) = \cos(\ln x) \Rightarrow f'(x) = \frac{-\sin(\ln x)}{x} = 0 \Rightarrow x = 1$; $f'(x) > 0$ for $\frac{1}{2} \le x < 1$ and $f'(x) < 0$ for $1 < x \le 2$ $\Rightarrow$ there is a relative maximum at $x = 1$ with $f(1) = \cos(\ln 1) = \cos 0 = 1$; $f\left(\frac{1}{2}\right) = \cos\left(\ln\left(\frac{1}{2}\right)\right)$ $= \cos(-\ln 2) = \cos(\ln 2)$ and $f(2) = \cos(\ln 2)$. Therefore, the absolute minimum occurs at $x = \frac{1}{2}$ and $x = 2$ with $f\left(\frac{1}{2}\right) = f(2) = \cos(\ln 2)$, and the absolute maximum occurs at $x = 1$ with $f(1) = 1$.

70. (a) $f(x) = x - \ln x \Rightarrow f'(x) = 1 - \frac{1}{x}$; if $x > 1$, then $f'(x) > 0$ which means that $f(x)$ is increasing
(b) $f(1) = 1 - \ln 1 = 1 \Rightarrow f(x) = x - \ln x > 0$, if $x > 1$ by part (a) $\Rightarrow x > \ln x$ if $x > 1$

71. $\int_1^5 (\ln 2x - \ln x)\,dx = \int_1^5 (-\ln x + \ln 2 + \ln x)\,dx = (\ln 2)\int_1^5 dx = (\ln 2)(5 - 1) = \ln 2^4 = \ln 16$

72. $A = \int_{-\pi/4}^{0} -\tan x\,dx + \int_0^{\pi/3} \tan x\,dx = \int_{-\pi/4}^{0} \frac{-\sin x}{\cos x}\,dx - \int_0^{\pi/3} \frac{-\sin x}{\cos x}\,dx = [\ln|\cos x|]_{-\pi/4}^{0} - [\ln|\cos x|]_0^{\pi/3}$
$= \left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right) - \left(\ln \frac{1}{2} - \ln 1\right) = \ln\sqrt{2} + \ln 2 = \frac{3}{2}\ln 2$

73. $V = \pi\int_0^3 \left(\frac{2}{\sqrt{y+1}}\right)^2 dy = 4\pi\int_0^3 \frac{1}{y+1}\,dy = 4\pi\,[\ln|y+1|]_0^3 = 4\pi(\ln 4 - \ln 1) = 4\pi \ln 4$

74. $V = \pi\int_{\pi/6}^{\pi/2} \cot x\,dx = \pi\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x}\,dx = \pi\,[\ln(\sin x)]_{\pi/6}^{\pi/2} = \pi\left(\ln 1 - \ln \frac{1}{2}\right) = \pi \ln 2$

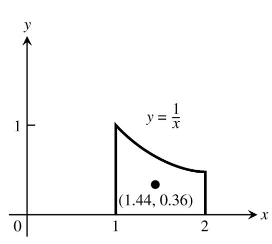
75. $V = 2\pi\int_{1/2}^{2} x\left(\frac{1}{x^2}\right) dx = 2\pi\int_{1/2}^{2} \frac{1}{x}\,dx = 2\pi\,[\ln|x|]_{1/2}^{2} = 2\pi\left(\ln 2 - \ln \frac{1}{2}\right) = 2\pi(2\ln 2) = \pi \ln 2^4 = \pi \ln 16$

76. $V = \pi\int_0^3 \left(\frac{9x}{\sqrt{x^3+9}}\right)^2 dx = 27\pi\int_0^3 dx = 27\pi\,[\ln(x^3+9)]_0^3 = 27\pi(\ln 36 - \ln 9) = 27\pi(\ln 4 + \ln 9 - \ln 9)$
$= 27\pi \ln 4 = 54\pi \ln 2$

77. (a) $y = \frac{x^2}{8} - \ln x \Rightarrow 1 + (y')^2 = 1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2 = 1 + \left(\frac{x^2-4}{4x}\right)^2 = \left(\frac{x^2+4}{4x}\right)^2 \Rightarrow L = \int_4^8 \sqrt{1 + (y')^2}\,dx$
$= \int_4^8 \frac{x^2+4}{4x}\,dx = \int_4^8 \left(\frac{x}{4} + \frac{1}{x}\right) dx = \left[\frac{x^2}{8} + \ln|x|\right]_4^8 = (8 + \ln 8) - (2 + \ln 4) = 6 + \ln 2$
(b) $x = \left(\frac{y}{4}\right)^2 - 2\ln\left(\frac{y}{4}\right) \Rightarrow \frac{dx}{dy} = \frac{y}{8} - \frac{2}{y} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{y}{8} - \frac{2}{y}\right)^2 = 1 + \left(\frac{y^2-16}{8y}\right)^2 = \left(\frac{y^2+16}{8y}\right)^2$
$\Rightarrow L = \int_4^{12} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_4^{12} \frac{y^2+16}{8y}\,dy = \int_4^{12} \left(\frac{y}{8} + \frac{2}{y}\right) dy = \left[\frac{y^2}{16} + 2\ln y\right]_4^{12} = (9 + 2\ln 12) - (1 + 2\ln 4)$
$= 8 + 2\ln 3 = 8 + \ln 9$

78. $L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \ln|x| + C = \ln x + C$ since $x > 0 \Rightarrow 0 = \ln 1 + C \Rightarrow C = 0 \Rightarrow y = \ln x$

79. (a) $M_y = \int_1^2 x\left(\frac{1}{x}\right) dx = 1$, $M_x = \int_1^2 \left(\frac{1}{2x}\right)\left(\frac{1}{x}\right) dx = \frac{1}{2}\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{2x}\right]_1^2 = \frac{1}{4}$, $M = \int_1^2 \frac{1}{x}\,dx = [\ln|x|]_1^2 = \ln 2$
$\Rightarrow \bar{x} = \frac{M_y}{M} = \frac{1}{\ln 2} \approx 1.44$ and $\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{1}{4}\right)}{\ln 2} \approx 0.36$
(b)



80. (a) $M_y = \int_1^{16} x\left(\frac{1}{\sqrt{x}}\right) dx = \int_1^{16} x^{1/2}\, dx = \frac{2}{3}\left[x^{3/2}\right]_1^{16} = 42$; $M_x = \int_1^{16}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right) dx = \frac{1}{2}\int_1^{16}\frac{1}{x}\, dx$
$= \frac{1}{2}\left[\ln|x|\right]_1^{16} = \ln 4$, $M = \int_1^{16}\frac{1}{\sqrt{x}}\, dx = \left[2x^{1/2}\right]_1^{16} = 6 \Rightarrow \bar{x} = \frac{M_y}{M} = 7$ and $\bar{y} = \frac{M_x}{M} = \frac{\ln 4}{6}$

(b) $M_y = \int_1^{16} x\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 4\int_1^{16} dx = 60$, $M_x = \int_1^{16}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 2\int_1^{16} x^{-3/2}\, dx$
$= -4\left[x^{-1/2}\right]_1^{16} = 3$, $M = \int_1^{16}\left(\frac{1}{\sqrt{x}}\right)\left(\frac{4}{\sqrt{x}}\right) dx = 4\int_1^{16}\frac{1}{x}\, dx = \left[4\ln|x|\right]_1^{16} = 4\ln 16 \Rightarrow \bar{x} = \frac{M_y}{M} = \frac{15}{\ln 16}$ and
$\bar{y} = \frac{M_x}{M} = \frac{3}{4\ln 16}$

81. $f(x) = \ln(x^3 - 1)$, domain of f: $(1, \infty) \Rightarrow f'(x) = \frac{3x^2}{x^3-1}$; $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$, not in the domain;
$f'(x) =$ undefined $\Rightarrow x^3 - 1 = 0 \Rightarrow x = 1$, not in domain. On $(1, \infty)$, $f'(x) > 0 \Rightarrow$ f is increasing on $(1, \infty)$
$\Rightarrow$ f is one-to-one

82. $g(x) = \sqrt{x^2 + \ln x}$, domain of g: $x > 0.652919 \Rightarrow g'(x) = \frac{2x + \frac{1}{x}}{2\sqrt{x^2+\ln x}} = \frac{2x^2+1}{2x\sqrt{x^2+\ln x}}$; $g'(x) = 0 \Rightarrow 2x^2 + 1 = 0 \Rightarrow$ no real solutions; $g'(x) =$ undefined $\Rightarrow 2x\sqrt{x^2 + \ln x} = 0 \Rightarrow x = 0$ or $x \approx 0.652919$, neither in domain. On $x > 0.652919$, $g'(x) > 0 \Rightarrow$ g is increasing for $x > 0.652919 \Rightarrow$ g is one-to-one

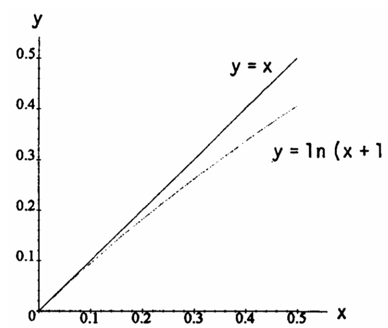
83. $\frac{dy}{dx} = 1 + \frac{1}{x}$ at $(1, 3) \Rightarrow y = x + \ln|x| + C$; $y = 3$ at $x = 1 \Rightarrow C = 2 \Rightarrow y = x + \ln|x| + 2$

84. $\frac{d^2y}{dx^2} = \sec^2 x \Rightarrow \frac{dy}{dx} = \tan x + C$ and $1 = \tan 0 + C \Rightarrow \frac{dy}{dx} = \tan x + 1 \Rightarrow y = \int(\tan x + 1)\, dx$
$= \ln|\sec x| + x + C_1$ and $0 = \ln|\sec 0| + 0 + C_1 \Rightarrow C_1 = 0 \Rightarrow y = \ln|\sec x| + x$

85. (a) $L(x) = f(0) + f'(0)\cdot x$, and $f(x) = \ln(1 + x) \Rightarrow f'(x)\big|_{x=0} = \frac{1}{1+x}\Big|_{x=0} = 1 \Rightarrow L(x) = \ln 1 + 1\cdot x \Rightarrow L(x) = x$

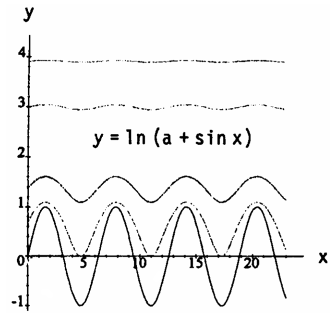
(b) Let $f(x) = \ln(x + 1)$. Since $f''(x) = -\frac{1}{(x+1)^2} < 0$ on $[0, 0.1]$, the graph of f is concave down on this interval and the largest error in the linear approximation will occur when $x = 0.1$. This error is $0.1 - \ln(1.1) \approx 0.00469$ to five decimal places.

(c) The approximation $y = x$ for $\ln(1 + x)$ is best for smaller positive values of x; in particular for $0 \le x \le 0.1$ in the graph. As x increases, so does the error $x - \ln(1 + x)$. From the graph an upper bound for the error is $0.5 - \ln(1 + 0.5) \approx 0.095$; i.e., $|E(x)| \le 0.095$ for $0 \le x \le 0.5$. Note from the graph that $0.1 - \ln(1 + 0.1) \approx 0.00469$ estimates the error in replacing $\ln(1 + x)$ by x over $0 \le x \le 0.1$. This is consistent with the estimate given in part (b) above.



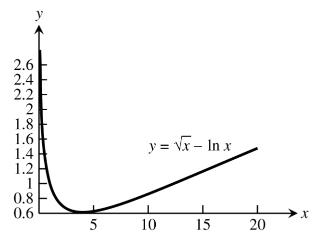
86. For all positive values of x, $\frac{d}{dx}\left[\ln\frac{a}{x}\right] = \frac{1}{\frac{a}{x}}\cdot -\frac{a}{x^2} = -\frac{1}{x}$ and $\frac{d}{dx}\left[\ln a - \ln x\right] = 0 - \frac{1}{x} = -\frac{1}{x}$. Since $\ln\frac{a}{x}$ and $\ln a - \ln x$ have the same derivative, then $\ln\frac{a}{x} = \ln a - \ln x + C$ for some constant C. Since this equation holds for all positve values of x, it must be true for $x = 1 \Rightarrow \ln\frac{1}{x} = \ln 1 - \ln x + C = 0 - \ln x + C \Rightarrow \ln\frac{1}{x} = -\ln x + C$. By part 3 we know that $\ln\frac{1}{x} = -\ln x \Rightarrow C = 0 \Rightarrow \ln\frac{a}{x} = \ln a - \ln x$.

87. (a)



(b) $y' = \frac{\cos x}{a+\sin x}$. Since $|\sin x|$ and $|\cos x|$ are less than or equal to 1, we have for $a > 1$ $\frac{-1}{a-1} \le y' \le \frac{1}{a-1}$ for all x. Thus, $\lim\limits_{a\to+\infty} y' = 0$ for all $x \Rightarrow$ the graph of y looks more and more horizontal as $a \to +\infty$.

88. (a) The graph of $y = \sqrt{x} - \ln x$ appears to be concave upward for all $x > 0$.



(b) $y = \sqrt{x} - \ln x \Rightarrow y' = \frac{1}{2\sqrt{x}} - \frac{1}{x} \Rightarrow y'' = -\frac{1}{4x^{3/2}} + \frac{1}{x^2} = \frac{1}{x^2}\left(-\frac{\sqrt{x}}{4} + 1\right) = 0 \Rightarrow \sqrt{x} = 4 \Rightarrow x = 16.$ Thus, $y'' > 0$ if $0 < x < 16$ and $y'' < 0$ if $x > 16$ so a point of inflection exists at $x = 16$. The graph of $y = \sqrt{x} - \ln x$ closely resembles a straight line for $x \ge 10$ and it is impossible to discuss the point of inflection visually from the graph.

7.3 EXPONENTIAL FUNCTIONS

1. (a) $e^{-0.3t} = 27 \Rightarrow \ln e^{-0.3t} = \ln 3^3 \Rightarrow (-0.3t)\ln e = 3\ln 3 \Rightarrow -0.3t = 3\ln 3 \Rightarrow t = -10\ln 3$
   (b) $e^{kt} = \frac{1}{2} \Rightarrow \ln e^{kt} = \ln 2^{-1} = kt\ln e = -\ln 2 \Rightarrow t = -\frac{\ln 2}{k}$
   (c) $e^{(\ln 0.2)t} = 0.4 \Rightarrow \left(e^{\ln 0.2}\right)^t = 0.4 \Rightarrow 0.2^t = 0.4 \Rightarrow \ln 0.2^t = \ln 0.4 \Rightarrow t\ln 0.2 = \ln 0.4 \Rightarrow t = \frac{\ln 0.4}{\ln 0.2}$

2. (a) $e^{-0.01t} = 1000 \Rightarrow \ln e^{-0.01t} = \ln 1000 \Rightarrow (-0.01t)\ln e = \ln 1000 \Rightarrow -0.01t = \ln 1000 \Rightarrow t = -100\ln 1000$
   (b) $e^{kt} = \frac{1}{10} \Rightarrow \ln e^{kt} = \ln 10^{-1} = kt\ln e = -\ln 10 \Rightarrow kt = -\ln 10 \Rightarrow t = -\frac{\ln 10}{k}$
   (c) $e^{(\ln 2)t} = \frac{1}{2} \Rightarrow \left(e^{\ln 2}\right)^t = 2^{-1} \Rightarrow 2^t = 2^{-1} \Rightarrow t = -1$

3. $e^{\sqrt{t}} = x^2 \Rightarrow \ln e^{\sqrt{t}} = \ln x^2 \Rightarrow \sqrt{t} = 2\ln x \Rightarrow t = 4(\ln x)^2$

4. $e^{x^2} e^{2x+1} = e^t \Rightarrow e^{x^2+2x+1} = e^t \Rightarrow \ln e^{x^2+2x+1} = \ln e^t \Rightarrow t = x^2 + 2x + 1$

5. $y = e^{-5x} \Rightarrow y' = e^{-5x}\frac{d}{dx}(-5x) \Rightarrow y' = -5e^{-5x}$

6. $y = e^{2x/3} \Rightarrow y' = e^{2x/3}\frac{d}{dx}\left(\frac{2x}{3}\right) \Rightarrow y' = \frac{2}{3}e^{2x/3}$

7. $y = e^{5-7x} \Rightarrow y' = e^{5-7x}\frac{d}{dx}(5-7x) \Rightarrow y' = -7e^{5-7x}$

8. $y = e^{\left(4\sqrt{x}+x^2\right)} \Rightarrow y' = e^{\left(4\sqrt{x}+x^2\right)}\frac{d}{dx}\left(4\sqrt{x}+x^2\right) \Rightarrow y' = \left(\frac{2}{\sqrt{x}} + 2x\right)e^{\left(4\sqrt{x}+x^2\right)}$

9. $y = xe^x - e^x \Rightarrow y' = (e^x + xe^x) - e^x = xe^x$

10. $y = (1+2x)\,e^{-2x} \Rightarrow y' = 2e^{-2x} + (1+2x)e^{-2x}\,\frac{d}{dx}(-2x) \Rightarrow y' = 2e^{-2x} - 2(1+2x)\,e^{-2x} = -4xe^{-2x}$

11. $y = (x^2-2x+2)\,e^x \Rightarrow y' = (2x-2)e^x + (x^2-2x+2)\,e^x = x^2e^x$

12. $y = (9x^2-6x+2)\,e^{3x} \Rightarrow y' = (18x-6)e^{3x} + (9x^2-6x+2)\,e^{3x}\,\frac{d}{dx}(3x) \Rightarrow y' = (18x-6)e^{3x} + 3\,(9x^2-6x+2)\,e^{3x}$
$= 27x^2e^{3x}$

13. $y = e^{\theta}(\sin\theta+\cos\theta) \Rightarrow y' = e^{\theta}(\sin\theta+\cos\theta) + e^{\theta}(\cos\theta-\sin\theta) = 2e^{\theta}\cos\theta$

14. $y = \ln\left(3\theta e^{-\theta}\right) = \ln 3 + \ln\theta + \ln e^{-\theta} = \ln 3 + \ln\theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$

15. $y = \cos\left(e^{-\theta^2}\right) \Rightarrow \frac{dy}{d\theta} = -\sin\left(e^{-\theta^2}\right)\frac{d}{d\theta}\left(e^{-\theta^2}\right) = \left(-\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right)\frac{d}{d\theta}(-\theta^2) = 2\theta e^{-\theta^2}\sin\left(e^{-\theta^2}\right)$

16. $y = \theta^3e^{-2\theta}\cos 5\theta \Rightarrow \frac{dy}{d\theta} = (3\theta^2)\left(e^{-2\theta}\cos 5\theta\right) + (\theta^3\cos 5\theta)\,e^{-2\theta}\,\frac{d}{d\theta}(-2\theta) - 5(\sin 5\theta)\left(\theta^3e^{-2\theta}\right)$
$= \theta^2e^{-2\theta}\,(3\cos 5\theta - 2\theta\cos 5\theta - 5\theta\sin 5\theta)$

17. $y = \ln\left(3te^{-t}\right) = \ln 3 + \ln t + \ln e^{-t} = \ln 3 + \ln t - t \Rightarrow \frac{dy}{dt} = \frac{1}{t} - 1 = \frac{1-t}{t}$

18. $y = \ln\left(2e^{-t}\sin t\right) = \ln 2 + \ln e^{-t} + \ln\sin t = \ln 2 - t + \ln\sin t \Rightarrow \frac{dy}{dt} = -1 + \left(\frac{1}{\sin t}\right)\frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t}$
$= \frac{\cos t - \sin t}{\sin t}$

19. $y = \ln\frac{e^{\theta}}{1+e^{\theta}} = \ln e^{\theta} - \ln\left(1+e^{\theta}\right) = \theta - \ln\left(1+e^{\theta}\right) \Rightarrow \frac{dy}{d\theta} = 1 - \left(\frac{1}{1+e^{\theta}}\right)\frac{d}{d\theta}\left(1+e^{\theta}\right) = 1 - \frac{e^{\theta}}{1+e^{\theta}} = \frac{1}{1+e^{\theta}}$

20. $y = \ln\frac{\sqrt{\theta}}{1+\sqrt{\theta}} = \ln\sqrt{\theta} - \ln\left(1+\sqrt{\theta}\right) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\sqrt{\theta}}\right)\frac{d}{d\theta}\left(\sqrt{\theta}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\frac{d}{d\theta}\left(1+\sqrt{\theta}\right)$
$= \left(\frac{1}{\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) = \frac{\left(1+\sqrt{\theta}\right)-\sqrt{\theta}}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\,(1+\theta^{1/2})}$

21. $y = e^{(\cos t+\ln t)} = e^{\cos t}e^{\ln t} = te^{\cos t} \Rightarrow \frac{dy}{dt} = e^{\cos t} + te^{\cos t}\,\frac{d}{dt}(\cos t) = (1 - t\sin t)\,e^{\cos t}$

22. $y = e^{\sin t}\,(\ln t^2+1) \Rightarrow \frac{dy}{dt} = e^{\sin t}(\cos t)\,(\ln t^2+1) + \frac{2}{t}\,e^{\sin t} = e^{\sin t}\left[(\ln t^2+1)\,(\cos t) + \frac{2}{t}\right]$

23. $\int_0^{\ln x}\sin e^t\,dt \Rightarrow y' = \left(\sin e^{\ln x}\right)\cdot\frac{d}{dx}(\ln x) = \frac{\sin x}{x}$

24. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}}\ln t\,dt \Rightarrow y' = (\ln e^{2x})\cdot\frac{d}{dx}(e^{2x}) - \left(\ln e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(e^{4\sqrt{x}}\right) = (2x)\,(2e^{2x}) - \left(4\sqrt{x}\right)\left(e^{4\sqrt{x}}\right)\cdot\frac{d}{dx}\left(4\sqrt{x}\right)$
$= 4xe^{2x} - 4\sqrt{x}\,e^{4\sqrt{x}}\left(\frac{2}{\sqrt{x}}\right) = 4xe^{2x} - 8e^{4\sqrt{x}}$

25. $\ln y = e^y\sin x \Rightarrow \left(\frac{1}{y}\right)y' = (y'e^y)\,(\sin x) + e^y\cos x \Rightarrow y'\left(\frac{1}{y} - e^y\sin x\right) = e^y\cos x$
$\Rightarrow y'\left(\frac{1-ye^y\sin x}{y}\right) = e^y\cos x \Rightarrow y' = \frac{ye^y\cos x}{1-ye^y\sin x}$

26. $\ln xy = e^{x+y} \Rightarrow \ln x + \ln y = e^{x+y} \Rightarrow \frac{1}{x} + \left(\frac{1}{y}\right)y' = (1+y')\,e^{x+y} \Rightarrow y'\left(\frac{1}{y} - e^{x+y}\right) = e^{x+y} - \frac{1}{x}$
$\Rightarrow y'\left(\frac{1-ye^{x+y}}{y}\right) = \frac{xe^{x+y}-1}{x} \Rightarrow y' = \frac{y\,(xe^{x+y}-1)}{x\,(1-ye^{x+y})}$

27. $e^{2x} = \sin(x+3y) \Rightarrow 2e^{2x} = (1+3y')\cos(x+3y) \Rightarrow 1+3y' = \frac{2e^{2x}}{\cos(x+3y)} \Rightarrow 3y' = \frac{2e^{2x}}{\cos(x+3y)} - 1 \Rightarrow y' = \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}$

28. $\tan y = e^x + \ln x \;\Rightarrow\; (\sec^2 y)\, y' = e^x + \frac{1}{x} \;\Rightarrow\; y' = \frac{(xe^x+1)\cos^2 y}{x}$

29. $\int (e^{3x} + 5e^{-x})\, dx = \frac{e^{3x}}{3} - 5e^{-x} + C$

30. $\int (2e^x - 3e^{-2x})\, dx = 2e^x + \frac{3}{2}e^{-2x} + C$

31. $\int_{\ln 2}^{\ln 3} e^x\, dx = [e^x]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$

32. $\int_{-\ln 2}^{0} e^{-x}\, dx = [-e^{-x}]_{-\ln 2}^{0} = -e^0 + e^{\ln 2} = -1 + 2 = 1$

33. $\int 8e^{(x+1)}\, dx = 8e^{(x+1)} + C$

34. $\int 2e^{(2x-1)}\, dx = e^{(2x-1)} + C$

35. $\int_{\ln 4}^{\ln 9} e^{x/2}\, dx = \left[2e^{x/2}\right]_{\ln 4}^{\ln 9} = 2\left[e^{(\ln 9)/2} - e^{(\ln 4)/2}\right] = 2\left(e^{\ln 3} - e^{\ln 2}\right) = 2(3-2) = 2$

36. $\int_0^{\ln 16} e^{x/4}\, dx = \left[4e^{x/4}\right]_0^{\ln 16} = 4\left(e^{(\ln 16)/4} - e^0\right) = 4\left(e^{\ln 2} - 1\right) = 4(2-1) = 4$

37. Let $u = r^{1/2} \;\Rightarrow\; du = \frac{1}{2}r^{-1/2}\, dr \;\Rightarrow\; 2\, du = r^{-1/2}\, dr$;
$\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\, dr = \int e^{r^{1/2}} \cdot r^{-1/2}\, dr = 2\int e^u\, du = 2e^u + C = 2e^{r^{1/2}} + C = 2e^{\sqrt{r}} + C$

38. Let $u = -r^{1/2} \;\Rightarrow\; du = -\frac{1}{2}r^{-1/2}\, dr \;\Rightarrow\; -2\, du = r^{-1/2}\, dr$;
$\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\, dr = \int e^{-r^{1/2}} \cdot r^{-1/2}\, dr = -2\int e^u\, du = -2e^{-r^{1/2}} + C = -2e^{-\sqrt{r}} + C$

39. Let $u = -t^2 \;\Rightarrow\; du = -2t\, dt \;\Rightarrow\; -du = 2t\, dt$;
$\int 2te^{-t^2}\, dt = -\int e^u\, du = -e^u + C = -e^{-t^2} + C$

40. Let $u = t^4 \;\Rightarrow\; du = 4t^3\, dt \;\Rightarrow\; \frac{1}{4}\, du = t^3\, dt$;
$\int t^3 e^{t^4}\, dt = \frac{1}{4}\int e^u\, du = \frac{1}{4}e^{t^4} + C$

41. Let $u = \frac{1}{x} \;\Rightarrow\; du = -\frac{1}{x^2}\, dx \;\Rightarrow\; -du = \frac{1}{x^2}\, dx$;
$\int \frac{e^{1/x}}{x^2}\, dx = \int -e^u\, du = -e^u + C = -e^{1/x} + C$

42. Let $u = -x^{-2} \;\Rightarrow\; du = 2x^{-3}\, dx \;\Rightarrow\; \frac{1}{2}\, du = x^{-3}\, dx$;
$\int \frac{e^{-1/x^2}}{x^3}\, dx = \int e^{-x^{-2}} \cdot x^{-3}\, dx = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-x^{-2}} + C = \frac{1}{2}e^{-1/x^2} + C$

43. Let $u = \tan\theta \;\Rightarrow\; du = \sec^2\theta\, d\theta$; $\theta = 0 \;\Rightarrow\; u = 0$, $\theta = \frac{\pi}{4} \;\Rightarrow\; u = 1$;
$\int_0^{\pi/4} \left(1 + e^{\tan\theta}\right)\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^2\theta\, d\theta + \int_0^1 e^u\, du = [\tan\theta]_0^{\pi/4} + [e^u]_0^1 = \left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right] + (e^1 - e^0)$
$= (1-0) + (e-1) = e$

44. Let $u = \cot\theta \;\Rightarrow\; du = -\csc^2\theta\, d\theta$; $\theta = \frac{\pi}{4} \;\Rightarrow\; u = 1$, $\theta = \frac{\pi}{2} \;\Rightarrow\; u = 0$;
$\int_{\pi/4}^{\pi/2} \left(1 + e^{\cot\theta}\right)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta - \int_1^0 e^u\, du = [-\cot\theta]_{\pi/4}^{\pi/2} - [e^u]_1^0 = \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right)\right] - (e^0 - e^1)$
$= (0+1) - (1-e) = e$

45. Let $u = \sec \pi t \Rightarrow du = \pi \sec \pi t \tan \pi t\, dt \Rightarrow \frac{du}{\pi} = \sec \pi t \tan \pi t\, dt$;
$\int e^{\sec(\pi t)} \sec(\pi t) \tan(\pi t)\, dt = \frac{1}{\pi} \int e^u\, du = \frac{e^u}{\pi} + C = \frac{e^{\sec(\pi t)}}{\pi} + C$

46. Let $u = \csc(\pi + t) \Rightarrow du = -\csc(\pi + t) \cot(\pi + t)\, dt$;
$\int e^{\csc(\pi+t)} \csc(\pi + t) \cot(\pi + t)\, dt = -\int e^u\, du = -e^u + C = -e^{\csc(\pi+t)} + C$

47. Let $u = e^v \Rightarrow du = e^v\, dv \Rightarrow 2\, du = 2e^v\, dv$; $v = \ln \frac{\pi}{6} \Rightarrow u = \frac{\pi}{6}$, $v = \ln \frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$;
$\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v\, dv = 2 \int_{\pi/6}^{\pi/2} \cos u\, du = [2 \sin u]_{\pi/6}^{\pi/2} = 2\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right)\right] = 2\left(1 - \frac{1}{2}\right) = 1$

48. Let $u = e^{x^2} \Rightarrow du = 2xe^{x^2}\, dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{\ln \pi} \Rightarrow u = e^{\ln \pi} = \pi$;
$\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos\left(e^{x^2}\right) dx = \int_1^{\pi} \cos u\, du = [\sin u]_1^{\pi} = \sin(\pi) - \sin(1) = -\sin(1) \approx -0.84147$

49. Let $u = 1 + e^r \Rightarrow du = e^r\, dr$;
$\int \frac{e^r}{1+e^r}\, dr = \int \frac{1}{u}\, du = \ln|u| + C = \ln(1 + e^r) + C$

50. $\int \frac{1}{1+e^x}\, dx = \int \frac{e^{-x}}{e^{-x}+1}\, dx$;
let $u = e^{-x} + 1 \Rightarrow du = -e^{-x}\, dx \Rightarrow -du = e^{-x}\, dx$;
$\int \frac{e^{-x}}{e^{-x}+1}\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C = -\ln(e^{-x} + 1) + C$

51. $\frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2)\, dt$;
let $u = e^t - 2 \Rightarrow du = e^t\, dt \Rightarrow y = \int \sin u\, du = -\cos u + C = -\cos(e^t - 2) + C$; $y(\ln 2) = 0$
$\Rightarrow -\cos\left(e^{\ln 2} - 2\right) + C = 0 \Rightarrow -\cos(2 - 2) + C = 0 \Rightarrow C = \cos 0 = 1$; thus, $y = 1 - \cos(e^t - 2)$

52. $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t} \sec^2(\pi e^{-t})\, dt$;
let $u = \pi e^{-t} \Rightarrow du = -\pi e^{-t}\, dt \Rightarrow -\frac{1}{\pi}\, du = e^{-t}\, dt \Rightarrow y = -\frac{1}{\pi} \int \sec^2 u\, du = -\frac{1}{\pi} \tan u + C$
$= -\frac{1}{\pi} \tan(\pi e^{-t}) + C$; $y(\ln 4) = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi e^{-\ln 4}\right) + C = \frac{2}{\pi} \Rightarrow -\frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right) + C = \frac{2}{\pi}$
$\Rightarrow -\frac{1}{\pi}(1) + C = \frac{2}{\pi} \Rightarrow C = \frac{3}{\pi}$; thus, $y = \frac{3}{\pi} - \frac{1}{\pi} \tan(\pi e^{-t})$

53. $\frac{d^2y}{dx^2} = 2e^{-x} \Rightarrow \frac{dy}{dx} = -2e^{-x} + C$; $x = 0$ and $\frac{dy}{dx} = 0 \Rightarrow 0 = -2e^0 + C \Rightarrow C = 2$; thus $\frac{dy}{dx} = -2e^{-x} + 2$
$\Rightarrow y = 2e^{-x} + 2x + C_1$; $x = 0$ and $y = 1 \Rightarrow 1 = 2e^0 + C_1 \Rightarrow C_1 = -1 \Rightarrow y = 2e^{-x} + 2x - 1 = 2(e^{-x} + x) - 1$

54. $\frac{d^2y}{dt^2} = 1 - e^{2t} \Rightarrow \frac{dy}{dt} = t - \frac{1}{2}e^{2t} + C$; $t = 1$ and $\frac{dy}{dt} = 0 \Rightarrow 0 = 1 - \frac{1}{2}e^2 + C \Rightarrow C = \frac{1}{2}e^2 - 1$; thus
$\frac{dy}{dt} = t - \frac{1}{2}e^{2t} + \frac{1}{2}e^2 - 1 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t + C_1$; $t = 1$ and $y = -1 \Rightarrow -1 = \frac{1}{2} - \frac{1}{4}e^2 + \frac{1}{2}e^2 - 1 + C_1$
$\Rightarrow C_1 = -\frac{1}{2} - \frac{1}{4}e^2 \Rightarrow y = \frac{1}{2}t^2 - \frac{1}{4}e^{2t} + \left(\frac{1}{2}e^2 - 1\right)t - \left(\frac{1}{2} + \frac{1}{4}e^2\right)$

55. $y = 2^x \Rightarrow y' = 2^x \ln 2$

56. $y = 3^{-x} \Rightarrow y' = 3^{-x}(\ln 3)(-1) = -3^{-x} \ln 3$

57. $y = 5^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 5^{\sqrt{s}}(\ln 5)\left(\frac{1}{2}s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right)5^{\sqrt{s}}$

58. $y = 2^{s^2} \Rightarrow \frac{dy}{ds} = 2^{s^2}(\ln 2)2s = (\ln 2^2)\left(s2^{s^2}\right) = (\ln 4)s2^{s^2}$

59. $y = x^{\pi} \Rightarrow y' = \pi x^{(\pi-1)}$

60. $y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1-e)\,t^{-e}$

61. $y = (\cos\theta)^{\sqrt{2}} \Rightarrow \frac{dy}{d\theta} = -\sqrt{2}\,(\cos\theta)^{\left(\sqrt{2}-1\right)}(\sin\theta)$

62. $y = (\ln\theta)^{\pi} \Rightarrow \frac{dy}{d\theta} = \pi(\ln\theta)^{(\pi-1)}\left(\frac{1}{\theta}\right) = \frac{\pi(\ln\theta)^{(\pi-1)}}{\theta}$

63. $y = 7^{\sec\theta}\ln 7 \Rightarrow \frac{dy}{d\theta} = (7^{\sec\theta}\ln 7)(\ln 7)(\sec\theta\tan\theta) = 7^{\sec\theta}(\ln 7)^2(\sec\theta\tan\theta)$

64. $y = 3^{\tan\theta}\ln 3 \Rightarrow \frac{dy}{d\theta} = (3^{\tan\theta}\ln 3)(\ln 3)\sec^2\theta = 3^{\tan\theta}(\ln 3)^2\sec^2\theta$

65. $y = 2^{\sin 3t} \Rightarrow \frac{dy}{dt} = (2^{\sin 3t}\ln 2)(\cos 3t)(3) = (3\cos 3t)\left(2^{\sin 3t}\right)(\ln 2)$

66. $y = 5^{-\cos 2t} \Rightarrow \frac{dy}{dt} = (5^{-\cos 2t}\ln 5)(\sin 2t)(2) = (2\sin 2t)\left(5^{-\cos 2t}\right)(\ln 5)$

67. $y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{5\theta}\right)(5) = \frac{1}{\theta\ln 2}$

68. $y = \log_3(1+\theta\ln 3) = \frac{\ln(1+\theta\ln 3)}{\ln 3} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 3}\right)\left(\frac{1}{1+\theta\ln 3}\right)(\ln 3) = \frac{1}{1+\theta\ln 3}$

69. $y = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2\,\frac{\ln x}{\ln 4} = 3\,\frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x\ln 4}$

70. $y = \frac{x\ln e}{\ln 25} - \frac{\ln x}{2\ln 5} = \frac{x}{2\ln 5} - \frac{\ln x}{2\ln 5} = \left(\frac{1}{2\ln 5}\right)(x-\ln x) \Rightarrow y' = \left(\frac{1}{2\ln 5}\right)\left(1-\frac{1}{x}\right) = \frac{x-1}{2x\ln 5}$

71. $y = x^3\log_{10}x = x^3\left(\frac{\ln x}{\ln 10}\right) = \frac{1}{\ln 10}x^3\ln x \Rightarrow y' = \frac{1}{\ln 10}\left(x^3\cdot\frac{1}{x} + 3x^2\ln x\right) = \frac{1}{\ln 10}x^2 + 3x^2\frac{\ln x}{\ln 10} = \frac{1}{\ln 10}x^2 + 3x^2\log_{10}x$

72. $y = \log_3 r\cdot\log_9 r = \left(\frac{\ln r}{\ln 3}\right)\left(\frac{\ln r}{\ln 9}\right) = \frac{\ln^2 r}{(\ln 3)(\ln 9)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 3)(\ln 9)}\right](2\ln r)\left(\frac{1}{r}\right) = \frac{2\ln r}{r(\ln 3)(\ln 9)}$

73. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3)\ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$
$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

74. $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2}\right)\left[\frac{\ln\left(\frac{7x}{3x+2}\right)}{\ln 5}\right] = \frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)$
$= \frac{1}{2}\ln 7x - \frac{1}{2}\ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2\cdot 7x} - \frac{3}{2\cdot(3x+2)} = \frac{(3x+2)-3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$

75. $y = \theta\sin(\log_7\theta) = \theta\sin\left(\frac{\ln\theta}{\ln 7}\right) \Rightarrow \frac{dy}{d\theta} = \sin\left(\frac{\ln\theta}{\ln 7}\right) + \theta\left[\cos\left(\frac{\ln\theta}{\ln 7}\right)\right]\left(\frac{1}{\theta\ln 7}\right) = \sin(\log_7\theta) + \frac{1}{\ln 7}\cos(\log_7\theta)$

76. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right) = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \ln e^{\theta} - \ln 2^{\theta}}{\ln 7} = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \theta - \theta\ln 2}{\ln 7}$
$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta}{(\sin\theta)(\ln 7)} - \frac{\sin\theta}{(\cos\theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7}\right)(\cot\theta - \tan\theta - 1 - \ln 2)$

77. $y = \log_{10}e^x = \frac{\ln e^x}{\ln 10} = \frac{x}{\ln 10} \Rightarrow y' = \frac{1}{\ln 10}$

78. $y = \frac{\theta\cdot 5^{\theta}}{2-\log_5\theta} = \frac{\theta\cdot 5^{\theta}}{2-\frac{\ln\theta}{\ln 5}} \Rightarrow y' = \frac{\left(2-\frac{\ln\theta}{\ln 5}\right)\left(\theta\cdot 5^{\theta}\ln 5 + 5^{\theta}(1)\right) - \left(\theta\cdot 5^{\theta}\right)\left(-\frac{1}{\theta\ln 5}\right)}{\left(2-\frac{\ln\theta}{\ln 5}\right)^2} = \frac{5^{\theta}\ln 5(2-\log_5\theta)(\theta\ln 5+1) + 5^{\theta}}{\ln 5(2-\log_5\theta)^2}$

79. $y = 3^{\log_2 t} = 3^{(\ln t)/(\ln 2)} \Rightarrow \frac{dy}{dt} = \left[3^{(\ln t)/(\ln 2)}(\ln 3)\right]\left(\frac{1}{t \ln 2}\right) = \frac{1}{t}(\log_2 3)\, 3^{\log_2 t}$

80. $y = 3 \log_8 (\log_2 t) = \frac{3 \ln(\log_2 t)}{\ln 8} = \frac{3 \ln\left(\frac{\ln t}{\ln 2}\right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8}\right)\left[\frac{1}{(\ln t)/(\ln 2)}\right]\left(\frac{1}{t \ln 2}\right) = \frac{3}{t(\ln t)(\ln 8)} = \frac{1}{t(\ln t)(\ln 2)}$

81. $y = \log_2 (8t^{\ln 2}) = \frac{\ln 8 + \ln(t^{\ln 2})}{\ln 2} = \frac{3 \ln 2 + (\ln 2)(\ln t)}{\ln 2} = 3 + \ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$

82. $y = \frac{t \ln\left(\left(e^{\ln 3}\right)^{\sin t}\right)}{\ln 3} = \frac{t \ln(3^{\sin t})}{\ln 3} = \frac{t(\sin t)(\ln 3)}{\ln 3} = t \sin t \Rightarrow \frac{dy}{dt} = \sin t + t \cos t$

83. $\int 5^x\, dx = \frac{5^x}{\ln 5} + C$

84. Let $u = 3 - 3^x \Rightarrow du = -3^x \ln 3\, dx \Rightarrow -\frac{1}{\ln 3} du = 3^x dx$;
$\int \frac{3^x}{3 - 3^x} dx = -\frac{1}{\ln 3}\int \frac{1}{u} du = -\frac{1}{\ln 3}\ln|u| + C = -\frac{\ln|3 - 3^x|}{\ln 3} + C$

85. $\int_0^1 2^{-\theta}\, d\theta = \int_0^1 \left(\frac{1}{2}\right)^{\theta} d\theta = \left[\frac{\left(\frac{1}{2}\right)^{\theta}}{\ln\left(\frac{1}{2}\right)}\right]_0^1 = \frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} - \frac{1}{\ln\left(\frac{1}{2}\right)} = -\frac{\frac{1}{2}}{\ln\left(\frac{1}{2}\right)} = \frac{-1}{2(\ln 1 - \ln 2)} = \frac{1}{2 \ln 2}$

86. $\int_{-2}^0 5^{-\theta}\, d\theta = \int_{-2}^0 \left(\frac{1}{5}\right)^{\theta} d\theta = \left[\frac{\left(\frac{1}{5}\right)^{\theta}}{\ln\left(\frac{1}{5}\right)}\right]_{-2}^0 = \frac{1}{\ln\left(\frac{1}{5}\right)} - \frac{\left(\frac{1}{5}\right)^{-2}}{\ln\left(\frac{1}{5}\right)} = \frac{1}{\ln\left(\frac{1}{5}\right)}(1 - 25) = \frac{-24}{\ln 1 - \ln 5} = \frac{24}{\ln 5}$

87. Let $u = x^2 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2} du = x\, dx$; $x = 1 \Rightarrow u = 1$, $x = \sqrt{2} \Rightarrow u = 2$;
$\int_1^{\sqrt{2}} x 2^{(x^2)}\, dx = \int_1^2 \left(\frac{1}{2}\right) 2^u\, du = \frac{1}{2}\left[\frac{2^u}{\ln 2}\right]_1^2 = \left(\frac{1}{2 \ln 2}\right)(2^2 - 2^1) = \frac{1}{\ln 2}$

88. Let $u = x^{1/2} \Rightarrow du = \frac{1}{2} x^{-1/2}\, dx \Rightarrow 2\, du = \frac{dx}{\sqrt{x}}$; $x = 1 \Rightarrow u = 1$, $x = 4 \Rightarrow u = 2$;
$\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\, dx = \int_1^4 2^{x^{1/2}} \cdot x^{-1/2}\, dx = 2\int_1^2 2^u\, du = \left[\frac{2^{(u+1)}}{\ln 2}\right]_1^2 = \left(\frac{1}{\ln 2}\right)(2^3 - 2^2) = \frac{4}{\ln 2}$

89. Let $u = \cos t \Rightarrow du = -\sin t\, dt \Rightarrow -du = \sin t\, dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{2} \Rightarrow u = 0$;
$\int_0^{\pi/2} 7^{\cos t} \sin t\, dt = -\int_1^0 7^u\, du = \left[-\frac{7^u}{\ln 7}\right]_1^0 = \left(\frac{-1}{\ln 7}\right)(7^0 - 7) = \frac{6}{\ln 7}$

90. Let $u = \tan t \Rightarrow du = \sec^2 t\, dt$; $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{4} \Rightarrow u = 1$;
$\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\, dt = \int_0^1 \left(\frac{1}{3}\right)^u du = \left[\frac{\left(\frac{1}{3}\right)^u}{\ln\left(\frac{1}{3}\right)}\right]_0^1 = \left(-\frac{1}{\ln 3}\right)\left[\left(\frac{1}{3}\right)^1 - \left(\frac{1}{3}\right)^0\right] = \frac{2}{3 \ln 3}$

91. Let $u = x^{2x} \Rightarrow \ln u = 2x \ln x \Rightarrow \frac{1}{u}\frac{du}{dx} = 2 \ln x + (2x)\left(\frac{1}{x}\right) \Rightarrow \frac{du}{dx} = 2u(\ln x + 1) \Rightarrow \frac{1}{2} du = x^{2x}(1 + \ln x)\, dx$;
$x = 2 \Rightarrow u = 2^4 = 16$, $x = 4 \Rightarrow u = 4^8 = 65{,}536$;
$\int_2^4 x^{2x}(1 + \ln x)\, dx = \frac{1}{2}\int_{16}^{65{,}536} du = \frac{1}{2}[u]_{16}^{65{,}536} = \frac{1}{2}(65{,}536 - 16) = \frac{65{,}520}{2} = 32{,}760$

92. Let $u = 1 + 2^{x^2} \Rightarrow du = 2^{x^2}(2x)\ln 2\, dx \Rightarrow \frac{1}{2 \ln 2} du = 2^{x^2} x\, dx$
$\int \frac{x\, 2^{x^2}}{1 + 2^{x^2}} dx = \frac{1}{2 \ln 2}\int \frac{1}{u} du = \frac{1}{2 \ln 2}\ln|u| + C = \frac{\ln\left(1 + 2^{x^2}\right)}{2 \ln 2} + C$

93. $\int 3x^{\sqrt{3}}\, dx = \frac{3x^{(\sqrt{3}+1)}}{\sqrt{3}+1} + C$

94. $\int x^{(\sqrt{2}-1)}\, dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$

95. $\int_0^3 \left(\sqrt{2}+1\right) x^{\sqrt{2}}\, dx = \left[x^{\left(\sqrt{2}+1\right)}\right]_0^3 = 3^{\left(\sqrt{2}+1\right)}$

96. $\int_1^e x^{(\ln 2)-1}\, dx = \left[\frac{x^{\ln 2}}{\ln 2}\right]_1^e = \frac{e^{\ln 2} - 1^{\ln 2}}{\ln 2} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$

97. $\int \frac{\log_{10} x}{x}\, dx = \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx\right]$
$\rightarrow \int \left(\frac{\ln x}{\ln 10}\right)\left(\frac{1}{x}\right) dx = \frac{1}{\ln 10}\int u\, du = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{2}u^2\right) + C = \frac{(\ln x)^2}{2\ln 10} + C$

98. $\int_1^4 \frac{\log_2 x}{x}\, dx = \int_1^4 \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx;\ x = 1 \Rightarrow u = 0,\ x = 4 \Rightarrow u = \ln 4\right]$
$\rightarrow \int_1^4 \left(\frac{\ln x}{\ln 2}\right)\left(\frac{1}{x}\right) dx = \int_0^{\ln 4}\left(\frac{1}{\ln 2}\right) u\, du = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2}u^2\right]_0^{\ln 4} = \left(\frac{1}{\ln 2}\right)\left[\frac{1}{2}(\ln 4)^2\right] = \frac{(\ln 4)^2}{2\ln 2} = \frac{(\ln 4)^2}{\ln 4} = \ln 4$

99. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\, dx = \int_1^4 \left(\frac{\ln 2}{x}\right)\left(\frac{\ln x}{\ln 2}\right) dx = \int_1^4 \frac{\ln x}{x}\, dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}\left[(\ln 4)^2 - (\ln 1)^2\right] = \frac{1}{2}(\ln 4)^2 = \frac{1}{2}(2\ln 2)^2 = 2(\ln 2)^2$

100. $\int_1^e \frac{2\ln 10\,(\log_{10} x)}{x}\, dx = \int_1^e \frac{(\ln 10)(2\ln x)}{(\ln 10)}\left(\frac{1}{x}\right) dx = \left[(\ln x)^2\right]_1^e = (\ln e)^2 - (\ln 1)^2 = 1$

101. $\int_0^2 \frac{\log_2 (x+2)}{x+2}\, dx = \frac{1}{\ln 2}\int_0^2 [\ln(x+2)]\left(\frac{1}{x+2}\right) dx = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln(x+2))^2}{2}\right]_0^2 = \left(\frac{1}{\ln 2}\right)\left[\frac{(\ln 4)^2}{2} - \frac{(\ln 2)^2}{2}\right]$
$= \left(\frac{1}{\ln 2}\right)\left[\frac{4(\ln 2)^2}{2} - \frac{(\ln 2)^2}{2}\right] = \frac{3}{2}\ln 2$

102. $\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}\, dx = \frac{10}{\ln 10}\int_{1/10}^{10}[\ln(10x)]\left(\frac{1}{10x}\right) dx = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln(10x))^2}{20}\right]_{1/10}^{10} = \left(\frac{10}{\ln 10}\right)\left[\frac{(\ln 100)^2}{20} - \frac{(\ln 1)^2}{2}\right]$
$= \left(\frac{10}{\ln 10}\right)\left[\frac{4(\ln 10)^2}{20}\right] = 2\ln 10$

103. $\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\, dx = \frac{2}{\ln 10}\int_0^9 \ln(x+1)\left(\frac{1}{x+1}\right) dx = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln(x+1))^2}{2}\right]_0^9 = \left(\frac{2}{\ln 10}\right)\left[\frac{(\ln 10)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 10$

104. $\int_2^3 \frac{2\log_2(x-1)}{x-1}\, dx = \frac{2}{\ln 2}\int_2^3 \ln(x-1)\left(\frac{1}{x-1}\right) dx = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln(x-1))^2}{2}\right]_2^3 = \left(\frac{2}{\ln 2}\right)\left[\frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2}\right] = \ln 2$

105. $\int \frac{dx}{x\log_{10} x} = \int \left(\frac{\ln 10}{\ln x}\right)\left(\frac{1}{x}\right) dx = (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx$; $\left[u = \ln x \Rightarrow du = \frac{1}{x}\, dx\right]$
$\rightarrow (\ln 10)\int \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx = (\ln 10)\int \frac{1}{u}\, du = (\ln 10)\ln|u| + C = (\ln 10)\ln|\ln x| + C$

106. $\int \frac{dx}{x(\log_8 x)^2} = \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} = (\ln 8)^2 \int \frac{(\ln x)^{-2}}{x}\, dx = (\ln 8)^2\, \frac{(\ln x)^{-1}}{-1} + C = -\frac{(\ln 8)^2}{\ln x} + C$

107. $\int_1^{\ln x} \frac{1}{t}\, dt = [\ln|t|]_1^{\ln x} = \ln|\ln x| - \ln 1 = \ln(\ln x),\ x > 1$

108. $\int_1^{e^x} \frac{1}{t}\, dt = [\ln|t|]_1^{e^x} = \ln e^x - \ln 1 = x\ln e = x$

109. $\int_1^{1/x} \frac{1}{t}\, dt = [\ln|t|]_1^{1/x} = \ln\left|\frac{1}{x}\right| - \ln 1 = (\ln 1 - \ln|x|) - \ln 1 = -\ln x,\ x > 0$

110. $\frac{1}{\ln a}\int_1^x \frac{1}{t}\, dt = \left[\frac{1}{\ln a}\ln|t|\right]_1^x = \frac{\ln x}{\ln a} - \frac{\ln 1}{\ln a} = \log_a x,\ x > 0$

111. $y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x\ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x\cdot\frac{1}{(x+1)} \Rightarrow y' = (x+1)^x\left[\frac{x}{x+1} + \ln(x+1)\right]$

112. $y = x^2 + x^{2x} \Rightarrow y - x^2 = x^{2x} \Rightarrow \ln(y - x^2) = \ln x^{2x} = 2x \ln x \Rightarrow \frac{1}{y - x^2}(y' - 2x) = 2x \cdot \frac{1}{x} + 2 \cdot \ln x = 2 + 2\ln x$
$\Rightarrow y' - 2x = (y - x^2)(2 + 2\ln x) \Rightarrow y' = ((x^2 + x^{2x}) - x^2)(2 + 2\ln x) + 2x = 2(x + x^{2x} + x^{2x}\ln x)$

113. $y = \left(\sqrt{t}\right)^t = \left(t^{1/2}\right)^t = t^{t/2} \Rightarrow \ln y = \ln t^{t/2} = \left(\frac{t}{2}\right)\ln t \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}\right)(\ln t) + \left(\frac{t}{2}\right)\left(\frac{1}{t}\right) = \frac{\ln t}{2} + \frac{1}{2}$
$\Rightarrow \frac{dy}{dt} = \left(\sqrt{t}\right)^t\left(\frac{\ln t}{2} + \frac{1}{2}\right)$

114. $y = t^{\sqrt{t}} = t^{(t^{1/2})} \Rightarrow \ln y = \ln t^{(t^{1/2})} = \left(t^{1/2}\right)(\ln t) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}t^{-1/2}\right)(\ln t) + t^{1/2}\left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}} \Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}}\right)t^{\sqrt{t}}$

115. $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{y'}{y} = \ln(\sin x) + x\left(\frac{\cos x}{\sin x}\right) \Rightarrow y' = (\sin x)^x[\ln(\sin x) + x\cot x]$

116. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x}$
$\Rightarrow y' = x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]$

117. $y = \sin x^x \Rightarrow y' = \cos x^x \frac{d}{dx}(x^x)$; if $u = x^x \Rightarrow \ln u = \ln x^x = x\ln x \Rightarrow \frac{u'}{u} = x \cdot \frac{1}{x} + 1 \cdot \ln x = 1 + \ln x$
$\Rightarrow u' = x^x(1 + \ln x) \Rightarrow y' = \cos x^x \cdot x^x(1 + \ln x) = x^x\cos x^x(1 + \ln x)$

118. $y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right)\ln(\ln x) + (\ln x)\left(\frac{1}{\ln x}\right)\frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x}$
$\Rightarrow y' = \left(\frac{\ln(\ln x) + 1}{x}\right)(\ln x)^{\ln x}$

119. $f(x) = e^x - 2x \Rightarrow f'(x) = e^x - 2$; $f'(x) = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$; $f(0) = 1$, the absolute maximum; $f(\ln 2) = 2 - 2\ln 2 \approx 0.613706$, the absolute minimum; $f(1) = e - 2 \approx 0.71828$, a relative or local maximum since $f''(x) = e^x$ is always positive.

120. The function $f(x) = 2e^{\sin(x/2)}$ has a maximum whenever $\sin\frac{x}{2} = 1$ and a minimum whenever $\sin\frac{x}{2} = -1$. Therefore the maximums occur at $x = \pi + 2k(2\pi)$ and the minimums occur at $x = 3\pi + 2k(2\pi)$, where k is any integer. The maximum is $2e \approx 5.43656$ and the minimum is $\frac{2}{e} \approx 0.73576$.

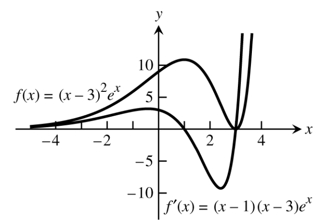
121. $f(x) = xe^{-x} \Rightarrow f'(x) = xe^{-x}(-1) + e^{-x} = e^{-x} - xe^{-x} \Rightarrow f''(x) = -e^{-x} - (xe^{-x}(-1) + e^{-x}) = xe^{-x} - 2e^{-x}$
(a) $f'(x) = 0 \Rightarrow e^{-x} - xe^{-x} = e^{-x}(1 - x) = 0 \Rightarrow e^{-x} = 0$ or $1 - x = 0 \Rightarrow x = 1$, $f(1) = (1)e^{-1} = \frac{1}{e}$; using second derivative test, $f''(1) = (1)e^{-1} - 2e^{-1} = -\frac{1}{e} < 0 \Rightarrow$ absolute maximum at $\left(1, \frac{1}{e}\right)$
(b) $f''(x) = 0 \Rightarrow xe^{-x} - 2e^{-x} = e^{-x}(x - 2) = 0 \Rightarrow e^{-x} = 0$ or $x - 2 = 0 \Rightarrow x = 2$, $f(2) = (2)e^{-2} = \frac{2}{e^2}$; since $f''(1) < 0$ and $f''(3) = e^{-3}(3 - 2) = \frac{1}{e^3} > 0 \Rightarrow$ point of inflection at $\left(2, \frac{2}{e^2}\right)$

122. $f(x) = \frac{e^x}{1 + e^{2x}} \Rightarrow f'(x) = \frac{(1 + e^{2x})e^x - e^x(2e^{2x})}{(1 + e^{2x})^2} = \frac{e^x - e^{3x}}{(1 + e^{2x})^2} \Rightarrow f''(x) = \frac{(1 + e^{2x})^2(e^x - 3e^{3x}) - (e^x - e^{3x})2(1 + e^{2x})(2e^{2x})}{\left[(1 + e^{2x})^2\right]^2}$
$= \frac{e^x(1 - 6e^{2x} + e^{4x})}{(1 + e^{2x})^3}$
(a) $f'(x) = 0 \Rightarrow e^x - e^{3x} = 0 \Rightarrow e^x(1 - e^{2x}) = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$; $f(0) = \frac{e^0}{1 + e^{2(0)}} = \frac{1}{2}$;
$f'(x) =$ undefined $\Rightarrow (1 + e^{2x})^2 = 0 \Rightarrow e^{2x} = -1 \Rightarrow$ no real solutions. Using the second derivative test, $f''(0) = \frac{e^0(1 - 6e^{2(0)} + e^{4(0)})}{(1 + e^{2(0)})^3} = \frac{-4}{8} < 0 \Rightarrow$ absolute maximum at $\left(0, \frac{1}{2}\right)$
(b) $f''(x) = 0 \Rightarrow e^x(1 - 6e^{2x} + e^{4x}) \Rightarrow e^x = 0$ or $1 - 6e^{2x} + e^{4x} = 0 \Rightarrow e^{2x} = \frac{-(-6) \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}$,
$\Rightarrow x = \frac{\ln\left(3 + 2\sqrt{2}\right)}{2}$ or $x = \frac{\ln\left(3 - 2\sqrt{2}\right)}{2}$. $f\left(\frac{\ln\left(3 + 2\sqrt{2}\right)}{2}\right) = \frac{\sqrt{3 + 2\sqrt{2}}}{4 + 2\sqrt{2}}$ and $f\left(\frac{\ln\left(3 - 2\sqrt{2}\right)}{2}\right) = \frac{\sqrt{3 - 2\sqrt{2}}}{4 - 2\sqrt{2}}$;

since $f''(-1) > 0,\ f''(0) < 0$, and $f''(1) > 0 \Rightarrow$ points of inflection at $\left(\frac{\ln\left(3+2\sqrt{2}\right)}{2}, \frac{\sqrt{3+2\sqrt{2}}}{4+2\sqrt{2}}\right)$ and $\left(\frac{\ln\left(3-2\sqrt{2}\right)}{2}, \frac{\sqrt{3-2\sqrt{2}}}{4-2\sqrt{2}}\right)$.

123. $f(x) = x^2 \ln \frac{1}{x} \Rightarrow f'(x) = 2x \ln \frac{1}{x} + x^2 \left(\frac{1}{\frac{1}{x}}\right)(-x^{-2}) = 2x \ln \frac{1}{x} - x = -x(2 \ln x + 1)$; $f'(x) = 0 \Rightarrow x = 0$ or $\ln x = -\frac{1}{2}$. Since $x = 0$ is not in the domain of f, $x = e^{-1/2} = \frac{1}{\sqrt{e}}$. Also, $f'(x) > 0$ for $0 < x < \frac{1}{\sqrt{e}}$ and $f'(x) < 0$ for $x > \frac{1}{\sqrt{e}}$. Therefore, $f\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln \sqrt{e} = \frac{1}{e} \ln e^{1/2} = \frac{1}{2e} \ln e = \frac{1}{2e}$ is the absolute maximum value of f assumed at $x = \frac{1}{\sqrt{e}}$.

124. $f(x) = (x-3)^2 e^x \Rightarrow f'(x) = 2(x-3)e^x + (x-3)^2 e^x = (x-3)e^x(2 + x - 3) = (x-1)(x-3)e^x$; thus $f'(x) > 0$ for $x < 1$ or $x > 3$, and $f'(x) < 0$ for $1 < x < 3 \Rightarrow f(1) = 4e \approx 10.87$ is a local maximum and $f(3) = 0$ is a local minimum. Since $f(x) \ge 0$ for all x, $f(3) = 0$ is also an absolute minimum.



125. $\int_0^{\ln 3} (e^{2x} - e^x)\,dx = \left[\frac{e^{2x}}{2} - e^x\right]_0^{\ln 3} = \left(\frac{e^{2\ln 3}}{2} - e^{\ln 3}\right) - \left(\frac{e^0}{2} - e^0\right) = \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{8}{2} - 2 = 2$

126. $\int_0^{2\ln 2} \left(e^{x/2} - e^{-x/2}\right) dx = \left[2e^{x/2} + 2e^{-x/2}\right]_0^{2\ln 2} = \left(2e^{\ln 2} + 2e^{-\ln 2}\right) - \left(2e^0 + 2e^0\right) = (4+1) - (2+2) = 5 - 4 = 1$

127. $L = \int_0^1 \sqrt{1 + \frac{e^x}{4}}\,dx \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2} \Rightarrow y = e^{x/2} + C$; $y(0) = 0 \Rightarrow 0 = e^0 + C \Rightarrow C = -1 \Rightarrow y = e^{x/2} - 1$

128. $S = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right) \sqrt{1 + \left(\frac{e^y - e^{-y}}{2}\right)^2}\,dy = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right) \sqrt{1 + \frac{1}{4}\left(e^{2y} - 2 + e^{-2y}\right)}\,dy$
$= 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right) \sqrt{\left(\frac{e^y + e^{-y}}{2}\right)^2}\,dy = 2\pi \int_0^{\ln 2} \left(\frac{e^y + e^{-y}}{2}\right)^2 dy = \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + 2 + e^{-2y}\right)dy$
$= \frac{\pi}{2}\left[\frac{1}{2}e^{2y} + 2y - \frac{1}{2}e^{-2y}\right]_0^{\ln 2} = \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2} + 2\ln 2 - \frac{1}{2}e^{-2\ln 2}\right) - \left(\frac{1}{2} + 0 - \frac{1}{2}\right)\right]$
$= \frac{\pi}{2}\left(\frac{1}{2} \cdot 4 + 2\ln 2 - \frac{1}{2} \cdot \frac{1}{4}\right) = \frac{\pi}{2}\left(2 - \frac{1}{8} + 2\ln 2\right) = \pi\left(\frac{15}{16} + \ln 2\right)$

129. $y = \frac{1}{2}(e^x + e^{-x}) \Rightarrow \frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$; $L = \int_0^1 \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2}\,dx = \int_0^1 \sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}}\,dx$
$= \int_0^1 \sqrt{\frac{e^{2x}}{4} + \frac{1}{2} + \frac{e^{-2x}}{4}}\,dx = \int_0^1 \sqrt{\left(\frac{1}{2}(e^x + e^{-x})\right)^2}\,dx = \int_0^1 \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}\left[e^x - e^{-x}\right]_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right) - 0 = \frac{e^2 - 1}{2e}$

130. $y = \ln(e^x - 1) - \ln(e^x + 1) \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{e^{2x} - 1}$; $L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2}\,dx = \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}}\,dx$
$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{(e^{2x} - 1)^2}}\,dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{(e^{2x} - 1)^2}}\,dx = \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}}\,dx = \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1}\,dx = \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x} + 1}{e^x}}{\frac{e^{2x} - 1}{e^x}}\,dx$
$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$; $\Big[$let $u = e^x - e^{-x} \Rightarrow du = (e^x + e^{-x})dx$, $x = \ln 2 \Rightarrow u = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2}$, $x = \ln 3$
$\Rightarrow u = e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{3} = \frac{8}{3}\Big] \to \int_{3/2}^{8/3} \frac{1}{u}\,du = \left[\ln|u|\right]_{3/2}^{8/3} = \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right) = \ln\left(\frac{16}{9}\right)$

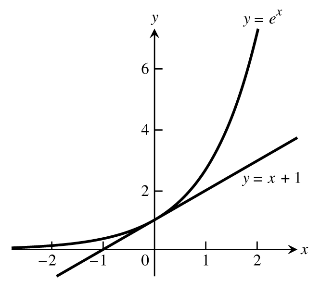
131. $y = \ln\cos x \Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$; $L = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\,dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$
$= \int_0^{\pi/4} \sec x\,dx = \left[\ln|\sec x + \tan x|\right]_0^{\pi/4} = \left(\ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right|\right) - (0) = \ln\left(\sqrt{2} + 1\right)$

132. $y = \ln \csc x \Rightarrow \frac{dy}{dx} = \frac{-\csc x \cot x}{\csc x} = -\cot x$; $L = \int_{\pi/6}^{\pi/4} \sqrt{1 + (-\cot x)^2}\, dx = \int_{\pi/6}^{\pi/4} \sqrt{1 + \cot^2 x}\, dx = \int_{\pi/6}^{\pi/4} \sqrt{\csc^2 x}\, dx$
$= \int_{\pi/6}^{\pi/4} \csc x\, dx = \left[-\ln|\csc x + \cot x|\right]_{\pi/6}^{\pi/4} = \left(-\ln\left|\csc\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right)\right|\right) + \left(\ln\left|\csc\left(\frac{\pi}{6}\right) + \cot\left(\frac{\pi}{6}\right)\right|\right)$
$= -\ln\left(\sqrt{2} + 1\right) + \ln\left(2 + \sqrt{3}\right) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right)$

133. (a) $\frac{d}{dx}(x \ln x - x + C) = x \cdot \frac{1}{x} + \ln x - 1 + 0 = \ln x$
(b) average value $= \frac{1}{e-1}\int_1^e \ln x\, dx = \frac{1}{e-1}\left[x \ln x - x\right]_1^e = \frac{1}{e-1}\left[(e \ln e - e) - (1 \ln 1 - 1)\right] = \frac{1}{e-1}(e - e + 1) = \frac{1}{e-1}$

134. average value $= \frac{1}{2-1}\int_1^2 \frac{1}{x}\, dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2$

135. (a) $f(x) = e^x \Rightarrow f'(x) = e^x$; $L(x) = f(0) + f'(0)(x - 0) \Rightarrow L(x) = 1 + x$
(b) $f(0) = 1$ and $L(0) = 1 \Rightarrow$ error $= 0$; $f(0.2) = e^{0.2} \approx 1.22140$ and $L(0.2) = 1.2 \Rightarrow$ error ≈ 0.02140
(c) Since $y'' = e^x > 0$, the tangent line approximation always lies below the curve $y = e^x$. Thus $L(x) = x + 1$ never overestimates e^x.

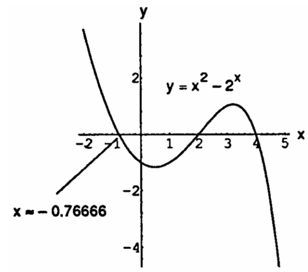


136. (a) $y = e^x \Rightarrow y'' = e^x > 0$ for all $x \Rightarrow$ the graph of $y = e^x$ is always concave upward
(b) area of the trapezoid ABCD $< \int_{\ln a}^{\ln b} e^x\, dx <$ area of the trapezoid AEFD $\Rightarrow \frac{1}{2}(AB + CD)(\ln b - \ln a)$
$< \int_{\ln a}^{\ln b} e^x\, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$. Now $\frac{1}{2}(AB + CD)$ is the height of the midpoint
$M = e^{(\ln a + \ln b)/2}$ since the curve containing the points B and C is linear $\Rightarrow e^{(\ln a + \ln b)/2}(\ln b - \ln a)$
$< \int_{\ln a}^{\ln b} e^x\, dx < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a)$
(c) $\int_{\ln a}^{\ln b} e^x\, dx = \left[e^x\right]_{\ln a}^{\ln b} = e^{\ln b} - e^{\ln a} = b - a$, so part (b) implies that
$e^{(\ln a + \ln b)/2}(\ln b - \ln a) < b - a < \left(\frac{e^{\ln a} + e^{\ln b}}{2}\right)(\ln b - \ln a) \Rightarrow e^{(\ln a + \ln b)/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$
$\Rightarrow e^{\ln a/2} \cdot e^{\ln b/2} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow \sqrt{e^{\ln a}}\sqrt{e^{\ln b}} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} \Rightarrow \sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$

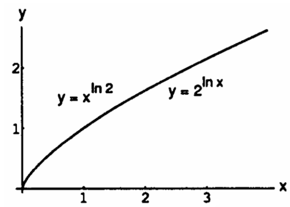
137. $A = \int_{-2}^{2} \frac{2x}{1+x^2}\, dx = 2\int_0^2 \frac{2x}{1+x^2}\, dx$; $[u = 1 + x^2 \Rightarrow du = 2x\, dx;\ x = 0 \Rightarrow u = 1,\ x = 2 \Rightarrow u = 5]$
$\rightarrow A = 2\int_1^5 \frac{1}{u}\, du = 2\left[\ln|u|\right]_1^5 = 2(\ln 5 - \ln 1) = 2 \ln 5$

138. $A = \int_{-1}^{1} 2^{(1-x)}\, dx = 2\int_{-1}^{1}\left(\frac{1}{2}\right)^x dx = 2\left[\frac{\left(\frac{1}{2}\right)^x}{\ln\left(\frac{1}{2}\right)}\right]_{-1}^{1} = -\frac{2}{\ln 2}\left(\frac{1}{2} - 2\right) = \left(-\frac{2}{\ln 2}\right)\left(-\frac{3}{2}\right) = \frac{3}{\ln 2}$

139. From zooming in on the graph at the right, we estimate the third root to be $x \approx -0.76666$

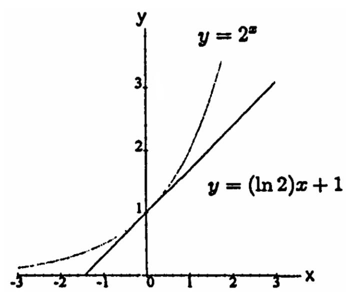


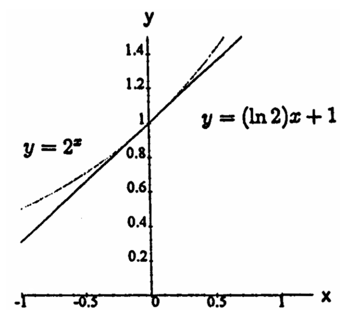
140. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = \left(e^{\ln 2}\right)^{\ln x} = 2^{\ln x}$.



141. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2$; $L(x) = \left(2^0 \ln 2\right) x + 2^0 = x \ln 2 + 1 \approx 0.69x + 1$

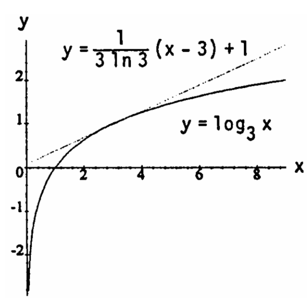
(b)

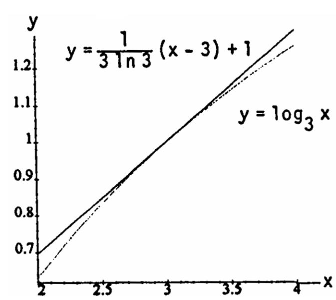




142. (a) $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x \ln 3}$, and $f(3) = \frac{\ln 3}{\ln 3} \Rightarrow L(x) = \frac{1}{3 \ln 3}(x - 3) + \frac{\ln 3}{\ln 3} = \frac{x}{3 \ln 3} - \frac{1}{\ln 3} + 1 \approx 0.30x + 0.09$

(b)





143. (a) The point of tangency is $(p, \ln p)$ and $m_{\text{tangent}} = \frac{1}{p}$ since $\frac{dy}{dx} = \frac{1}{x}$. The tangent line passes through $(0, 0) \Rightarrow$ the equation of the tangent line is $y = \frac{1}{p}x$. The tangent line also passes through $(p, \ln p) \Rightarrow \ln p = \frac{1}{p}p = 1 \Rightarrow p = e$, and the tangent line equation is $y = \frac{1}{e}x$.

(b) $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$ for $x \neq 0 \Rightarrow y = \ln x$ is concave downward over its domain. Therefore, $y = \ln x$ lies below the graph of $y = \frac{1}{e}x$ for all $x > 0$, $x \neq e$, and $\ln x < \frac{x}{e}$ for $x > 0$, $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

144. Using Newton's Method: $f(x) = \ln(x) - 1 \Rightarrow f'(x) = \frac{1}{x} \Rightarrow x_{n+1} = x_n - \frac{\ln(x_n) - 1}{\frac{1}{xn}} \Rightarrow x_{n+1} = x_n\left[2 - \ln(x_n)\right]$.
Then, $x_1 = 2$, $x_2 = 2.61370564$, $x_3 = 2.71624393$, and $x_5 = 2.71828183$. Many other methods may be used. For example, graph $y = \ln x - 1$ and determine the zero of y.

7.4 EXPONENTIAL CHANGE AND SEPARABLE DIFFERENTIAL EQUATIONS

1. (a) $y = e^{-x} \Rightarrow y' = -e^{-x} \Rightarrow 2y' + 3y = 2(-e^{-x}) + 3e^{-x} = e^{-x}$
(b) $y = e^{-x} + e^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}e^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}e^{-3x/2}\right) + 3\left(e^{-x} + e^{-3x/2}\right) = e^{-x}$
(c) $y = e^{-x} + Ce^{-3x/2} \Rightarrow y' = -e^{-x} - \frac{3}{2}Ce^{-3x/2} \Rightarrow 2y' + 3y = 2\left(-e^{-x} - \frac{3}{2}Ce^{-3x/2}\right) + 3\left(e^{-x} + Ce^{-3x/2}\right) = e^{-x}$

2. (a) $y = -\frac{1}{x} \Rightarrow y' = \frac{1}{x^2} = \left(-\frac{1}{x}\right)^2 = y^2$
(b) $y = -\frac{1}{x+3} \Rightarrow y' = \frac{1}{(x+3)^2} = \left[-\frac{1}{(x+3)}\right]^2 = y^2$
(c) $y = \frac{1}{x+C} \Rightarrow y' = \frac{1}{(x+C)^2} = \left[-\frac{1}{x+C}\right]^2 = y^2$

3. $y = \frac{1}{x}\int_1^x \frac{e^t}{t}\,dt \Rightarrow y' = -\frac{1}{x^2}\int_1^x \frac{e^t}{t}\,dt + \left(\frac{1}{x}\right)\left(\frac{e^x}{x}\right) \Rightarrow x^2y' = -\int_1^x \frac{e^t}{t}\,dt + e^x = -x\left(\frac{1}{x}\int_1^x \frac{e^t}{t}\,dt\right) + e^x = -xy + e^x$
$\Rightarrow x^2y' + xy = e^x$

4. $y = \frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt \Rightarrow y' = -\frac{1}{2}\left[\frac{4x^3}{\left(\sqrt{1+x^4}\right)^3}\right]\int_1^x \sqrt{1+t^4}\,dt + \frac{1}{\sqrt{1+x^4}}\left(\sqrt{1+x^4}\right)$
$\Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right)\left(\frac{1}{\sqrt{1+x^4}}\int_1^x \sqrt{1+t^4}\,dt\right) + 1 \Rightarrow y' = \left(\frac{-2x^3}{1+x^4}\right)y + 1 \Rightarrow y' + \frac{2x^3}{1+x^4}\cdot y = 1$

5. $y = e^{-x}\tan^{-1}(2e^x) \Rightarrow y' = -e^{-x}\tan^{-1}(2e^x) + e^{-x}\left[\frac{1}{1+(2e^x)^2}\right](2e^x) = -e^{-x}\tan^{-1}(2e^x) + \frac{2}{1+4e^{2x}}$
$\Rightarrow y' = -y + \frac{2}{1+4e^{2x}} \Rightarrow y' + y = \frac{2}{1+4e^{2x}}$; $y(-\ln 2) = e^{-(-\ln 2)}\tan^{-1}(2e^{-\ln 2}) = 2\tan^{-1} 1 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

6. $y = (x-2)e^{-x^2} \Rightarrow y' = e^{-x^2} + \left(-2xe^{-x^2}\right)(x-2) \Rightarrow y' = e^{-x^2} - 2xy$; $y(2) = (2-2)e^{-2^2} = 0$

7. $y = \frac{\cos x}{x} \Rightarrow y' = \frac{-x\sin x - \cos x}{x^2} \Rightarrow y' = -\frac{\sin x}{x} - \frac{1}{x}\left(\frac{\cos x}{x}\right) \Rightarrow y' = -\frac{\sin x}{x} - \frac{y}{x} \Rightarrow xy' = -\sin x - y \Rightarrow xy' + y = -\sin x$;
$y\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/2)}{(\pi/2)} = 0$

8. $y = \frac{x}{\ln x} \Rightarrow y' = \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \Rightarrow y' = \frac{1}{\ln x} - \frac{1}{(\ln x)^2} \Rightarrow x^2y' = \frac{x^2}{\ln x} - \frac{x^2}{(\ln x)^2} \Rightarrow x^2y' = xy - y^2$; $y(e) = \frac{e}{\ln e} = e$.

9. $2\sqrt{xy}\,\frac{dy}{dx} = 1 \Rightarrow 2x^{1/2}y^{1/2}\,dy = dx \Rightarrow 2y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \int 2y^{1/2}\,dy = \int x^{-1/2}\,dx \Rightarrow 2\left(\frac{2}{3}y^{3/2}\right) = 2x^{1/2} + C_1$
$\Rightarrow \frac{2}{3}y^{3/2} - x^{1/2} = C$, where $C = \frac{1}{2}C_1$

10. $\frac{dy}{dx} = x^2\sqrt{y} \Rightarrow dy = x^2y^{1/2}\,dx \Rightarrow y^{-1/2}\,dy = x^2\,dx \Rightarrow \int y^{-1/2}\,dy = \int x^2\,dx \Rightarrow 2y^{1/2} = \frac{x^3}{3} + C \Rightarrow 2y^{1/2} - \frac{1}{3}x^3 = C$

11. $\frac{dy}{dx} = e^{x-y} \Rightarrow dy = e^xe^{-y}\,dx \Rightarrow e^y\,dy = e^x\,dx \Rightarrow \int e^y\,dy = \int e^x\,dx \Rightarrow e^y = e^x + C \Rightarrow e^y - e^x = C$

12. $\frac{dy}{dx} = 3x^2e^{-y} \Rightarrow dy = 3x^2e^{-y}dx \Rightarrow e^y\,dy = 3x^2dx \Rightarrow \int e^y\,dy = \int 3x^2dx \Rightarrow e^y = x^3 + C \Rightarrow e^y - x^3 = C$

13. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow dy = \left(\sqrt{y}\cos^2\sqrt{y}\right)dx \Rightarrow \frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = dx \Rightarrow \int \frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = \int dx$. In the integral on the left-hand side, substitute $u = \sqrt{y} \Rightarrow du = \frac{1}{2\sqrt{y}}dy \Rightarrow 2\,du = \frac{1}{\sqrt{y}}dy$, and we have $\int \sec^2 u\,du = \int dx \Rightarrow 2\tan u = x + C$
$\Rightarrow -x + 2\tan\sqrt{y} = C$

14. $\sqrt{2xy}\,\frac{dy}{dx} = 1 \Rightarrow dy = \frac{1}{\sqrt{2xy}}\,dx \Rightarrow \sqrt{2}\sqrt{y}dy = \frac{1}{\sqrt{x}}\,dx \Rightarrow \sqrt{2}\,y^{1/2}\,dy = x^{-1/2}\,dx \Rightarrow \sqrt{2}\int y^{1/2}\,dy = \int x^{-1/2}\,dx$
$\Rightarrow \sqrt{2}\,\frac{y^{3/2}}{\frac{3}{2}}\,dy = \frac{x^{1/2}}{\frac{1}{2}} + C_1 \Rightarrow \sqrt{2}\,y^{3/2} = 3\sqrt{x} + \frac{3}{2}C_1 \Rightarrow \sqrt{2}\left(\sqrt{y}\right)^3 - 3\sqrt{x} = C$, where $C = \frac{3}{2}C_1$

15. $\sqrt{x}\,\frac{dy}{dx} = e^{y+\sqrt{x}} \Rightarrow \frac{dy}{dx} = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}} \Rightarrow dy = \frac{e^y e^{\sqrt{x}}}{\sqrt{x}}dx \Rightarrow e^{-y}\,dy = \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \Rightarrow \int e^{-y}\,dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. In the integral on the right-hand side, substitute $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$, and we have $\int e^{-y}\,dy = 2\int e^u\,du \Rightarrow -e^{-y} = 2e^u + C_1$
$\Rightarrow -e^{-y} = 2e^{\sqrt{x}} + C$, where $C = -C_1$

16. $(\sec x)\,\frac{dy}{dx} = e^{y+\sin x} \Rightarrow \frac{dy}{dx} = e^{y+\sin x}\cos x \Rightarrow dy = (e^y e^{\sin x}\cos x)dx \Rightarrow e^{-y}\,dy = e^{\sin x}\cos x\,dx$
$\Rightarrow \int e^{-y}\,dy = \int e^{\sin x}\cos x\,dx \Rightarrow -e^{-y} = e^{\sin x} + C_1 \Rightarrow e^{-y} + e^{\sin x} = C$, where $C = -C_1$

17. $\frac{dy}{dx} = 2x\sqrt{1-y^2} \Rightarrow dy = 2x\sqrt{1-y^2}dx \Rightarrow \frac{dy}{\sqrt{1-y^2}} = 2x\,dx \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \Rightarrow \sin^{-1} y = x^2 + C$ since $|y| < 1$
$\Rightarrow y = \sin(x^2 + C)$

18. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}} \Rightarrow dy = \frac{e^{2x-y}}{e^{x+y}}dx \Rightarrow dy = \frac{e^{2x}e^{-y}}{e^x e^y}dx = \frac{e^x}{e^{2y}}dx \Rightarrow e^{2y}\,dy = e^x\,dx \Rightarrow \int e^{2y}\,dy = \int e^x\,dx \Rightarrow \frac{e^{2y}}{2} = e^x + C_1$
$\Rightarrow e^{2y} - 2e^x = C$ where $C = 2C_1$

19. $y^2\frac{dy}{dx} = 3x^2y^3 - 6x^2 \Rightarrow y^2\,dy = 3x^2(y^3 - 2)dx \Rightarrow \frac{y^2}{y^3-2}\,dy = 3x^2dx \Rightarrow \int \frac{y^2}{y^3-2}\,dy = \int 3x^2dx \Rightarrow \frac{1}{3}\ln|y^3 - 2| = x^3 + C$

20. $\frac{dy}{dx} = x\,y + 3x - 2y - 6 = (y+3)(x-2) \Rightarrow \frac{1}{y+3}dy = (x-2)dx \Rightarrow \int \frac{1}{y+3}dy = \int (x-2)dx$
$\Rightarrow \ln|y+3| = \frac{1}{2}x^2 - 2x + C$

21. $\frac{1}{x}\frac{dy}{dx} = ye^{x^2} + 2\sqrt{y}\,e^{x^2} = e^{x^2}\left(y + 2\sqrt{y}\right) \Rightarrow \frac{1}{y+2\sqrt{y}}dy = x\,e^{x^2}dx \Rightarrow \int \frac{1}{y+2\sqrt{y}}dy = \int x\,e^{x^2}dx$
$\Rightarrow \int \frac{1}{\sqrt{y}(\sqrt{y}+2)}dy = \int x\,e^{x^2}dx \Rightarrow 2\ln|\sqrt{y}+2| = \frac{1}{2}e^{x^2} + C \Rightarrow 4\ln|\sqrt{y}+2| = e^{x^2} + C \Rightarrow 4\ln\left(\sqrt{y}+2\right) = e^{x^2} + C$

22. $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1 = (e^{-y}+1)(e^x+1) \Rightarrow \frac{1}{e^{-y}+1}dy = (e^x+1)dx \Rightarrow \int \frac{1}{e^{-y}+1}dy = \int (e^x+1)dx$
$\Rightarrow \int \frac{e^y}{1+e^y}dy = \int (e^x+1)dx \Rightarrow \ln|1+e^y| = e^x + x + C \Rightarrow \ln(1+e^y) = e^x + x + C$

23. (a) $y = y_0e^{kt} \Rightarrow 0.99y_0 = y_0e^{1000k} \Rightarrow k = \frac{\ln 0.99}{1000} \approx -0.00001$
(b) $0.9 = e^{(-0.00001)t} \Rightarrow (-0.00001)t = \ln(0.9) \Rightarrow t = \frac{\ln(0.9)}{-0.00001} \approx 10{,}536$ years
(c) $y = y_0e^{(20{,}000)k} \approx y_0e^{-0.2} = y_0(0.82) \Rightarrow 82\%$

24. (a) $\frac{dp}{dh} = kp \Rightarrow p = p_0e^{kh}$ where $p_0 = 1013$; $90 = 1013e^{20k} \Rightarrow k = \frac{\ln(90) - \ln(1013)}{20} \approx -0.121$
(b) $p = 1013e^{-6.05} \approx 2.389$ millibars
(c) $900 = 1013e^{(-0.121)h} \Rightarrow -0.121h = \ln\left(\frac{900}{1013}\right) \Rightarrow h = \frac{\ln(1013) - \ln(900)}{0.121} \approx 0.977$ km

25. $\frac{dy}{dt} = -0.6y \Rightarrow y = y_0e^{-0.6t}$; $y_0 = 100 \Rightarrow y = 100e^{-0.6t} \Rightarrow y = 100e^{-0.6} \approx 54.88$ grams when $t = 1$ hr

26. $A = A_0e^{kt} \Rightarrow 800 = 1000e^{10k} \Rightarrow k = \frac{\ln(0.8)}{10} \Rightarrow A = 1000e^{(\ln(0.8)/10)t}$, where A represents the amount of sugar that remains after time t. Thus after another 14 hrs, $A = 1000e^{(\ln(0.8)/10)24} \approx 585.35$ kg

27. $L(x) = L_0e^{-kx} \Rightarrow \frac{L_0}{2} = L_0e^{-18k} \Rightarrow \ln\frac{1}{2} = -18k \Rightarrow k = \frac{\ln 2}{18} \approx 0.0385 \Rightarrow L(x) = L_0e^{-0.0385x}$; when the intensity is one-tenth of the surface value, $\frac{L_0}{10} = L_0e^{-0.0385x} \Rightarrow \ln 10 = 0.0385x \Rightarrow x \approx 59.8$ ft

28. $V(t) = V_0e^{-t/40} \Rightarrow 0.1V_0 = V_0e^{-t/40}$ when the voltage is 10% of its original value $\Rightarrow t = -40\ln(0.1) \approx 92.1$ sec

29. $y = y_0e^{kt}$ and $y_0 = 1 \Rightarrow y = e^{kt} \Rightarrow$ at $y = 2$ and $t = 0.5$ we have $2 = e^{0.5k} \Rightarrow \ln 2 = 0.5k \Rightarrow k = \frac{\ln 2}{0.5} = \ln 4$. Therefore, $y = e^{(\ln 4)t} \Rightarrow y = e^{24\ln 4} = 4^{24} = 2.81474978 \times 10^{14}$ at the end of 24 hrs

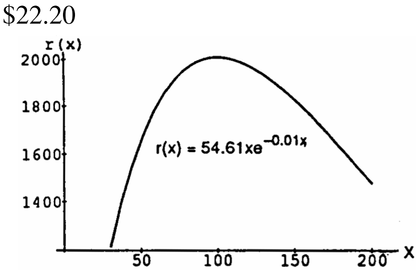
30. $y = y_0e^{kt}$ and $y(3) = 10{,}000 \Rightarrow 10{,}000 = y_0e^{3k}$; also $y(5) = 40{,}000 = y_0e^{5k}$. Therefore $y_0e^{5k} = 4y_0e^{3k}$ $\Rightarrow e^{5k} = 4e^{3k} \Rightarrow e^{2k} = 4 \Rightarrow k = \ln 2$. Thus, $y = y_0e^{(\ln 2)t} \Rightarrow 10{,}000 = y_0e^{3\ln 2} = y_0e^{\ln 8} \Rightarrow 10{,}000 = 8y_0$ $\Rightarrow y_0 = \frac{10{,}000}{8} = 1250$

31. (a) $10{,}000e^{k(1)} = 7500 \Rightarrow e^k = 0.75 \Rightarrow k = \ln 0.75$ and $y = 10{,}000e^{(\ln 0.75)t}$. Now $1000 = 10{,}000e^{(\ln 0.75)t}$ $\Rightarrow \ln 0.1 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.1}{\ln 0.75} \approx 8.00$ years (to the nearest hundredth of a year)
(b) $1 = 10{,}000e^{(\ln 0.75)t} \Rightarrow \ln 0.0001 = (\ln 0.75)t \Rightarrow t = \frac{\ln 0.0001}{\ln 0.75} \approx 32.02$ years (to the nearest hundredth of a year)

32. (a) There are $(60)(60)(24)(365) = 31{,}536{,}000$ seconds in a year. Thus, assuming exponential growth, $P = 257{,}313{,}431e^{kt}$ and $257{,}313{,}432 = 257{,}313{,}431e^{(14k/31{,}536{,}000)} \Rightarrow \ln\left(\frac{257{,}313{,}432}{257{,}313{,}431}\right) = \frac{14k}{31{,}536{,}000} \Rightarrow k \approx 0.0087542$
(b) $P = 257{,}313{,}431e^{(0.0087542)(15)} \approx 293{,}420{,}847$ (to the nearest integer). Answers will vary considerably with the number of decimal places retained.

33. $0.9P_0 = P_0e^k \Rightarrow k = \ln 0.9$; when the well's output falls to one-fifth of its present value $P = 0.2P_0$ $\Rightarrow 0.2P_0 = P_0e^{(\ln 0.9)t} \Rightarrow 0.2 = e^{(\ln 0.9)t} \Rightarrow \ln(0.2) = (\ln 0.9)t \Rightarrow t = \frac{\ln 0.2}{\ln 0.9} \approx 15.28$ yr

34. (a) $\frac{dp}{dx} = -\frac{1}{100}p \Rightarrow \frac{dp}{p} = -\frac{1}{100}dx \Rightarrow \ln p = -\frac{1}{100}x + C \Rightarrow p = e^{(-0.01x+C)} = e^Ce^{-0.01x} = C_1e^{-0.01x}$; $p(100) = 20.09 \Rightarrow 20.09 = C_1e^{(-0.01)(100)} \Rightarrow C_1 = 20.09e \approx 54.61 \Rightarrow p(x) = 54.61e^{-0.01x}$ (in dollars)
(b) $p(10) = 54.61e^{(-0.01)(10)} = \$49.41$, and $p(90) = 54.61e^{(-0.01)(90)} = \$22.20$
(c) $r(x) = xp(x) \Rightarrow r'(x) = p(x) + xp'(x)$; $p'(x) = -.5461e^{-0.01x} \Rightarrow r'(x)$ $= (54.61 - .5461x)e^{-0.01x}$. Thus, $r'(x) = 0$ $\Rightarrow 54.61 = .5461x \Rightarrow x = 100$. Since $r' > 0$ for any $x < 100$ and $r' < 0$ for $x > 100$, then $r(x)$ must be a maximum at $x = 100$.



35. $A = A_0e^{kt}$ and $A_0 = 10 \Rightarrow A = 10\,e^{kt}$, $5 = 10\,e^{k(24360)} \Rightarrow k = \frac{\ln(0.5)}{24360} \approx -0.000028454 \Rightarrow A = 10\,e^{-0.000028454t}$, then $0.2(10) = 10\,e^{-0.000028254t} \Rightarrow t = \frac{\ln 0.2}{-0.000028454} \approx 56563$ years

36. $A = A_0e^{kt}$ and $\frac{1}{2}A_0 = A_0e^{139k} \Rightarrow \frac{1}{2} = e^{139k} \Rightarrow k = \frac{\ln(0.5)}{139} \approx -0.00499$; then $0.05A_0 = A_0e^{-0.00499t}$ $\Rightarrow t = \frac{\ln 0.05}{-0.00499} \approx 600$ days

37. $y = y_0e^{-kt} = y_0e^{-(k)(3/k)} = y_0e^{-3} = \frac{y_0}{e^3} < \frac{y_0}{20} = (0.05)(y_0) \Rightarrow$ after three mean lifetimes less than 5% remains

38. (a) $A = A_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-2.645k} \Rightarrow k = \frac{\ln 2}{2.645} \approx 0.262$

(b) $\frac{1}{k} \approx 3.816$ years

(c) $(0.05)A = A \exp\left(-\frac{\ln 2}{2.645} t\right) \Rightarrow -\ln 20 = \left(-\frac{\ln 2}{2.645}\right) t \Rightarrow t = \frac{2.645 \ln 20}{\ln 2} \approx 11.431$ years

39. $T - T_s = (T_0 - T_s) e^{-kt}$, $T_0 = 90°C$, $T_s = 20°C$, $T = 60°C \Rightarrow 60 - 20 = 70e^{-10k} \Rightarrow \frac{4}{7} = e^{-10k} \Rightarrow k = \frac{\ln\left(\frac{7}{4}\right)}{10} \approx 0.05596$

(a) $35 - 20 = 70e^{-0.05596t} \Rightarrow t \approx 27.5$ min is the total time $\Rightarrow$ it will take $27.5 - 10 = 17.5$ minutes longer to reach 35°C

(b) $T - T_s = (T_0 - T_s) e^{-kt}$, $T_0 = 90°C$, $T_s = -15°C \Rightarrow 35 + 15 = 105e^{-0.05596t} \Rightarrow t \approx 13.26$ min

40. $T - 65° = (T_0 - 65°) e^{-kt} \Rightarrow 35° - 65° = (T_0 - 65°) e^{-10k}$ and $50° - 65° = (T_0 - 65°) e^{-20k}$. Solving $-30° = (T_0 - 65°) e^{-10k}$ and $-15° = (T_0 - 65°) e^{-20k}$ simultaneously $\Rightarrow (T_0 - 65°) e^{-10k} = 2(T_0 - 65°) e^{-20k}$ $\Rightarrow e^{10k} = 2 \Rightarrow k = \frac{\ln 2}{10}$ and $-30° = \frac{T_0 - 65°}{e^{10k}} \Rightarrow -30° \left[e^{10\left(\frac{\ln 2}{10}\right)}\right] = T_0 - 65° \Rightarrow T_0 = 65° - 30° \left(e^{\ln 2}\right) = 65° - 60° = 5°$

41. $T - T_s = (T_0 - T_s) e^{-kt} \Rightarrow 39 - T_s = (46 - T_s) e^{-10k}$ and $33 - T_s = (46 - T_s) e^{-20k} \Rightarrow \frac{39 - T_s}{46 - T_s} = e^{-10k}$ and $\frac{33 - T_s}{46 - T_s} = e^{-20k} = \left(e^{-10k}\right)^2 \Rightarrow \frac{33 - T_s}{46 - T_s} = \left(\frac{39 - T_s}{46 - T_s}\right)^2 \Rightarrow (33 - T_s)(46 - T_s) = (39 - T_s)^2 \Rightarrow 1518 - 79T_s + T_s^2$ $= 1521 - 78T_s + T_s^2 \Rightarrow -T_s = 3 \Rightarrow T_s = -3°C$

42. Let x represent how far above room temperature the silver will be 15 min from now, y how far above room temperature the silver will be 120 min from now, and t_0 the time the silver will be 10°C above room temperature. We then have the following time-temperature table:

time in min.	0	20 (Now)	35	140	t_0
temperature	$T_s + 70°$	$T_s + 60°$	$T_s + x$	$T_s + y$	$T_s + 10°$

$T - T_s = (T_0 - T_s) e^{-kt} \Rightarrow (60 + T_s) - T_s = [(70 + T_s) - T_s] e^{-20k} \Rightarrow 60 = 70e^{-20k} \Rightarrow k = \left(-\frac{1}{20}\right) \ln\left(\frac{6}{7}\right) \approx 0.00771$

(a) $T - T_s = (T_0 - T_s) e^{-0.00771t} \Rightarrow (T_s + x) - T_s = [(70 + T_s) - T_s] e^{-(0.00771)(35)} \Rightarrow x = 70e^{-0.26985} \approx 53.44°C$

(b) $T - T_s = (T_0 - T_s) e^{-0.00771t} \Rightarrow (T_s + y) - T_s = [(70 + T_s) - T_s] e^{-(0.00771)(140)} \Rightarrow y = 70e^{-1.0794} \approx 23.79°C$

(c) $T - T_s = (T_0 - T_s) e^{-0.00771t} \Rightarrow (T_s + 10) - T_s = [(70 + T_s) - T_s] e^{-(0.00771)t_0} \Rightarrow 10 = 70e^{-0.00771t_0}$ $\Rightarrow \ln\left(\frac{1}{7}\right) = -0.00771t_0 \Rightarrow t_0 = \left(-\frac{1}{0.00771}\right) \ln\left(\frac{1}{7}\right) = 252.39 \Rightarrow 252.39 - 20 \approx 232$ minutes from now the silver will be 10°C above room temperature

43. From Example 4, the half-life of carbon-14 is 5700 yr $\Rightarrow \frac{1}{2} c_0 = c_0 e^{-k(5700)} \Rightarrow k = \frac{\ln 2}{5700} \approx 0.0001216 \Rightarrow c = c_0 e^{-0.0001216t}$ $\Rightarrow (0.445)c_0 = c_0 e^{-0.0001216t} \Rightarrow t = \frac{\ln(0.445)}{-0.0001216} \approx 6659$ years

44. From Exercise 43, $k \approx 0.0001216$ for carbon-14.

(a) $c = c_0 e^{-0.0001216t} \Rightarrow (0.17)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 14{,}571.44$ years $\Rightarrow$ 12,571 BC

(b) $(0.18)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 14{,}101.41$ years $\Rightarrow$ 12,101 BC

(c) $(0.16)c_0 = c_0 e^{-0.0001216t} \Rightarrow t \approx 15{,}069.98$ years $\Rightarrow$ 13,070 BC

45. From Exercise 43, $k \approx 0.0001216$ for carbon-14 $\Rightarrow y = y_0 e^{-0.0001216t}$. When $t = 5000$ $\Rightarrow y = y_0 e^{-0.0001216(5000)} \approx 0.5444y_0 \Rightarrow \frac{y}{y_0} \approx 0.5444 \Rightarrow$ approximately 54.44% remains

46. From Exercise 43, $k \approx 0.0001216$ for carbon-14. Thus, $c = c_0 e^{-0.0001216t} \Rightarrow (0.995)c_0 = c_0 e^{-0.0001216t}$ $\Rightarrow t = \frac{\ln(0.995)}{-0.0001216} \approx 41$ years old

7.5 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

1. l'Hôpital: $\lim\limits_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$ or $\lim\limits_{x \to 2} \frac{x-2}{x^2-4} = \lim\limits_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim\limits_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$

2. l'Hôpital: $\lim\limits_{x \to 0} \frac{\sin 5x}{x} = \frac{5\cos 5x}{1}\Big|_{x=0} = 5$ or $\lim\limits_{x \to 0} \frac{\sin 5x}{x} = 5 \lim\limits_{5x \to 0} \frac{\sin 5x}{5x} = 5 \cdot 1 = 5$

3. l'Hôpital: $\lim\limits_{x \to \infty} \frac{5x^2-3x}{7x^2+1} = \lim\limits_{x \to \infty} \frac{10x-3}{14x} = \lim\limits_{x \to \infty} \frac{10}{14} = \frac{5}{7}$ or $\lim\limits_{x \to \infty} \frac{5x^2-3x}{7x^2+1} = \lim\limits_{x \to \infty} \frac{5-\frac{3}{x}}{7+\frac{1}{x^2}} = \frac{5}{7}$

4. l'Hôpital: $\lim\limits_{x \to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x \to 1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$ or $\lim\limits_{x \to 1} \frac{x^3-1}{4x^3-x-3} = \lim\limits_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)}$
$= \lim\limits_{x \to 1} \frac{(x^2+x+1)}{(4x^2+4x+3)} = \frac{3}{11}$

5. l'Hôpital: $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x \to 0} \frac{\sin x}{2x} = \lim\limits_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$ or $\lim\limits_{x \to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x \to 0} \left[\frac{(1-\cos x)}{x^2}\left(\frac{1+\cos x}{1+\cos x}\right)\right]$
$= \lim\limits_{x \to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim\limits_{x \to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{1}{1+\cos x}\right)\right] = \frac{1}{2}$

6. l'Hôpital: $\lim\limits_{x \to \infty} \frac{2x^2+3x}{x^3+x+1} = \lim\limits_{x \to \infty} \frac{4x+3}{3x^2+1} = \lim\limits_{x \to \infty} \frac{4}{6x} = 0$ or $\lim\limits_{x \to \infty} \frac{2x^2+3x}{x^3+x+1} = \lim\limits_{x \to \infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0}{1} = 0$

7. $\lim\limits_{x \to 2} \frac{x-2}{x^2-4} = \lim\limits_{x \to 2} \frac{1}{2x} = \frac{1}{4}$

8. $\lim\limits_{x \to -5} \frac{x^2-25}{x+5} = \lim\limits_{x \to -5} \frac{2x}{1} = -10$

9. $\lim\limits_{t \to -3} \frac{t^3-4t+15}{t^2-t-12} = \lim\limits_{t \to -3} \frac{3t^2-4}{2t-1} = \frac{3(-3)^2-4}{2(-3)-1} = -\frac{23}{7}$

10. $\lim\limits_{t \to 1} \frac{t^3-1}{4t^3-t-3} = \lim\limits_{t \to 1} \frac{3t^2}{12t^2-1} = \frac{3}{11}$

11. $\lim\limits_{x \to \infty} \frac{5x^3-2x}{7x^3+3} = \lim\limits_{x \to \infty} \frac{15x^2-2}{21x^2} = \lim\limits_{x \to \infty} \frac{30x}{42x} = \lim\limits_{x \to \infty} \frac{30}{42} = \frac{5}{7}$

12. $\lim\limits_{x \to \infty} \frac{x-8x^2}{12x^2+5x} = \lim\limits_{x \to \infty} \frac{1-16x}{24x+5} = \lim\limits_{x \to \infty} \frac{-16}{24} = -\frac{2}{3}$

13. $\lim\limits_{t \to 0} \frac{\sin t^2}{t} = \lim\limits_{t \to 0} \frac{(\cos t^2)(2t)}{1} = 0$

14. $\lim\limits_{t \to 0} \frac{\sin 5t}{2t} = \lim\limits_{t \to 0} \frac{5\cos 5t}{2} = \frac{5}{2}$

15. $\lim\limits_{x \to 0} \frac{8x^2}{\cos x - 1} = \lim\limits_{x \to 0} \frac{16x}{-\sin x} = \lim\limits_{x \to 0} \frac{16}{-\cos x} = \frac{16}{-1} = -16$

16. $\lim\limits_{x \to 0} \frac{\sin x - x}{x^3} = \lim\limits_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim\limits_{x \to 0} \frac{-\sin x}{6x} = \lim\limits_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}$

17. $\lim\limits_{\theta \to \pi/2} \frac{2\theta-\pi}{\cos(2\pi-\theta)} = \lim\limits_{\theta \to \pi/2} \frac{2}{\sin(2\pi-\theta)} = \frac{2}{\sin\left(\frac{3\pi}{2}\right)} = -2$

18. $\lim\limits_{\theta \to -\pi/3} \frac{3\theta+\pi}{\sin\left(\theta+\frac{\pi}{3}\right)} = \lim\limits_{\theta \to -\pi/3} \frac{3}{\cos\left(\theta+\frac{\pi}{3}\right)} = 3$

19. $\lim\limits_{\theta \to \pi/2} \frac{1-\sin\theta}{1+\cos 2\theta} = \lim\limits_{\theta \to \pi/2} \frac{-\cos\theta}{-2\sin 2\theta} = \lim\limits_{\theta \to \pi/2} \frac{\sin\theta}{-4\cos 2\theta} = \frac{1}{(-4)(-1)} = \frac{1}{4}$

20. $\lim\limits_{x \to 1} \frac{x-1}{\ln x - \sin(\pi x)} = \lim\limits_{x \to 1} \frac{1}{\frac{1}{x} - \pi\cos(\pi x)} = \frac{1}{1+\pi}$

21. $\lim\limits_{x \to 0} \frac{x^2}{\ln(\sec x)} = \lim\limits_{x \to 0} \frac{2x}{\left(\frac{\sec x \tan x}{\sec x}\right)} = \lim\limits_{x \to 0} \frac{2x}{\tan x} = \lim\limits_{x \to 0} \frac{2}{\sec^2 x} = \frac{2}{1^2} = 2$

22. $\lim\limits_{x \to \pi/2} \frac{\ln(\csc x)}{\left(x - \left(\frac{\pi}{2}\right)\right)^2} = \lim\limits_{x \to \pi/2} \frac{-\left(\frac{\csc x \cot x}{\csc x}\right)}{2\left(x - \left(\frac{\pi}{2}\right)\right)} = \lim\limits_{x \to \pi/2} \frac{-\cot x}{2\left(x - \left(\frac{\pi}{2}\right)\right)} = \lim\limits_{x \to \pi/2} \frac{\csc^2 x}{2} = \frac{1^2}{2} = \frac{1}{2}$

23. $\lim\limits_{t \to 0} \frac{t(1-\cos t)}{t - \sin t} = \lim\limits_{t \to 0} \frac{(1-\cos t) + t(\sin t)}{1 - \cos t} = \lim\limits_{t \to 0} \frac{\sin t + (\sin t + t\cos t)}{\sin t} = \lim\limits_{t \to 0} \frac{\cos t + \cos t + \cos t - t\sin t}{\cos t} = \frac{1+1+1-0}{1} = 3$

24. $\lim\limits_{t \to 0} \frac{t\sin t}{1 - \cos t} = \lim\limits_{t \to 0} \frac{\sin t + t\cos t}{\sin t} = \lim\limits_{t \to 0} \frac{\cos t + (\cos t - t\sin t)}{\cos t} = \frac{1 + (1-0)}{1} = 2$

25. $\lim\limits_{x \to (\pi/2)^-} \left(x - \frac{\pi}{2}\right)\sec x = \lim\limits_{x \to (\pi/2)^-} \frac{\left(x - \frac{\pi}{2}\right)}{\cos x} = \lim\limits_{x \to (\pi/2)^-} \left(\frac{1}{-\sin x}\right) = \frac{1}{-1} = -1$

26. $\lim\limits_{x \to (\pi/2)^-} \left(\frac{\pi}{2} - x\right)\tan x = \lim\limits_{x \to (\pi/2)^-} \frac{\left(\frac{\pi}{2} - x\right)}{\cot x} = \lim\limits_{x \to (\pi/2)^-} \left(\frac{-1}{-\csc^2 x}\right) = \lim\limits_{x \to (\pi/2)^-} \sin^2 x = 1$

27. $\lim\limits_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta} = \lim\limits_{\theta \to 0} \frac{3^{\sin\theta}(\ln 3)(\cos\theta)}{1} = \frac{(3^0)(\ln 3)(1)}{1} = \ln 3$

28. $\lim\limits_{\theta \to 0} \frac{\left(\frac{1}{2}\right)^{\theta} - 1}{\theta} = \lim\limits_{\theta \to 0} \frac{\left(\ln\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)^{\theta}}{1} = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$

29. $\lim\limits_{x \to 0} \frac{x\,2^x}{2^x - 1} = \lim\limits_{x \to 0} \frac{(1)(2^x) + (x)(\ln 2)(2^x)}{(\ln 2)(2^x)} = \frac{1 \cdot 2^0 + 0}{(\ln 2) \cdot 2^0} = \frac{1}{\ln 2}$

30. $\lim\limits_{x \to 0} \frac{3^x - 1}{2^x - 1} = \lim\limits_{x \to 0} \frac{3^x \ln 3}{2^x \ln 2} = \frac{3^0 \cdot \ln 3}{2^0 \cdot \ln 2} = \frac{\ln 3}{\ln 2}$

31. $\lim\limits_{x \to \infty} \frac{\ln(x+1)}{\log_2 x} = \lim\limits_{x \to \infty} \frac{\ln(x+1)}{\left(\frac{\ln x}{\ln 2}\right)} = (\ln 2)\lim\limits_{x \to \infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)} = (\ln 2)\lim\limits_{x \to \infty} \frac{x}{x+1} = (\ln 2)\lim\limits_{x \to \infty} \frac{1}{1} = \ln 2$

32. $\lim\limits_{x \to \infty} \frac{\log_2 x}{\log_3(x+3)} = \lim\limits_{x \to \infty} \frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln(x+3)}{\ln 3}\right)} = \left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x \to \infty} \frac{\ln x}{\ln(x+3)} = \left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x+3}\right)} = \left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x \to \infty} \frac{x+3}{x}$
$= \left(\frac{\ln 3}{\ln 2}\right)\lim\limits_{x \to \infty} \frac{1}{1} = \frac{\ln 3}{\ln 2}$

33. $\lim\limits_{x \to 0^+} \frac{\ln(x^2+2x)}{\ln x} = \lim\limits_{x \to 0^+} \frac{\left(\frac{2x+2}{x^2+2x}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to 0^+} \frac{2x^2+2x}{x^2+2x} = \lim\limits_{x \to 0^+} \frac{4x+2}{2x+2} = \lim\limits_{x \to 0^+} \frac{2}{2} = 1$

34. $\lim\limits_{x \to 0^+} \frac{\ln(e^x - 1)}{\ln x} = \lim\limits_{x \to 0^+} \frac{\left(\frac{e^x}{e^x - 1}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to 0^+} \frac{xe^x}{e^x - 1} = \lim\limits_{x \to 0^+} \frac{e^x + xe^x}{e^x} = \frac{1+0}{1} = 1$

35. $\lim\limits_{y \to 0} \frac{\sqrt{5y+25} - 5}{y} = \lim\limits_{y \to 0} \frac{(5y+25)^{1/2} - 5}{y} = \lim\limits_{y \to 0} \frac{\left(\frac{1}{2}\right)(5y+25)^{-1/2}(5)}{1} = \lim\limits_{y \to 0} \frac{5}{2\sqrt{5y+25}} = \frac{1}{2}$

36. $\lim\limits_{y \to 0} \frac{\sqrt{ay+a^2} - a}{y} = \lim\limits_{y \to 0} \frac{(ay+a^2)^{1/2} - a}{y} = \lim\limits_{y \to 0} \frac{\left(\frac{1}{2}\right)(ay+a^2)^{-1/2}(a)}{1} = \lim\limits_{y \to 0} \frac{a}{2\sqrt{ay+a^2}} = \frac{1}{2},\ a > 0$

37. $\lim\limits_{x \to \infty} [\ln 2x - \ln(x+1)] = \lim\limits_{x \to \infty} \ln\left(\frac{2x}{x+1}\right) = \ln\left(\lim\limits_{x \to \infty} \frac{2x}{x+1}\right) = \ln\left(\lim\limits_{x \to \infty} \frac{2}{1}\right) = \ln 2$

38. $\lim_{x \to 0^+} (\ln x - \ln \sin x) = \lim_{x \to 0^+} \ln\left(\frac{x}{\sin x}\right) = \ln\left(\lim_{x \to 0^+} \frac{x}{\sin x}\right) = \ln\left(\lim_{x \to 0^+} \frac{1}{\cos x}\right) = \ln 1 = 0$

39. $\lim_{x\to 0^+} \frac{(\ln x)^2}{\ln(\sin x)} = \lim_{x\to 0^+} \frac{2(\ln x)\left(\frac{1}{x}\right)}{\frac{\cos x}{\sin x}} = \lim_{x\to 0^+} \frac{2(\ln x)(\sin x)}{x \cos x} = \lim_{x\to 0^+} \left[\frac{2(\ln x)}{\cos x} \cdot \frac{\sin x}{x}\right] = -\infty \cdot 1 = -\infty$

40. $\lim_{x \to 0^+} \left(\frac{3x+1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0^+} \left(\frac{(3x+1)(\sin x) - x}{x \sin x}\right) = \lim_{x \to 0^+} \frac{3 \sin x + (3x+1)(\cos x) - 1}{\sin x + x \cos x}$
$= \lim_{x \to 0^+} \left(\frac{3 \cos x + 3 \cos x + (3x+1)(-\sin x)}{\cos x + \cos x - x \sin x}\right) = \frac{3+3+(1)(0)}{1+1-0} = \frac{6}{2} = 3$

41. $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x \to 1^+} \left(\frac{\ln x - (x-1)}{(x-1)(\ln x)}\right) = \lim_{x \to 1^+} \left(\frac{\frac{1}{x} - 1}{(\ln x) + (x-1)\left(\frac{1}{x}\right)}\right) = \lim_{x \to 1^+} \left(\frac{1-x}{(x \ln x) + x - 1}\right)$
$= \lim_{x \to 1^+} \left(\frac{-1}{(\ln x + 1) + 1}\right) = \frac{-1}{(0+1)+1} = -\frac{1}{2}$

42. $\lim_{x \to 0^+} (\csc x - \cot x + \cos x) = \lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} + \cos x\right) = \lim_{x \to 0^+} \left(\frac{(1-\cos x) + (\sin x)(\cos x)}{\sin x}\right)$
$= \lim_{x \to 0^+} \left(\frac{\sin x + \cos^2 x - \sin^2 x}{\cos x}\right) = \frac{0+1-0}{1} = 1$

43. $\lim_{\theta \to 0} \frac{\cos\theta - 1}{e^\theta - \theta - 1} = \lim_{\theta \to 0} \frac{-\sin\theta}{e^\theta - 1} = \lim_{\theta \to 0} \frac{-\cos\theta}{e^\theta} = -1$

44. $\lim_{h \to 0} \frac{e^h - (1+h)}{h^2} = \lim_{h \to 0} \frac{e^h - 1}{2h} = \lim_{h \to 0} \frac{e^h}{2} = \frac{1}{2}$

45. $\lim_{t \to \infty} \frac{e^t + t^2}{e^t - t} = \lim_{t \to \infty} \frac{e^t + 2t}{e^t} = \lim_{t \to \infty} \frac{e^t + 2}{e^t} = \lim_{t \to \infty} \frac{e^t}{e^t} = 1$

46. $\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0$

47. $\lim_{x\to 0} \frac{x - \sin x}{x \tan x} = \lim_{x\to 0} \frac{1 - \cos x}{x \sec^2 x + \tan x} = \lim_{x\to 0} \frac{\sin x}{2x \sec^2 x \tan x + 2\sec^2 x} = \frac{0}{2} = 0$

48. $\lim_{x\to 0} \frac{(e^x - 1)^2}{x \sin x} = \lim_{x\to 0} \frac{2(e^x - 1)e^x}{x \cos x + \sin x} = \lim_{x\to 0} \frac{2e^{2x} - 2e^x}{x \cos x + \sin x} = \lim_{x\to 0} \frac{4e^{2x} - 2e^x}{-x \sin x + 2\cos x} = \frac{2}{2} = 1$

49. $\lim_{\theta\to 0} \frac{\theta - \sin\theta\cos\theta}{\tan\theta - \theta} = \lim_{\theta\to 0} \frac{1 + \sin^2\theta - \cos^2\theta}{\sec^2\theta - 1} = \lim_{\theta\to 0} \frac{2\sin^2\theta}{\tan^2\theta} = \lim_{\theta\to 0} 2\cos^2\theta = 2$

50. $\lim_{x\to 0} \frac{\sin 3x - 3x + x^2}{\sin x \sin 2x} = \lim_{x\to 0} \frac{3\cos 3x - 3 + 2x}{2\sin x \cos 2x + \cos x \sin 2x} = \lim_{x\to 0} \frac{3\cos 3x - 3 + 2x}{\sin x \cos 2x + \sin 3x} = \lim_{x\to 0} \frac{-9\sin 3x + 2}{-2\sin x \sin 2x + \cos x \cos 2x + 3\cos 3x} = \frac{2}{4} = \frac{1}{2}$

51. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(1-x)} \Rightarrow \ln f(x) = \ln\left(x^{1/(1-x)}\right) = \frac{\ln x}{1-x}$. Now $\lim_{x \to 1^+} \ln f(x) = \lim_{x \to 1^+} \frac{\ln x}{1-x} = \lim_{x \to 1^+} \frac{\left(\frac{1}{x}\right)}{-1} = -1$. Therefore $\lim_{x \to 1^+} x^{1/(1-x)} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

52. The limit leads to the indeterminate form 1^∞. Let $f(x) = x^{1/(x-1)} \Rightarrow \ln f(x) = \ln\left(x^{1/(x-1)}\right) = \frac{\ln x}{x-1}$. Now $\lim_{x \to 1^+} \ln f(x) = \lim_{x \to 1^+} \frac{\ln x}{x-1} = \lim_{x \to 1^+} \frac{\left(\frac{1}{x}\right)}{1} = 1$. Therefore $\lim_{x \to 1^+} x^{1/(x-1)} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} e^{\ln f(x)} = e^1 = e$

53. The limit leads to the indeterminate form ∞^0. Let $f(x) = (\ln x)^{1/x} \Rightarrow \ln f(x) = \ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x}$. Now $\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln(\ln x)}{x} = \lim_{x \to \infty} \frac{\left(\frac{1}{x \ln x}\right)}{1} = 0$. Therefore $\lim_{x \to \infty} (\ln x)^{1/x} = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\ln f(x)} = e^0 = 1$

54. The limit leads to the indeterminate form 1^{∞}. Let $f(x) = (\ln x)^{1/(x-e)} \Rightarrow \ln f(x) = \frac{\ln(\ln x)}{x-e} = \lim_{x \to e^+} \ln f(x)$
$= \lim_{x \to e^+} \frac{\ln(\ln x)}{x-e} = \lim_{x \to e^+} \frac{\left(\frac{1}{x \ln x}\right)}{1} = \frac{1}{e}$. Therefore $(\ln x)^{1/(x-e)} = \lim_{x \to e^+} f(x) = \lim_{x \to e^+} e^{\ln f(x)} = e^{1/e}$

55. The limit leads to the indeterminate form 0^0. Let $f(x) = x^{-1/\ln x} \Rightarrow \ln f(x) = -\frac{\ln x}{\ln x} = -1$. Therefore
$\lim_{x \to 0^+} x^{-1/\ln x} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$

56. The limit leads to the indeterminate form ∞^0. Let $f(x) = x^{1/\ln x} \Rightarrow \ln f(x) = \frac{\ln x}{\ln x} = 1$. Therefore $\lim_{x \to \infty} x^{1/\ln x}$
$= \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\ln f(x)} = e^1 = e$

57. The limit leads to the indeterminate form ∞^0. Let $f(x) = (1+2x)^{1/(2\ln x)} \Rightarrow \ln f(x) = \frac{\ln(1+2x)}{2\ln x}$
$\Rightarrow \lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x \to \infty} \frac{x}{1+2x} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2}$. Therefore $\lim_{x \to \infty} (1+2x)^{1/(2\ln x)}$
$= \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\ln f(x)} = e^{1/2}$

58. The limit leads to the indeterminate form 1^{∞}. Let $f(x) = (e^x + x)^{1/x} \Rightarrow \ln f(x) = \frac{\ln(e^x + x)}{x}$
$\Rightarrow \lim_{x \to 0} \ln f(x) = \lim_{x \to 0} \frac{\ln(e^x+x)}{x} = \lim_{x \to 0} \frac{e^x+1}{e^x+x} = 2$. Therefore $\lim_{x \to 0} (e^x+x)^{1/x} = \lim_{x \to 0} f(x) = \lim_{x \to 0} e^{\ln f(x)} = e^2$

59. The limit leads to the indeterminate form 0^0. Let $f(x) = x^x \Rightarrow \ln f(x) = x \ln x \Rightarrow \ln f(x) = \frac{\ln x}{\left(\frac{1}{x}\right)}$
$= \lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to 0^+} (-x) = 0$. Therefore $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} f(x)$
$= \lim_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$

60. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(1 + \frac{1}{x}\right)^x \Rightarrow \ln f(x) = \frac{\ln(1+x^{-1})}{x^{-1}} \Rightarrow \lim_{x \to 0^+} \ln f(x)$
$= \lim_{x \to 0^+} \frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} = \lim_{x \to 0^+} \frac{1}{1+x^{-1}} = \lim_{x \to 0^+} \frac{x}{x+1} = 0$. Therefore $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to 0^+} f(x)$
$= \lim_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$

61. The limit leads to the indeterminate form 1^{∞}. Let $f(x) = \left(\frac{x+2}{x-1}\right)^x \Rightarrow \ln f(x) = \ln\left(\frac{x+2}{x-1}\right)^x = x \ln\left(\frac{x+2}{x-1}\right) \Rightarrow \lim_{x\to\infty} \ln f(x)$
$= \lim_{x\to\infty} x \ln\left(\frac{x+2}{x-1}\right) = \lim_{x\to\infty} \left(\frac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}\right) = \lim_{x\to\infty} \left(\frac{\ln(x+2) - \ln(x-1)}{\frac{1}{x}}\right) = \lim_{x\to\infty} \left(\frac{\frac{1}{x+2} - \frac{1}{x-1}}{-\frac{1}{x^2}}\right) = \lim_{x\to\infty} \left(\frac{\frac{-3}{(x+2)(x-1)}}{-\frac{1}{x^2}}\right)$
$= \lim_{x\to\infty} \left(\frac{3x^2}{(x+2)(x-1)}\right) = \lim_{x\to\infty} \left(\frac{6x}{2x+1}\right) = \lim_{x\to\infty} \left(\frac{6}{2}\right) = 3$. Therefore, $\lim_{x\to\infty} \left(\frac{x+2}{x-1}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^3$

62. The limit leads to the indeterminate form ∞^0. Let $f(x) = \left(\frac{x^2+1}{x+2}\right)^{1/x} \Rightarrow \ln f(x) = \ln\left(\frac{x^2+1}{x+2}\right)^{1/x} = \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right)$
$\Rightarrow \lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{1}{x} \ln\left(\frac{x^2+1}{x+2}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x} = \lim_{x\to\infty} \frac{\ln(x^2+1) - \ln(x+2)}{x} = \lim_{x\to\infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1} = \lim_{x\to\infty} \frac{x^2+4x-1}{(x^2+1)(x+2)}$
$= \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = \lim_{x\to\infty} \frac{2x+4}{3x^2+4x+1} = \lim_{x\to\infty} \frac{2}{6x+4} = 0$. Therefore, $\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$

63. $\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \left(\frac{\ln x}{\frac{1}{x^2}}\right) = \lim_{x\to 0^+} \left(\frac{\frac{1}{x}}{-\frac{2}{x^3}}\right) = \lim_{x\to 0^+} \left(-\frac{x^3}{2x}\right) = \lim_{x\to 0^+} \left(-\frac{3x^2}{2}\right) = 0$

64. $\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \left(\frac{(\ln x)^2}{\frac{1}{x}}\right) = \lim_{x\to 0^+} \left(\frac{2(\ln x)\frac{1}{x}}{-\frac{1}{x^2}}\right) = \lim_{x\to 0^+} \left(\frac{2\ln x}{-\frac{1}{x}}\right) = \lim_{x\to 0^+} \left(\frac{\frac{2}{x}}{\frac{1}{x^2}}\right) = \lim_{x\to 0^+} \left(\frac{2x^2}{x}\right) = \lim_{x\to 0^+} (2x) = 0$

65. $\lim\limits_{x\to 0^+} x\tan\left(\frac{\pi}{2} - x\right) = \lim\limits_{x\to 0^+}\left(\frac{x}{\cot\left(\frac{\pi}{2}-x\right)}\right) = \lim\limits_{x\to 0^+}\left(\frac{1}{\csc^2\left(\frac{\pi}{2}-x\right)}\right) = \frac{1}{1} = 1$

66. $\lim\limits_{x\to 0^+} \sin x \cdot \ln x = \lim\limits_{x\to 0^+}\left(\frac{\ln x}{\csc x}\right) = \lim\limits_{x\to 0^+}\left(\frac{\frac{1}{x}}{-\csc x\cot x}\right) = \lim\limits_{x\to 0^+}\left(-\frac{\sin x\tan x}{x}\right) = \lim\limits_{x\to 0^+}\left(-\frac{\sin x\sec^2 x + \cos x\tan x}{1}\right) = \frac{0}{1} = 0$

67. $\lim\limits_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9x+1}{x+1}} = \sqrt{\lim\limits_{x\to\infty} \frac{9}{1}} = \sqrt{9} = 3$

68. $\lim\limits_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \sqrt{\frac{1}{\lim\limits_{x\to 0^+}\frac{\sin x}{x}}} = \sqrt{\frac{1}{1}} = 1$

69. $\lim\limits_{x\to \pi/2^-} \frac{\sec x}{\tan x} = \lim\limits_{x\to \pi/2^-} \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) = \lim\limits_{x\to \pi/2^-} \frac{1}{\sin x} = 1$

70. $\lim\limits_{x\to 0^+} \frac{\cot x}{\csc x} = \lim\limits_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim\limits_{x\to 0^+} \cos x = 1$

71. $\lim\limits_{x\to\infty} \frac{2^x - 3^x}{3^x + 4^x} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{3}\right)^x - 1}{1 + \left(\frac{4}{3}\right)^x} = 0$

72. $\lim\limits_{x\to -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim\limits_{x\to -\infty} \frac{1 + \left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x - 1} = \lim\limits_{x\to -\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \frac{1+0}{0-1} = -1$

73. $\lim\limits_{x\to\infty} \frac{e^{x^2}}{x e^x} = \lim\limits_{x\to\infty} \frac{e^{x^2 - x}}{x} = \lim\limits_{x\to\infty} \frac{e^{x(x-1)}}{x} = \lim\limits_{x\to\infty} \frac{e^{x(x-1)}(2x-1)}{1} = \infty$

74. $\lim\limits_{x\to 0^+} \frac{x}{e^{-1/x}} = \lim\limits_{x\to 0^+} \frac{e^{1/x}}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{e^{1/x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} e^{1/x} = \infty$

75. Part (b) is correct because part (a) is neither in the $\frac{0}{0}$ nor $\frac{\infty}{\infty}$ form and so l'Hôpital's rule may not be used.

76. Part (b) is correct; the step $\lim\limits_{x\to 0} \frac{2x-2}{2x-\cos x} = \lim\limits_{x\to 0} \frac{2}{2+\sin x}$ in part (a) is false because $\lim\limits_{x\to 0} \frac{2x-2}{2x-\cos x}$ is not an indeterminate quotient form.

77. Part (d) is correct, the other parts are indeterminate forms and cannot be calculated by the incorrect arithmetic

78. (a) We seek c in $(-2, 0)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(0)-f(-2)}{g(0)-g(-2)} = \frac{0+2}{0-4} = -\frac{1}{2}$. Since $f'(c) = 1$ and $g'(c) = 2c$ we have that $\frac{1}{2c} = -\frac{1}{2}$ $\Rightarrow c = -1$.

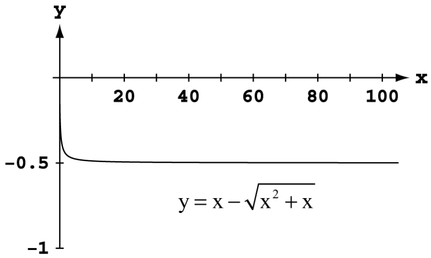
(b) We seek c in (a, b) so that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{b-a}{b^2-a^2} = \frac{1}{b+a}$. Since $f'(c) = 1$ and $g'(c) = 2c$ we have that $\frac{1}{2c} = \frac{1}{b+a}$ $\Rightarrow c = \frac{b+a}{2}$.

(c) We seek c in $(0, 3)$ so that $\frac{f'(c)}{g'(c)} = \frac{f(3)-f(0)}{g(3)-g(0)} = \frac{-3-0}{9-0} = -\frac{1}{3}$. Since $f'(c) = c^2 - 4$ and $g'(c) = 2c$ we have that $\frac{c^2-4}{2c} = -\frac{1}{3} \Rightarrow c = \frac{-1\pm\sqrt{37}}{3} \Rightarrow c = \frac{-1+\sqrt{37}}{3}$.

79. If f(x) is to be continuous at $x = 0$, then $\lim\limits_{x\to 0} f(x) = f(0) \Rightarrow c = f(0) = \lim\limits_{x\to 0} \frac{9x - 3\sin 3x}{5x^3} = \lim\limits_{x\to 0} \frac{9 - 9\cos 3x}{15x^2}$ $= \lim\limits_{x\to 0} \frac{27\sin 3x}{30x} = \lim\limits_{x\to 0} \frac{81\cos 3x}{30} = \frac{27}{10}$.

80. $\lim_{x\to0}\left(\frac{\tan 2x}{x^3}+\frac{a}{x^2}+\frac{\sin bx}{x}\right) = \lim_{x\to0}\left(\frac{\tan 2x+ax+x^2\sin bx}{x^3}\right) = \lim_{x\to0}\left(\frac{2\sec^2 2x+a+bx^2\cos bx+2x\sin bx}{3x^2}\right)$ will be in $\frac{0}{0}$ form if $\lim_{x\to0}(2\sec^2 2x + a + bx^2\cos bx + 2x\sin bx) = a+2 = 0 \Rightarrow a=-2;$ $\lim_{x\to0}\left(\frac{2\sec^2 2x-2+bx^2\cos bx+2x\sin bx}{3x^2}\right)$ $= \lim_{x\to0}\left(\frac{8\sec^2 2x\tan 2x - b^2x^2\sin bx+4bx\cos bx+2\sin bx}{6x}\right) = \lim_{x\to0}\left(\frac{32\sec^2 2x\tan^2 2x+16\sec^4 2x-b^3x^2\cos bx-6b^2x\sin bx+6b\cos bx}{6}\right)$ $= \frac{16+6b}{6} = 0 \Rightarrow 16+6b=0 \Rightarrow b=-\frac{8}{3}$

81. (a)

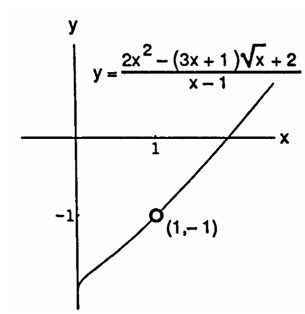


(b) The limit leads to the indeterminate form $\infty-\infty$:

$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) = \lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty}\left(\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}\right) = \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$ $= \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\frac{1}{x}}} = \frac{-1}{1+\sqrt{1+0}} = -\frac{1}{2}$

82. $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x}\right) = \lim_{x\to\infty} x\left(\frac{\sqrt{x^2+1}}{x}-\frac{\sqrt{x}}{x}\right) = \lim_{x\to\infty} x\left(\sqrt{\frac{x^2+1}{x^2}}-\sqrt{\frac{x}{x^2}}\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x^2}}-\sqrt{\frac{1}{x}}\right) = \infty$

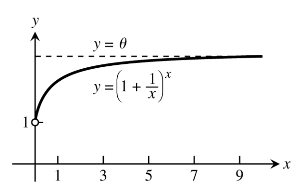
83. The graph indicates a limit near -1. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ $= \lim_{x\to1}\frac{2x^2-3x^{3/2}-x^{1/2}+2}{x-1} = \lim_{x\to1}\frac{4x-\frac{9}{2}x^{1/2}-\frac{1}{2}x^{-1/2}}{1}$ $= \frac{4-\frac{9}{2}-\frac{1}{2}}{1} = \frac{4-5}{1} = -1$



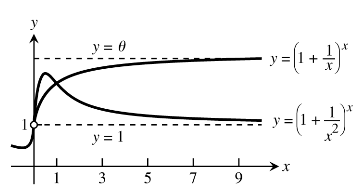
84. (a) The limit leads to the indeterminate form 1^∞. Let $f(x) = \left(1+\frac{1}{x}\right)^x \Rightarrow \ln f(x) = x\ln\left(1+\frac{1}{x}\right) \Rightarrow \lim_{x\to\infty}\ln f(x)$ $= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty}\frac{\ln(1+x^{-1})}{x^{-1}} = \lim_{x\to\infty}\frac{\left(\frac{-x^{-2}}{1+x^{-1}}\right)}{-x^{-2}} = \lim_{x\to\infty}\frac{1}{1+\left(\frac{1}{x}\right)} = \frac{1}{1+0} = 1$ $\Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^1 = e$

(b)

x	$\left(1+\frac{1}{x}\right)^x$
10	2.5937424601
100	2.70481382942
1000	2.71692393224
10,000	2.71814592683
100,000	2.71826823717



Both functions have limits as x approaches infinity. The function f has a maximum but no minimum while g has no extrema. The limit of f(x) leads to the indeterminate form 1^∞.



(c) Let $f(x) = \left(1 + \frac{1}{x^2}\right)^x \Rightarrow \ln f(x) = x \ln\left(1 + x^{-2}\right)$

$\Rightarrow \lim\limits_{x \to \infty} \ln f(x) = \lim\limits_{x \to \infty} \frac{\ln(1 + x^{-2})}{x^{-1}} = \lim\limits_{x \to \infty} \frac{\left(\frac{-2x^{-3}}{1 + x^{-2}}\right)}{-x^{-2}} = \lim\limits_{x \to \infty} \frac{2x^2}{(x^3 + x)} = \lim\limits_{x \to \infty} \frac{4x}{(3x^2 + 1)} = \lim\limits_{x \to \infty} \frac{4}{6x} = 0.$

Therefore $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x = \lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^0 = 1$

85. Let $f(k) = \left(1 + \frac{r}{k}\right)^k \Rightarrow \ln f(k) = \frac{\ln(1 + rk^{-1})}{k^{-1}} \Rightarrow \lim\limits_{k \to \infty} \frac{\ln(1 + rk^{-1})}{k^{-1}} = \lim\limits_{k \to \infty} \frac{\left(\frac{-rk^{-2}}{1 + rk^{-1}}\right)}{-k^{-2}} = \lim\limits_{k \to \infty} \frac{r}{1 + rk^{-1}}$

$= \lim\limits_{k \to \infty} \frac{rk}{k + r} = \lim\limits_{k \to \infty} \frac{r}{1} = r.$ Therefore $\lim\limits_{k \to \infty} \left(1 + \frac{r}{k}\right)^k = \lim\limits_{k \to \infty} f(k) = \lim\limits_{k \to \infty} e^{\ln f(k)} = e^r.$

86. (a) $y = x^{1/x} \Rightarrow \ln y = \frac{\ln x}{x} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x) - \ln x}{x^2} \Rightarrow y' = \left(\frac{1 - \ln x}{x^2}\right)\left(x^{1/x}\right)$. The sign pattern is

$y' = \underset{0}{|} + + + + + \underset{e}{|} - - - -$ which indicates a maximum value of $y = e^{1/e}$ when $x = e$

(b) $y = x^{1/x^2} \Rightarrow \ln y = \frac{\ln x}{x^2} \Rightarrow \frac{y'}{y} = \frac{\left(\frac{1}{x}\right)(x^2) - 2x \ln x}{x^4} \Rightarrow y' = \left(\frac{1 - 2\ln x}{x^3}\right)\left(x^{1/x^2}\right)$. The sign pattern is

$y' = \underset{0}{|} + + + \underset{\sqrt{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/2e}$ when $x = \sqrt{e}$

(c) $y = x^{1/x^n} \Rightarrow \ln y = \frac{\ln x}{x^n} = \frac{\left(\frac{1}{x}\right)(x^n) - (\ln x)(nx^{n-1})}{x^{2n}} \Rightarrow y' = \frac{x^{n-1}(1 - n \ln x)}{x^{2n}} \cdot x^{1/x^n}$. The sign pattern is

$y' = \underset{0}{|} + + + \underset{\sqrt[n]{e}}{|} - - - -$ which indicates a maximum of $y = e^{1/ne}$ when $x = \sqrt[n]{e}$

(d) $\lim\limits_{x \to \infty} x^{1/x^n} = \lim\limits_{x \to \infty} \left(e^{\ln x}\right)^{1/x^n} = \lim\limits_{x \to \infty} e^{(\ln x)/x^n} = \exp\left(\lim\limits_{x \to \infty} \frac{\ln x}{x^n}\right) = \exp\left(\lim\limits_{x \to \infty} \left(\frac{1}{nx^n}\right)\right) = e^0 = 1$

87. (a) $y = x\tan\left(\frac{1}{x}\right)$, $\lim\limits_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right) = \lim\limits_{x\to\infty}\left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = \lim\limits_{x\to\infty}\left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right) = \lim\limits_{x\to\infty}\sec^2\left(\frac{1}{x}\right) = 1;$ $\lim\limits_{x\to-\infty}\left(x\tan\left(\frac{1}{x}\right)\right)$

$= \lim\limits_{x\to-\infty}\left(\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = \lim\limits_{x\to-\infty}\left(\frac{\sec^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}\right) = \lim\limits_{x\to-\infty}\sec^2\left(\frac{1}{x}\right) = 1 \Rightarrow$ the horizontal asymptote is $y = 1$ as $x \to \infty$ and as $x \to -\infty$.

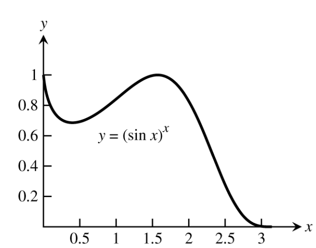
(b) $y = \frac{3x + e^{2x}}{2x + e^{3x}}$, $\lim\limits_{x\to\infty}\left(\frac{3x + e^{2x}}{2x + e^{3x}}\right) = \lim\limits_{x\to\infty}\left(\frac{3 + 2e^{2x}}{2 + 3e^{3x}}\right) = \lim\limits_{x\to\infty}\left(\frac{4e^{2x}}{9e^{3x}}\right) = \lim\limits_{x\to\infty}\left(\frac{4}{9e^x}\right) = 0;$ $\lim\limits_{x\to-\infty}\left(\frac{3x + e^{2x}}{2x + e^{3x}}\right)$

$= \lim\limits_{x\to-\infty}\left(\frac{3 + 2e^{2x}}{2 + 3e^{3x}}\right) = \frac{3}{2} \Rightarrow$ the horizontal asymptotes are $y = 0$ as $x \to \infty$ and $y = \frac{3}{2}$ as $x \to -\infty$.

88. $f'(0) = \lim\limits_{h\to0}\frac{f(0 + h) - f(0)}{h} = \lim\limits_{h\to0}\frac{e^{-1/h^2} - 0}{h} = \lim\limits_{h\to0}\frac{e^{-1/h^2}}{h} = \lim\limits_{h\to0}\left(\frac{\frac{1}{h}}{e^{1/h^2}}\right) = \lim\limits_{h\to0}\left(\frac{-\frac{1}{h^2}}{e^{1/h^2}\left(-\frac{2}{h^3}\right)}\right) = \lim\limits_{h\to0}\left(\frac{h}{2e^{1/h^2}}\right)$

$= \lim\limits_{h\to0}\left(\frac{h}{2}e^{-1/h^2}\right) = 0$

89. (a) We should assign the value 1 to $f(x) = (\sin x)^x$ to make it continuous at $x = 0$.

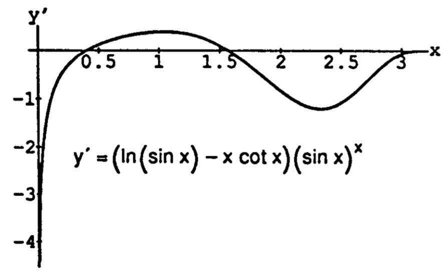


(b) $\ln f(x) = x \ln(\sin x) = \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} \Rightarrow \lim\limits_{x \to 0^+} \ln f(x) = \lim\limits_{x \to 0^+} \frac{\ln(\sin x)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to 0^+} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{\left(-\frac{1}{x^2}\right)}$

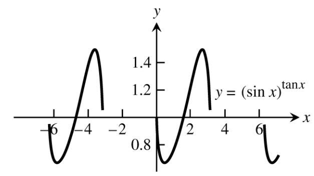
$= \lim\limits_{x \to 0} \frac{-x^2}{\tan x} = \lim\limits_{x \to 0} \frac{-2x}{\sec^2 x} = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = e^0 = 1$

(c) The maximum value of $f(x)$ is close to 1 near the point $x \approx 1.55$ (see the graph in part (a)).

(d) The root in question is near 1.57.



90. (a) When $\sin x < 0$ there are gaps in the sketch. The width of each gap is π.



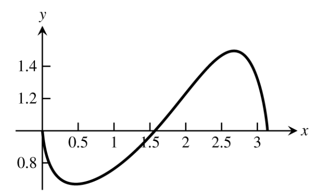
(b) Let $f(x) = (\sin x)^{\tan x} \Rightarrow \ln f(x) = (\tan x) \ln (\sin x)$

$\Rightarrow \lim\limits_{x \to \pi/2^-} \ln f(x) = \lim\limits_{x \to \pi/2^-} \frac{\ln (\sin x)}{\cot x}$

$= \lim\limits_{x \to \pi/2^-} \frac{\left(\frac{1}{\sin x}\right)(\cos x)}{-\csc^2 x} = \lim\limits_{x \to \pi/2^-} \frac{\cos x}{(-\csc x)} = 0$

$\Rightarrow \lim\limits_{x \to \pi/2^-} f(x) = e^0 = 1$. Similarly,

$\lim\limits_{x \to \pi/2^+} f(x) = e^0 = 1$. Therefore, $\lim\limits_{x \to \pi/2} f(x) = 1$.



(c) From the graph in part (b) we have a minimum of about 0.665 at $x \approx 0.47$ and the maximum is about 1.491 at $x \approx 2.66$.

7.6 INVERSE TRIGONOMETRIC FUNCTIONS

1. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$

2. (a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{6}$

3. (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$

4. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{3}$

5. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$

6. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$

7. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

8. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

9. $\sin\left(\cos^{-1} \frac{\sqrt{2}}{2}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

10. $\sec\left(\cos^{-1} \frac{1}{2}\right) = \sec\left(\frac{\pi}{3}\right) = 2$

11. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

12. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cot\left(-\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}}$

13. $\lim\limits_{x \to 1^-} \sin^{-1} x = \frac{\pi}{2}$

14. $\lim\limits_{x \to -1^+} \cos^{-1} x = \pi$

15. $\lim\limits_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

16. $\lim\limits_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

17. $\lim\limits_{x \to \infty} \sec^{-1} x = \frac{\pi}{2}$

18. $\lim\limits_{x \to -\infty} \sec^{-1} x = \lim\limits_{x \to -\infty} \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

19. $\lim\limits_{x \to \infty} \csc^{-1} x = \lim\limits_{x \to \infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$

20. $\lim\limits_{x \to -\infty} \csc^{-1} x = \lim\limits_{x \to -\infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$

21. $y = \cos^{-1}(x^2) \Rightarrow \frac{dy}{dx} = -\frac{2x}{\sqrt{1-(x^2)^2}} = \frac{-2x}{\sqrt{1-x^4}}$

22. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

23. $y = \sin^{-1}\sqrt{2}t \Rightarrow \frac{dy}{dt} = \frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}t\right)^2}} = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

24. $y = \sin^{-1}(1-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}$

25. $y = \sec^{-1}(2s+1) \Rightarrow \frac{dy}{ds} = \frac{2}{|2s+1|\sqrt{(2s+1)^2-1}} = \frac{2}{|2s+1|\sqrt{4s^2+4s}} = \frac{1}{|2s+1|\sqrt{s^2+s}}$

26. $y = \sec^{-1} 5s \Rightarrow \frac{dy}{ds} = \frac{5}{|5s|\sqrt{(5s)^2-1}} = \frac{1}{|s|\sqrt{25s^2-1}}$

27. $y = \csc^{-1}(x^2+1) \Rightarrow \frac{dy}{dx} = -\frac{2x}{|x^2+1|\sqrt{(x^2+1)^2-1}} = \frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$

28. $y = \csc^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{1}{2}\right)}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} = \frac{-1}{|x|\sqrt{\frac{x^2-4}{4}}} = \frac{-2}{|x|\sqrt{x^2-4}}$

29. $y = \sec^{-1}\left(\frac{1}{t}\right) = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

30. $y = \sin^{-1}\left(\frac{3}{t^2}\right) = \csc^{-1}\left(\frac{t^2}{3}\right) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right|\sqrt{\left(\frac{t^2}{3}\right)^2-1}} = \frac{-2t}{t^2\sqrt{\frac{t^4-9}{9}}} = \frac{-6}{t\sqrt{t^4-9}}$

31. $y = \cot^{-1}\sqrt{t} = \cot^{-1} t^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+(t^{1/2})^2} = \frac{-1}{2\sqrt{t}(1+t)}$

32. $y = \cot^{-1}\sqrt{t-1} = \cot^{-1}(t-1)^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)(t-1)^{-1/2}}{1+[(t-1)^{1/2}]^2} = \frac{-1}{2\sqrt{t-1}\,(1+t-1)} = \frac{-1}{2t\sqrt{t-1}}$

33. $y = \ln(\tan^{-1} x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1} x} = \frac{1}{(\tan^{-1} x)(1+x^2)}$

34. $y = \tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2} = \frac{1}{x[1+(\ln x)^2]}$

35. $y = \csc^{-1}(e^t) \Rightarrow \frac{dy}{dt} = -\frac{e^t}{|e^t|\sqrt{(e^t)^2-1}} = \frac{-1}{\sqrt{e^{2t}-1}}$

36. $y = \cos^{-1}(e^{-t}) \Rightarrow \frac{dy}{dt} = -\frac{-e^{-t}}{\sqrt{1-(e^{-t})^2}} = \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$

37. $y = s\sqrt{1-s^2} + \cos^{-1} s = s(1-s^2)^{1/2} + \cos^{-1} s \Rightarrow \frac{dy}{ds} = (1-s^2)^{1/2} + s\left(\frac{1}{2}\right)(1-s^2)^{-1/2}(-2s) - \frac{1}{\sqrt{1-s^2}}$
$= \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}} = \sqrt{1-s^2} - \frac{s^2+1}{\sqrt{1-s^2}} = \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$

38. $y = \sqrt{s^2-1} - \sec^{-1} s = (s^2-1)^{1/2} - \sec^{-1} s \Rightarrow \frac{dy}{dx} = \left(\frac{1}{2}\right)(s^2-1)^{-1/2}(2s) - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s}{\sqrt{s^2-1}} - \frac{1}{|s|\sqrt{s^2-1}}$
$= \frac{s|s|-1}{|s|\sqrt{s^2-1}}$

39. $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1} x = \tan^{-1}(x^2-1)^{1/2} + \csc^{-1} x \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x)}{1+\left[(x^2-1)^{1/2}\right]^2} - \frac{1}{|x|\sqrt{x^2-1}}$
$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0$, for $x > 1$

40. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1}(x^{-1}) - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{-x^{-2}}{1+(x^{-1})^2} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$

41. $y = x\sin^{-1} x + \sqrt{1-x^2} = x\sin^{-1} x + (1-x^2)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1} x + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)$
$= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$

42. $y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - x\left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right] = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2} = -\tan^{-1}\left(\frac{x}{2}\right)$

43. $\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + C$

44. $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{1-(2x)^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = 2x$ and $du = 2\,dx$
$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}(2x) + C$

45. $\int \frac{1}{17+x^2}\,dx = \int \frac{1}{\left(\sqrt{17}\right)^2+x^2}\,dx = \frac{1}{\sqrt{17}}\tan^{-1}\frac{x}{\sqrt{17}} + C$

46. $\int \frac{1}{9+3x^2}\,dx = \frac{1}{3}\int \frac{1}{\left(\sqrt{3}\right)^2+x^2}\,dx = \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C = \frac{\sqrt{3}}{9}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

47. $\int \frac{dx}{x\sqrt{25x^2-2}} = \int \frac{du}{u\sqrt{u^2-2}}$, where $u = 5x$ and $du = 5\,dx$
$= \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{u}{\sqrt{2}}\right| + C = \frac{1}{\sqrt{2}}\sec^{-1}\left|\frac{5x}{\sqrt{2}}\right| + C$

48. $\int \frac{dx}{x\sqrt{5x^2-4}} = \int \frac{du}{u\sqrt{u^2-4}}$, where $u = \sqrt{5}x$ and $du = \sqrt{5}\,dx$
$= \frac{1}{2}\sec^{-1}\left|\frac{u}{2}\right| + C = \frac{1}{2}\sec^{-1}\left|\frac{\sqrt{5}x}{2}\right| + C$

49. $\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}} = \left[4\sin^{-1}\frac{s}{2}\right]_0^1 = 4\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right) = 4\left(\frac{\pi}{6} - 0\right) = \frac{2\pi}{3}$

50. $\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9-4s^2}} = \frac{1}{2}\int_0^{3\sqrt{2}/4} \frac{du}{\sqrt{9-u^2}}$, where $u = 2s$ and $du = 2\,ds$; $s = 0 \Rightarrow u = 0$, $s = \frac{3\sqrt{2}}{4} \Rightarrow u = \frac{3\sqrt{2}}{2}$
$= \left[\frac{1}{2}\sin^{-1}\frac{u}{3}\right]_0^{3\sqrt{2}/2} = \frac{1}{2}\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1} 0\right) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$

51. $\int_0^2 \frac{dt}{8+2t^2} = \frac{1}{\sqrt{2}}\int_0^{2\sqrt{2}} \frac{du}{8+u^2}$, where $u = \sqrt{2}t$ and $du = \sqrt{2}\,dt$; $t = 0 \Rightarrow u = 0$, $t = 2 \Rightarrow u = 2\sqrt{2}$
$= \left[\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8}}\tan^{-1}\frac{u}{\sqrt{8}}\right]_0^{2\sqrt{2}} = \frac{1}{4}\left(\tan^{-1}\frac{2\sqrt{2}}{\sqrt{8}} - \tan^{-1} 0\right) = \frac{1}{4}\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{16}$

52. $\int_{-2}^{2} \frac{dt}{4+3t^2} = \frac{1}{\sqrt{3}} \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{du}{4+u^2}$, where $u = \sqrt{3}t$ and $du = \sqrt{3}\,dt$; $t = -2 \Rightarrow u = -2\sqrt{3}$, $t = 2 \Rightarrow u = 2\sqrt{3}$
$= \left[\frac{1}{\sqrt{3}} \cdot \frac{1}{2} \tan^{-1} \frac{u}{2}\right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{1}{2\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1}\left(-\sqrt{3}\right)\right] = \frac{1}{2\sqrt{3}} \left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{\pi}{3\sqrt{3}}$

53. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$, where $u = 2y$ and $du = 2\,dy$; $y = -1 \Rightarrow u = -2$, $y = -\frac{\sqrt{2}}{2} \Rightarrow u = -\sqrt{2}$
$= \left[\sec^{-1} |u|\right]_{-2}^{-\sqrt{2}} = \sec^{-1} \left|-\sqrt{2}\right| - \sec^{-1} |-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

54. $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$, where $u = 3y$ and $du = 3\,dy$; $y = -\frac{2}{3} \Rightarrow u = -2$, $y = -\frac{\sqrt{2}}{3} \Rightarrow u = -\sqrt{2}$
$= \left[\sec^{-1} |u|\right]_{-2}^{-\sqrt{2}} = \sec^{-1} \left|-\sqrt{2}\right| - \sec^{-1} |-2| = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

55. $\int \frac{3\,dr}{\sqrt{1-4(r-1)^2}} = \frac{3}{2} \int \frac{du}{\sqrt{1-u^2}}$, where $u = 2(r-1)$ and $du = 2\,dr$
$= \frac{3}{2} \sin^{-1} u + C = \frac{3}{2} \sin^{-1} 2(r-1) + C$

56. $\int \frac{6\,dr}{\sqrt{4-(r+1)^2}} = 6 \int \frac{du}{\sqrt{4-u^2}}$, where $u = r+1$ and $du = dr$
$= 6 \sin^{-1} \frac{u}{2} + C = 6 \sin^{-1} \left(\frac{r+1}{2}\right) + C$

57. $\int \frac{dx}{2+(x-1)^2} = \int \frac{du}{2+u^2}$, where $u = x-1$ and $du = dx$
$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x-1}{\sqrt{2}}\right) + C$

58. $\int \frac{dx}{1+(3x+1)^2} = \frac{1}{3} \int \frac{du}{1+u^2}$, where $u = 3x+1$ and $du = 3\,dx$
$= \frac{1}{3} \tan^{-1} u + C = \frac{1}{3} \tan^{-1} (3x+1) + C$

59. $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}} = \frac{1}{2} \int \frac{du}{u\sqrt{u^2-4}}$, where $u = 2x-1$ and $du = 2\,dx$
$= \frac{1}{2} \cdot \frac{1}{2} \sec^{-1} \left|\frac{u}{2}\right| + C = \frac{1}{4} \sec^{-1} \left|\frac{2x-1}{2}\right| + C$

60. $\int \frac{dx}{(x+3)\sqrt{(x+3)^2-25}} = \int \frac{du}{u\sqrt{u^2-25}}$, where $u = x+3$ and $du = dx$
$= \frac{1}{5} \sec^{-1} \left|\frac{u}{5}\right| + C = \frac{1}{5} \sec^{-1} \left|\frac{x+3}{5}\right| + C$

61. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1+(\sin\theta)^2} = 2 \int_{-1}^{1} \frac{du}{1+u^2}$, where $u = \sin\theta$ and $du = \cos\theta\,d\theta$; $\theta = -\frac{\pi}{2} \Rightarrow u = -1$, $\theta = \frac{\pi}{2} \Rightarrow u = 1$
$= \left[2\tan^{-1} u\right]_{-1}^{1} = 2\left(\tan^{-1} 1 - \tan^{-1}(-1)\right) = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

62. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1+(\cot x)^2} = -\int_{\sqrt{3}}^{1} \frac{du}{1+u^2}$, where $u = \cot x$ and $du = -\csc^2 x\,dx$; $x = \frac{\pi}{6} \Rightarrow u = \sqrt{3}$, $x = \frac{\pi}{4} \Rightarrow u = 1$
$= \left[-\tan^{-1} u\right]_{\sqrt{3}}^{1} = -\tan^{-1} 1 + \tan^{-1} \sqrt{3} = -\frac{\pi}{4} + \frac{\pi}{3} = \frac{\pi}{12}$

63. $\int_{0}^{\ln\sqrt{3}} \frac{e^x\,dx}{1+e^{2x}} = \int_{1}^{\sqrt{3}} \frac{du}{1+u^2}$, where $u = e^x$ and $du = e^x\,dx$; $x = 0 \Rightarrow u = 1$, $x = \ln\sqrt{3} \Rightarrow u = \sqrt{3}$
$= \left[\tan^{-1} u\right]_{1}^{\sqrt{3}} = \tan^{-1} \sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

64. $\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1+\ln^2 t)} = 4\int_0^{\pi/4} \frac{du}{1+u^2}$, where $u = \ln t$ and $du = \frac{1}{t}\,dt$; $t = 1 \Rightarrow u = 0$, $t = e^{\pi/4} \Rightarrow u = \frac{\pi}{4}$
$= \left[4 \tan^{-1} u\right]_0^{\pi/4} = 4\left(\tan^{-1} \frac{\pi}{4} - \tan^{-1} 0\right) = 4 \tan^{-1} \frac{\pi}{4}$

65. $\int \frac{y\,dy}{\sqrt{1-y^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$, where $u = y^2$ and $du = 2y\,dy$
$= \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} y^2 + C$

66. $\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}} = \int \frac{du}{\sqrt{1-u^2}}$, where $u = \tan y$ and $du = \sec^2 y\,dy$
$= \sin^{-1} u + C = \sin^{-1}(\tan y) + C$

67. $\int \frac{dx}{\sqrt{-x^2+4x-3}} = \int \frac{dx}{\sqrt{1-(x^2-4x+4)}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

68. $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + C$

69. $\int_{-1}^{0} \frac{6\,dt}{\sqrt{3-2t-t^2}} = 6\int_{-1}^{0} \frac{dt}{\sqrt{4-(t^2+2t+1)}} = 6\int_{-1}^{0} \frac{dt}{\sqrt{2^2-(t+1)^2}} = 6\left[\sin^{-1}\left(\frac{t+1}{2}\right)\right]_{-1}^{0}$
$= 6\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right] = 6\left(\frac{\pi}{6} - 0\right) = \pi$

70. $\int_{1/2}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}} = 3\int_{1/2}^{1} \frac{2\,dt}{\sqrt{4-(4t^2-4t+1)}} = 3\int_{1/2}^{1} \frac{2\,dt}{\sqrt{2^2-(2t-1)^2}} = 3\left[\sin^{-1}\left(\frac{2t-1}{2}\right)\right]_{1/2}^{1}$
$= 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\right] = 3\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{2}$

71. $\int \frac{dy}{y^2-2y+5} = \int \frac{dy}{4+y^2-2y+1} = \int \frac{dy}{2^2+(y-1)^2} = \frac{1}{2}\tan^{-1}\left(\frac{y-1}{2}\right) + C$

72. $\int \frac{dy}{y^2+6y+10} = \int \frac{dy}{1+(y^2+6y+9)} = \int \frac{dy}{1+(y+3)^2} = \tan^{-1}(y+3) + C$

73. $\int_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int_1^2 \frac{dx}{1+(x^2-2x+1)} = 8\int_1^2 \frac{dx}{1+(x-1)^2} = 8\left[\tan^{-1}(x-1)\right]_1^2 = 8\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

74. $\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{1+(x^2-6x+9)} = 2\int_2^4 \frac{dx}{1+(x-3)^2} = 2\left[\tan^{-1}(x-3)\right]_2^4 = 2\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

75. $\int \frac{x+4}{x^2+4}dx = \int \frac{x}{x^2+4}dx + \int \frac{4}{x^2+4}dx$; $\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du$ where $u = x^2+4 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$
$\Rightarrow \int \frac{x+4}{x^2+4}dx = \frac{1}{2}\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$

76. $\int \frac{t-2}{t^2-6t+10}dt = \int \frac{t-2}{(t-3)^2+1}dt$ $\left[\text{Let } w = t-3 \Rightarrow w+3 = t \Rightarrow dw = dt\right] \to \int \frac{w+1}{w^2+1}dw = \int \frac{w}{w^2+1}dw + \int \frac{1}{w^2+1}dw$;
$\int \frac{w}{w^2+1}dw = \frac{1}{2}\int \frac{1}{u}du$ where $u = w^2+1 \Rightarrow du = 2w\,dw \Rightarrow \frac{1}{2}du = w\,dw \Rightarrow \int \frac{w}{w^2+1}dw + \int \frac{1}{w^2+1}dw$
$= \frac{1}{2}\ln(w^2+1) + \tan^{-1}(w) + C = \frac{1}{2}\ln\left((t-3)^2+1\right) + \tan^{-1}(t-3) + C = \frac{1}{2}\ln(t^2-6t+10) + \tan^{-1}(t-3) + C$

77. $\int \frac{x^2+2x-1}{x^2+9}dx = \int \left(1 + \frac{2x-10}{x^2+9}\right)dx = \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx$; $\int \frac{2x}{x^2+9}dx = \int \frac{1}{u}du$ where $u = x^2+9$
$\Rightarrow du = 2x\,dx \Rightarrow \int dx + \int \frac{2x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx = x + \ln(x^2+9) - \frac{10}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

78. $\int \frac{t^3-2t^2+3t-4}{t^2+1}dt = \int \left(t-2+\frac{2t-2}{t^2+1}\right)dt = \int (t-2)dt + \int \frac{2t}{t^2+1}dt - 2\int \frac{1}{t^2+1}dt$; $\int \frac{2t}{t^2+1}dt = \int \frac{1}{u}du$ where $u = t^2+1$
$\Rightarrow du = 2t\,dt \Rightarrow \int (t-2)dt + \int \frac{2t}{t^2+1}dt - 2\int \frac{1}{t^2+1}dt = \frac{1}{2}t^2 - 2t + \ln(t^2+1) - 2\tan^{-1}(t) + C$

79. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{x^2+2x+1-1}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} = \int \frac{du}{u\sqrt{u^2-1}}$, where $u = x + 1$ and $du = dx$
$= \sec^{-1}|u| + C = \sec^{-1}|x+1| + C$

80. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-1}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}} = \int \frac{1}{u\sqrt{u^2-1}}\,du$, where $u = x - 2$ and $du = dx$
$= \sec^{-1}|u| + C = \sec^{-1}|x-2| + C$

81. $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx = \int e^u\,du$, where $u = \sin^{-1}x$ and $du = \frac{dx}{\sqrt{1-x^2}}$
$= e^u + C = e^{\sin^{-1}x} + C$

82. $\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx = -\int e^u\,du$, where $u = \cos^{-1}x$ and $du = \frac{-dx}{\sqrt{1-x^2}}$
$= -e^u + C = -e^{\cos^{-1}x} + C$

83. $\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}\,dx = \int u^2\,du$, where $u = \sin^{-1}x$ and $du = \frac{dx}{\sqrt{1-x^2}}$
$= \frac{u^3}{3} + C = \frac{(\sin^{-1}x)^3}{3} + C$

84. $\int \frac{\sqrt{\tan^{-1}x}}{1+x^2}\,dx = \int u^{1/2}\,du$, where $u = \tan^{-1}x$ and $du = \frac{dx}{1+x^2}$
$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan^{-1}x)^{3/2} + C = \frac{2}{3}\sqrt{(\tan^{-1}x)^3} + C$

85. $\int \frac{1}{(\tan^{-1}y)(1+y^2)}\,dy = \int \frac{\left(\frac{1}{1+y^2}\right)}{\tan^{-1}y}\,dy = \int \frac{1}{u}\,du$, where $u = \tan^{-1}y$ and $du = \frac{dy}{1+y^2}$
$= \ln|u| + C = \ln|\tan^{-1}y| + C$

86. $\int \frac{1}{(\sin^{-1}y)\sqrt{1+y^2}}\,dy = \int \frac{\left(\frac{1}{\sqrt{1-y^2}}\right)}{\sin^{-1}y}\,dy = \int \frac{1}{u}\,du$, where $u = \sin^{-1}y$ and $du = \frac{dy}{\sqrt{1-y^2}}$
$= \ln|u| + C = \ln|\sin^{-1}y| + C$

87. $\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1}x)}{x\sqrt{x^2-1}}\,dx = \int_{\pi/4}^{\pi/3} \sec^2 u\,du$, where $u = \sec^{-1}x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \sqrt{2} \Rightarrow u = \frac{\pi}{4}$, $x = 2 \Rightarrow u = \frac{\pi}{3}$
$= [\tan u]_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4} = \sqrt{3} - 1$

88. $\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1}x)}{x\sqrt{x^2-1}}\,dx = \int_{\pi/6}^{\pi/3} \cos u\,du$, where $u = \sec^{-1}x$ and $du = \frac{dx}{x\sqrt{x^2-1}}$; $x = \frac{2}{\sqrt{3}} \Rightarrow u = \frac{\pi}{6}$, $x = 2 \Rightarrow u = \frac{\pi}{3}$
$= [\sin u]_{\pi/6}^{\pi/3} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6} = \frac{\sqrt{3}-1}{2}$

89. $\int \frac{1}{\sqrt{x}(x+1)\left[(\tan^{-1}\sqrt{x})^2+9\right]}\,dx = 2\int \frac{1}{u^2+9}du$ where $u = \tan^{-1}\sqrt{x} \Rightarrow du = \frac{1}{1+(\sqrt{x})^2}\frac{1}{2\sqrt{x}}dx \Rightarrow 2du = \frac{1}{(1+x)\sqrt{x}}dx$
$= \frac{2}{3}\tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right) + C$

90. $\int \frac{e^x \sin^{-1}e^x}{\sqrt{1-e^{2x}}}\,dx = \int u\,du$ where $u = \sin^{-1}e^x \Rightarrow du = \frac{1}{\sqrt{1-e^{2x}}}e^x dx$
$= \frac{1}{2}(\sin^{-1}e^x)^2 + C$

91. $\lim_{x\to 0} \frac{\sin^{-1}5x}{x} = \lim_{x\to 0} \frac{\left(\frac{5}{\sqrt{1-25x^2}}\right)}{1} = 5$

92. $\lim\limits_{x \to 1^+} \frac{\sqrt{x^2-1}}{\sec^{-1} x} = \lim\limits_{x \to 1^+} \frac{(x^2-1)^{1/2}}{\sec^{-1} x} = \lim\limits_{x \to 1^+} \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x)}{\left(\frac{1}{|x|\sqrt{x^2-1}}\right)} = \lim\limits_{x \to 1^+} x\,|x| = 1$

93. $\lim\limits_{x \to \infty} x \tan^{-1}\left(\frac{2}{x}\right) = \lim\limits_{x \to \infty} \frac{\tan^{-1}(2x^{-1})}{x^{-1}} = \lim\limits_{x \to \infty} \frac{\left(\frac{-2x^{-2}}{1+4x^{-2}}\right)}{-x^{-2}} = \lim\limits_{x \to \infty} \frac{2}{1+4x^{-2}} = 2$

94. $\lim\limits_{x \to 0} \frac{2\tan^{-1} 3x^2}{7x^2} = \lim\limits_{x \to 0} \frac{\left(\frac{12x}{1+9x^4}\right)}{14x} = \lim\limits_{x \to 0} \frac{6}{7(1+9x^4)} = \frac{6}{7}$

95. $\lim\limits_{x\to 0} \frac{\tan^{-1} x^2}{x \sin^{-1} x} = \lim\limits_{x\to 0} \left(\frac{\frac{2x}{1+x^4}}{x\frac{1}{\sqrt{1-x^2}} + \sin^{-1} x}\right) = \lim\limits_{x\to 0} \left(\frac{\frac{-2\left(3x^4-1\right)}{\left(1+x^4\right)^2}}{\frac{-x^2+2}{\left(1-x^2\right)^{3/2}}}\right) = \frac{\frac{-2(0-1)}{1^2}}{\frac{-0+2}{(1-0)^{3/2}}} = \frac{2}{2} = 1$

96. $\lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x}{e^{2x}+x} = \lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1}}{2e^{2x}+1} = \lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}}{e^{2x}+1} + \frac{2e^{2x}}{\left(e^{2x}+1\right)^2}}{4e^{2x}} = \lim\limits_{x\to\infty} \frac{e^x \tan^{-1} e^x + \frac{e^{2x}\left(e^{2x}+3\right)}{\left(e^{2x}+1\right)^2}}{4e^{2x}}$

$= \lim\limits_{x\to\infty} \left[\frac{\tan^{-1} e^x}{4e^x} + \frac{(e^{2x}+3)}{4(e^{2x}+1)^2}\right] = \lim\limits_{x\to\infty} \left[\frac{\tan^{-1} e^x}{4e^x} + \frac{\left(1+3e^{-2x}\right)}{4(e^x+e^{-x})^2}\right] = 0+0 = 0$

97. $\lim\limits_{x\to 0^+} \frac{\left[\tan^{-1}\left(\sqrt{x}\right)\right]^2}{x\sqrt{x+1}} = \lim\limits_{x\to 0^+} \frac{\tan^{-1}\left(\sqrt{x}\right)\frac{1}{\sqrt{x}(1+x)}}{\frac{x}{2\sqrt{x+1}} + \sqrt{x+1}} = \lim\limits_{x\to 0^+} \frac{\frac{\tan^{-1}\left(\sqrt{x}\right)}{\sqrt{x}(1+x)}}{\frac{3x+2}{2\sqrt{x+1}}} = \lim\limits_{x\to 0^+} \left(\frac{2\tan^{-1}\left(\sqrt{x}\right)}{(3x+2)\sqrt{x}\sqrt{x+1}}\right) = \lim\limits_{x\to 0^+} \left(\frac{\frac{1}{\sqrt{x}(1+x)}}{\frac{12x^2+13x+2}{2\sqrt{x}\sqrt{x+1}}}\right)$

$= \lim\limits_{x\to 0^+} \left(\frac{2}{(12x^2+13x+2)\sqrt{x+1}}\right) = \frac{2}{2} = 1$

98. $\lim\limits_{x\to 0^+} \frac{\sin^{-1}(x^2)}{(\sin^{-1} x)^2} = \lim\limits_{x\to 0^+} \left(\frac{\frac{2x}{\sqrt{1-x^4}}}{2(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}}}\right) = \lim\limits_{x\to 0^+} \left(\frac{x}{\sin^{-1} x\sqrt{1+x^2}}\right) = \lim\limits_{x\to 0^+} \left(\frac{1}{\sin^{-1} x\cdot\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1-x^2}}\sqrt{1+x^2}}\right) =$

$= \lim\limits_{x\to 0^+} \left(\frac{\sqrt{1+x^2}\sqrt{1-x^2}}{1+x^2+x\sqrt{1-x^2}\sin^{-1} x}\right) = \frac{1}{1} = 1$

99. If $y = \ln x - \frac{1}{2}\ln(1+x^2) - \frac{\tan^{-1} x}{x} + C$, then $dy = \left[\frac{1}{x} - \frac{x}{1+x^2} - \frac{\left(\frac{x}{1+x^2}\right) - \tan^{-1} x}{x^2}\right] dx$

$= \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{1}{x(1+x^2)} + \frac{\tan^{-1} x}{x^2}\right) dx = \frac{x(1+x^2) - x^3 - x + (\tan^{-1} x)(1+x^2)}{x^2(1+x^2)} dx = \frac{\tan^{-1} x}{x^2} dx,$

which verifies the formula

100. If $y = \frac{x^4}{4}\cos^{-1} 5x + \frac{5}{4}\int \frac{x^4}{\sqrt{1-25x^2}} dx$, then $dy = \left[x^3\cos^{-1} 5x + \left(\frac{x^4}{4}\right)\left(\frac{-5}{\sqrt{1-25x^2}}\right) + \frac{5}{4}\left(\frac{x^4}{\sqrt{1-25x^2}}\right)\right] dx$
$= (x^3\cos^{-1} 5x)\, dx$, which verifies the formula

101. If $y = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1} x + C$, then
$dy = \left[(\sin^{-1} x)^2 + \frac{2x(\sin^{-1} x)}{\sqrt{1-x^2}} - 2 + \frac{-2x}{\sqrt{1-x^2}}\sin^{-1} x + 2\sqrt{1-x^2}\left(\frac{1}{\sqrt{1-x^2}}\right)\right] dx = (\sin^{-1} x)^2\, dx$, which verifies the formula

102. If $y = x\ln(a^2+x^2) - 2x + 2a\tan^{-1}\left(\frac{x}{a}\right) + C$, then $dy = \left[\ln(a^2+x^2) + \frac{2x^2}{a^2+x^2} - 2 + \frac{2}{1+\left(\frac{x^2}{a^2}\right)}\right] dx$
$= \left[\ln(a^2+x^2) + 2\left(\frac{a^2+x^2}{a^2+x^2}\right) - 2\right] dx = \ln(a^2+x^2)\, dx$, which verifies the formula

103. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow dy = \frac{dx}{\sqrt{1-x^2}} \Rightarrow y = \sin^{-1} x + C$; $x = 0$ and $y = 0 \Rightarrow 0 = \sin^{-1} 0 + C \Rightarrow C = 0 \Rightarrow y = \sin^{-1} x$

104. $\frac{dy}{dx} = \frac{1}{x^2+1} - 1 \Rightarrow dy = \left(\frac{1}{1+x^2} - 1\right) dx \Rightarrow y = \tan^{-1}(x) - x + C$; $x = 0$ and $y = 1 \Rightarrow 1 = \tan^{-1} 0 - 0 + C$
$\Rightarrow\ C = 1\ \Rightarrow\ y = \tan^{-1}(x) - x + 1$

105. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}\ \Rightarrow\ dy = \frac{dx}{x\sqrt{x^2-1}}\ \Rightarrow\ y = \sec^{-1}|x| + C$; $x = 2$ and $y = \pi\ \Rightarrow\ \pi = \sec^{-1} 2 + C\ \Rightarrow\ C = \pi - \sec^{-1} 2$
$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}\ \Rightarrow\ y = \sec^{-1}(x) + \frac{2\pi}{3}, x > 1$

106. $\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\ \Rightarrow\ dy = \left(\frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right) dx\ \Rightarrow\ y = \tan^{-1} x - 2\sin^{-1} x + C$; $x = 0$ and $y = 2$
$\Rightarrow\ 2 = \tan^{-1} 0 - 2\sin^{-1} 0 + C\ \Rightarrow\ C = 2\ \Rightarrow\ y = \tan^{-1} x - 2\sin^{-1} x + 2$

107. (a) The angle α is the large angle between the wall and the right end of the blackboard minus the small angle between the left end of the blackboard and the wall $\Rightarrow\ \alpha = \cot^{-1}\left(\frac{x}{15}\right) - \cot^{-1}\left(\frac{x}{3}\right)$.

(b) $\frac{d\alpha}{dt} = -\frac{\frac{1}{15}}{1+\left(\frac{x}{15}\right)^2} + \frac{\frac{1}{3}}{1+\left(\frac{x}{3}\right)^2} = -\frac{15}{225+x^2} + \frac{3}{9+x^2} = \frac{540-12x^2}{(225+x^2)(9+x^2)}$; $\frac{d\alpha}{dt} = 0 \Rightarrow 540 - 12x^2 = 0 \Rightarrow x = \pm 3\sqrt{5}$
Since $x > 0$, consider only $x = 3\sqrt{5} \Rightarrow \alpha\left(3\sqrt{5}\right) = \cot^{-1}\left(\frac{3\sqrt{5}}{15}\right) - \cot^{-1}\left(\frac{3\sqrt{5}}{3}\right) \approx 0.729728 \approx 41.8103°$. Using the first derivative test, $\left.\frac{d\alpha}{dt}\right|_{x=1} = \frac{132}{565} > 0$ and $\left.\frac{d\alpha}{dt}\right|_{x=10} = -\frac{132}{7085} < 0 \Rightarrow$ local maximum of $41.8103°$ when $x = 3\sqrt{5} \approx 6.7082$ ft.

108. $V = \pi \int_0^{\pi/3} [2^2 - (\sec y)^2]\, dy = \pi\,[4y - \tan y]_0^{\pi/3} = \pi\left(\frac{4\pi}{3} - \sqrt{3}\right)$

109. $V = \left(\frac{1}{3}\right)\pi r^2 h = \left(\frac{1}{3}\right)\pi(3\sin\theta)^2(3\cos\theta) = 9\pi(\cos\theta - \cos^3\theta)$, where $0 \le \theta \le \frac{\pi}{2}$
$\Rightarrow \frac{dV}{d\theta} = -9\pi(\sin\theta)(1 - 3\cos^2\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{3}}\ \Rightarrow$ the critical points are: $0, \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, and $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$; but $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is not in the domain. When $\theta = 0$, we have a minimum and when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $\approx 54.7°$, we have a maximum volume.

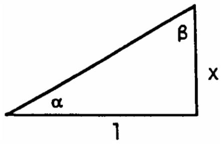
110. $65° + (90° - \beta) + (90° - \alpha) = 180°\ \Rightarrow\ \alpha = 65° - \beta = 65° - \tan^{-1}\left(\frac{21}{50}\right) \approx 65° - 22.78° \approx 42.22°$

111. Take each square as a unit square. From the diagram we have the following: the smallest angle α has a tangent of 1 $\Rightarrow\ \alpha = \tan^{-1} 1$; the middle angle β has a tangent of 2 $\Rightarrow\ \beta = \tan^{-1} 2$; and the largest angle γ has a tangent of 3 $\Rightarrow\ \gamma = \tan^{-1} 3$. The sum of these three angles is $\pi\ \Rightarrow\ \alpha + \beta + \gamma = \pi$
$\Rightarrow\ \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.

112. (a) From the symmetry of the diagram, we see that $\pi - \sec^{-1} x$ is the vertical distance from the graph of $y = \sec^{-1} x$ to the line $y = \pi$ and this distance is the same as the height of $y = \sec^{-1} x$ above the x-axis at $-x$; i.e., $\pi - \sec^{-1} x = \sec^{-1}(-x)$.

(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, where $-1 \le x \le 1\ \Rightarrow\ \cos^{-1}\left(-\frac{1}{x}\right) = \pi - \cos^{-1}\left(\frac{1}{x}\right)$, where $x \ge 1$ or $x \le -1$
$\Rightarrow\ \sec^{-1}(-x) = \pi - \sec^{-1} x$

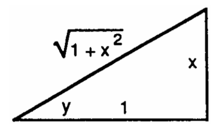
113. $\sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$; $\sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$; and $\sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$.
If $x \in (-1, 0)$ and $x = -a$, then $\sin^{-1}(x) + \cos^{-1}(x) = \sin^{-1}(-a) + \cos^{-1}(-a) = -\sin^{-1} a + (\pi - \cos^{-1} a)$
$= \pi - (\sin^{-1} a + \cos^{-1} a) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$ from Equations (3) and (4) in the text.

114. 

$\Rightarrow \tan\alpha = x$ and $\tan\beta = \frac{1}{x} \Rightarrow \frac{\pi}{2} = \alpha + \beta = \tan^{-1} x + \tan^{-1} \frac{1}{x}$.

115. $\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u \Rightarrow \frac{d}{dx}(\csc^{-1} u) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}} = -\frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}, |u| > 1$

116. $y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$
$\Rightarrow (\sec^2 y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\left(\sqrt{1+x^2}\right)^2}$
$= \frac{1}{1+x^2}$, as indicated by the triangle



117. $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} = \frac{1}{\sec(\sec^{-1} b)\tan(\sec^{-1} b)} = \frac{1}{b\left(\pm\sqrt{b^2-1}\right)}$.
Since the slope of $\sec^{-1} x$ is always positive, we the right sign by writing $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$.

118. $\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u \Rightarrow \frac{d}{dx}(\cot^{-1} u) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{1+u^2} = -\frac{\frac{du}{dx}}{1+u^2}$

119. The functions f and g have the same derivative (for $x \ge 0$), namely $\frac{1}{\sqrt{x}(x+1)}$. The functions therefore differ by a constant. To identify the constant we can set x equal to 0 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}(-1) = 2\tan^{-1}(0) + C \Rightarrow -\frac{\pi}{2} = 0 + C \Rightarrow C = -\frac{\pi}{2}$. For $x \ge 0$, we have $\sin^{-1}\left(\frac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$.

120. The functions f and g have the same derivative for $x > 0$, namely $\frac{-1}{1+x^2}$. The functions therefore differ by a constant for $x > 0$. To identify the constant we can set x equal to 1 in the equation $f(x) = g(x) + C$, obtaining $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \tan^{-1} 1 + C \Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C \Rightarrow C = 0$. For $x > 0$, we have $\sin^{-1}\frac{1}{\sqrt{x^2+1}} = \tan^{-1}\frac{1}{x}$.

121. $V = \pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = \pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\frac{1}{1+x^2}\,dx = \pi\left[\tan^{-1} x\right]_{-\sqrt{3}/3}^{\sqrt{3}} = \pi\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$
$= \pi\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi^2}{2}$

122. Consider $y = \sqrt{r^2 - x^2} \Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$; Since $\frac{dy}{dx}$ is undefined at $x = r$ and $x = -r$, we will find the length from $x = 0$ to $x = \frac{r}{\sqrt{2}}$ (in other words, the length of $\frac{1}{8}$ of a circle) $\Rightarrow L = \int_0^{r/\sqrt{2}}\sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}dx = \int_0^{r/\sqrt{2}}\sqrt{1 + \frac{x^2}{r^2-x^2}}dx$
$= \int_0^{r/\sqrt{2}}\sqrt{\frac{r^2}{r^2-x^2}}dx = \int_0^{r/\sqrt{2}}\frac{r}{\sqrt{r^2-x^2}}dx = \left[r\sin^{-1}\left(\frac{x}{r}\right)\right]_0^{r/\sqrt{2}} = r\sin^{-1}\left(\frac{r/\sqrt{2}}{r}\right) - r\sin^{-1}(0)$
$= r\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0 = r\left(\frac{\pi}{4}\right) = \frac{\pi r}{4}$. The total circumference of the circle is $C = 8L = 8\left(\frac{\pi r}{4}\right) = 2\pi r$.

123. (a) $A(x) = \frac{\pi}{4}(\text{diameter})^2 = \frac{\pi}{4}\left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{\pi}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^{1}\frac{\pi\,dx}{1+x^2}$
$= \pi\left[\tan^{-1} x\right]_{-1}^{1} = (\pi)(2)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2}$

(b) $A(x) = (\text{edge})^2 = \left[\frac{1}{\sqrt{1+x^2}} - \left(-\frac{1}{\sqrt{1+x^2}}\right)\right]^2 = \frac{4}{1+x^2} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-1}^{1}\frac{4\,dx}{1+x^2}$
$= 4\left[\tan^{-1} x\right]_{-1}^{1} = 4\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = 4\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 2\pi$

124. (a) $A(x) = \frac{\pi}{4}\,(\text{diameter})^2 = \frac{\pi}{4}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{\pi}{4}\left(\frac{4}{\sqrt{1-x^2}}\right) = \frac{\pi}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx$

$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\pi}{\sqrt{1-x^2}}\,dx = \pi\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = \pi\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$

(b) $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{1}{2}\left(\frac{2}{\sqrt[4]{1-x^2}} - 0\right)^2 = \frac{2}{\sqrt{1-x^2}} \Rightarrow V = \int_a^b A(x)\,dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}}\,dx$

$= 2\left[\sin^{-1} x\right]_{-\sqrt{2}/2}^{\sqrt{2}/2} = 2\left(\frac{\pi}{4}\cdot 2\right) = \pi$

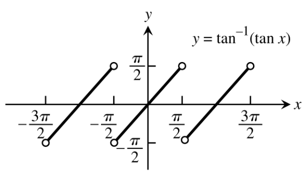
125. (a) $\sec^{-1} 1.5 = \cos^{-1}\frac{1}{1.5} \approx 0.84107$ (b) $\csc^{-1}(-1.5) = \sin^{-1}\left(-\frac{1}{1.5}\right) \approx -0.72973$

(c) $\cot^{-1} 2 = \frac{\pi}{2} - \tan^{-1} 2 \approx 0.46365$

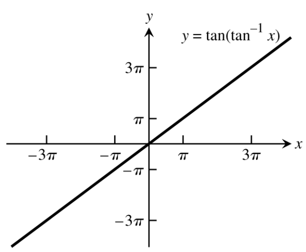
126. (a) $\sec^{-1}(-3) = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.91063$ (b) $\csc^{-1} 1.7 = \sin^{-1}\left(\frac{1}{1.7}\right) \approx 0.62887$

(c) $\cot^{-1}(-2) = \frac{\pi}{2} - \tan^{-1}(-2) \approx 2.67795$

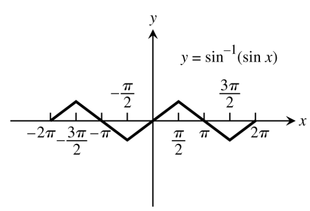
127. (a) Domain: all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer.
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



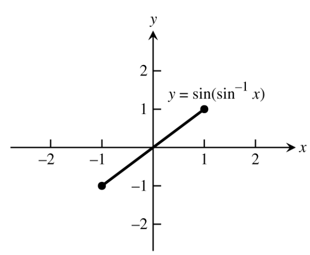
(b) Domain: $-\infty < x < \infty$; Range: $-\infty < y < \infty$
The graph of $y = \tan^{-1}(\tan x)$ is periodic, the graph of $y = \tan(\tan^{-1} x) = x$ for $-\infty \le x < \infty$.



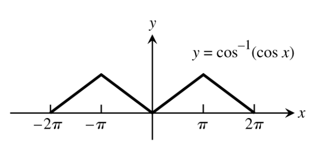
128. (a) Domain: $-\infty < x < \infty$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



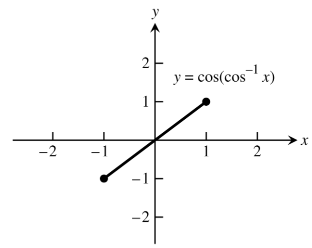
(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \sin^{-1}(\sin x)$ is periodic; the graph of $y = \sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$.



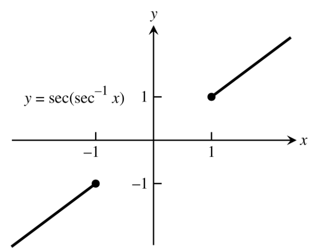
129. (a) Domain: $-\infty < x < \infty$; Range: $0 \le y \le \pi$



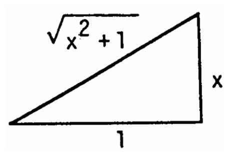
(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \cos^{-1}(\cos x)$ is periodic; the graph of $y = \cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$.

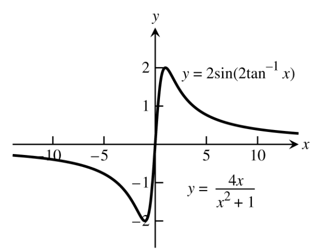


130. Since the domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$, we have $\sec(\sec^{-1} x) = x$ for $|x| \ge 1$. The graph of $y = \sec(\sec^{-1} x)$ is the line $y = x$ with the open line segment from $(-1, -1)$ to $(1, 1)$ removed.

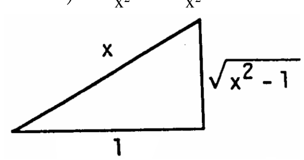


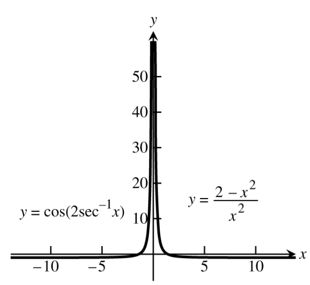
131. The graphs are identical for $y = 2\sin(2\tan^{-1} x)$
$= 4[\sin(\tan^{-1} x)][\cos(\tan^{-1} x)] = 4\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)$
$= \frac{4x}{x^2+1}$ from the triangle



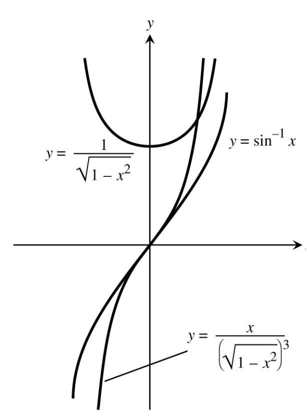


132. The graphs are identical for $y = \cos(2\sec^{-1} x)$
$= \cos^2(\sec^{-1} x) - \sin^2(\sec^{-1} x) = \frac{1}{x^2} - \frac{x^2-1}{x^2}$
$= \frac{2-x^2}{x^2}$ from the triangle

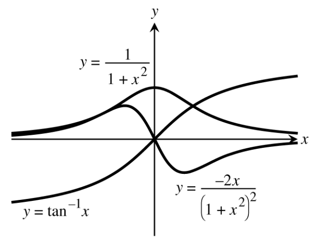




133. The values of f increase over the interval $[-1, 1]$ because $f' > 0$, and the graph of f steepens as the values of f' increase towards the ends of the interval. The graph of f is concave down to the left of the origin where $f'' < 0$, and concave up to the right of the origin where $f'' > 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local minimum value.



134. The values of f increase throughout the interval $(-\infty, \infty)$ because $f' > 0$, and they increase most rapidly near the origin where the values of f' are relatively large. The graph of f is concave up to the left of the origin where $f'' > 0$, and concave down to the right of the origin where $f'' < 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local maximum value.



7.7 HYPERBOLIC FUNCTIONS

1. $\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + \left(-\frac{3}{4}\right)^2} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5}$, $\coth x = \frac{1}{\tanh x} = -\frac{5}{3}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$, and $\operatorname{csch} x = \frac{1}{\sin x} = -\frac{4}{3}$

2. $\sinh x = \frac{4}{3} \Rightarrow \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{3}\right)} = \frac{4}{5}$, $\coth x = \frac{1}{\tanh x} = \frac{5}{4}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{3}{5}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{3}{4}$

3. $\cosh x = \frac{17}{15}$, $x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \sqrt{\frac{289}{225} - 1} = \sqrt{\frac{64}{225}} = \frac{8}{15}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{8}{15}\right)}{\left(\frac{17}{15}\right)}$ $= \frac{8}{17}$, $\coth x = \frac{1}{\tanh x} = \frac{17}{8}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{15}{17}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{15}{8}$

4. $\cosh x = \frac{13}{5}$, $x > 0 \Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{144}{25}} = \frac{12}{5}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(\frac{12}{5}\right)}{\left(\frac{13}{5}\right)} = \frac{12}{13}$, $\coth x = \frac{1}{\tanh x} = \frac{13}{12}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}$, and $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{5}{12}$

5. $2 \cosh (\ln x) = 2\left(\frac{e^{\ln x} + e^{-\ln x}}{2}\right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$

6. $\sinh (2 \ln x) = \frac{e^{2 \ln x} - e^{-2 \ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$

7. $\cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$

8. $\cosh 3x - \sinh 3x = \frac{e^{3x} + e^{-3x}}{2} - \frac{e^{3x} - e^{-3x}}{2} = e^{-3x}$

9. $(\sinh x + \cosh x)^4 = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4 = (e^x)^4 = e^{4x}$

10. $\ln (\cosh x + \sinh x) + \ln (\cosh x - \sinh x) = \ln (\cosh^2 x - \sinh^2 x) = \ln 1 = 0$

11. (a) $\sinh 2x = \sinh (x + x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$
    (b) $\cosh 2x = \cosh (x + x) = \cosh x \cosh x + \sinh x \sin x = \cosh^2 x + \sinh^2 x$

12. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left[(e^x + e^{-x}) + (e^x - e^{-x})\right]\left[(e^x + e^{-x}) - (e^x - e^{-x})\right] = \frac{1}{4}(2e^x)(2e^{-x})$ $= \frac{1}{4}(4e^0) = \frac{1}{4}(4) = 1$

13. $y = 6 \sinh \frac{x}{3} \Rightarrow \frac{dy}{dx} = 6\left(\cosh \frac{x}{3}\right)\left(\frac{1}{3}\right) = 2 \cosh \frac{x}{3}$

14. $y = \frac{1}{2} \sinh (2x + 1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}[\cosh (2x + 1)](2) = \cosh (2x + 1)$

15. $y = 2\sqrt{t} \tanh \sqrt{t} = 2t^{1/2} \tanh t^{1/2} \Rightarrow \frac{dy}{dt} = \left[\operatorname{sech}^2\left(t^{1/2}\right)\right]\left(\frac{1}{2} t^{-1/2}\right)\left(2t^{1/2}\right) + \left(\tanh t^{1/2}\right)\left(t^{-1/2}\right) = \operatorname{sech}^2 \sqrt{t} + \frac{\tanh \sqrt{t}}{\sqrt{t}}$

16. $y = t^2 \tanh \frac{1}{t} = t^2 \tanh t^{-1} \Rightarrow \frac{dy}{dt} = [\operatorname{sech}^2 (t^{-1})] (-t^{-2}) (t^2) + (2t) (\tanh t^{-1}) = -\operatorname{sech}^2 \frac{1}{t} + 2t \tanh \frac{1}{t}$

17. $y = \ln (\sinh z) \Rightarrow \frac{dy}{dz} = \frac{\cosh z}{\sinh z} = \coth z$

18. $y = \ln (\cosh z) \Rightarrow \frac{dy}{dz} = \frac{\sinh z}{\cosh z} = \tanh z$

19. $y = (\operatorname{sech} \theta)(1 - \ln \operatorname{sech} \theta) \Rightarrow \frac{dy}{d\theta} = \left(-\frac{-\operatorname{sech} \theta \tanh \theta}{\operatorname{sech} \theta}\right) (\operatorname{sech} \theta) + (-\operatorname{sech} \theta \tanh \theta)(1 - \ln \operatorname{sech} \theta)$
$= \operatorname{sech} \theta \tanh \theta - (\operatorname{sech} \theta \tanh \theta)(1 - \ln \operatorname{sech} \theta) = (\operatorname{sech} \theta \tanh \theta)[1 - (1 - \ln \operatorname{sech} \theta)] = (\operatorname{sech} \theta \tanh \theta)(\ln \operatorname{sech} \theta)$

20. $y = (\operatorname{csch} \theta)(1 - \ln \operatorname{csch} \theta) \Rightarrow \frac{dy}{d\theta} = (\operatorname{csch} \theta) \left(-\frac{-\operatorname{csch} \theta \coth \theta}{\operatorname{csch} \theta}\right) + (1 - \ln \operatorname{csch} \theta)(-\operatorname{csch} \theta \coth \theta)$
$= \operatorname{csch} \theta \coth \theta - (1 - \ln \operatorname{csch} \theta)(\operatorname{csch} \theta \coth \theta) = (\operatorname{csch} \theta \coth \theta)(1 - 1 + \ln \operatorname{csch} \theta) = (\operatorname{csch} \theta \coth \theta)(\ln \operatorname{csch} \theta)$

21. $y = \ln \cosh v - \frac{1}{2} \tanh^2 v \Rightarrow \frac{dy}{dv} = \frac{\sinh v}{\cosh v} - \left(\frac{1}{2}\right) (2 \tanh v) (\operatorname{sech}^2 v) = \tanh v - (\tanh v) (\operatorname{sech}^2 v)$
$= (\tanh v) (1 - \operatorname{sech}^2 v) = (\tanh v) (\tanh^2 v) = \tanh^3 v$

22. $y = \ln \sinh v - \frac{1}{2} \coth^2 v \Rightarrow \frac{dy}{dv} = \frac{\cosh v}{\sinh v} - \left(\frac{1}{2}\right) (2 \coth v) (-\operatorname{csch}^2 v) = \coth v + (\coth v) (\operatorname{csch}^2 v)$
$= (\coth v) (1 + \operatorname{csch}^2 v) = (\coth v) (\coth^2 v) = \coth^3 v$

23. $y = (x^2 + 1) \operatorname{sech} (\ln x) = (x^2 + 1) \left(\frac{2}{e^{\ln x} + e^{-\ln x}}\right) = (x^2 + 1) \left(\frac{2}{x + x^{-1}}\right) = (x^2 + 1) \left(\frac{2x}{x^2+1}\right) = 2x \Rightarrow \frac{dy}{dx} = 2$

24. $y = (4x^2 - 1) \operatorname{csch} (\ln 2x) = (4x^2 - 1) \left(\frac{2}{e^{\ln 2x} - e^{-\ln 2x}}\right) = (4x^2 - 1) \left(\frac{2}{2x - (2x)^{-1}}\right) = (4x^2 - 1) \left(\frac{4x}{4x^2-1}\right) = 4x \Rightarrow \frac{dy}{dx} = 4$

25. $y = \sinh^{-1} \sqrt{x} = \sinh^{-1} \left(x^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right) x^{-1/2}}{\sqrt{1 + (x^{1/2})^2}} = \frac{1}{2\sqrt{x}\,\sqrt{1+x}} = \frac{1}{2\sqrt{x(1+x)}}$

26. $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1} \left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{[2(x+1)^{1/2}]^2 - 1}} = \frac{1}{\sqrt{x+1}\,\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

27. $y = (1 - \theta) \tanh^{-1} \theta \Rightarrow \frac{dy}{d\theta} = (1 - \theta) \left(\frac{1}{1-\theta^2}\right) + (-1) \tanh^{-1} \theta = \frac{1}{1+\theta} - \tanh^{-1} \theta$

28. $y = (\theta^2 + 2\theta) \tanh^{-1} (\theta + 1) \Rightarrow \frac{dy}{d\theta} = (\theta^2 + 2\theta) \left[\frac{1}{1-(\theta+1)^2}\right] + (2\theta + 2) \tanh^{-1} (\theta + 1) = \frac{\theta^2 + 2\theta}{-\theta^2 - 2\theta} + (2\theta + 2) \tanh^{-1} (\theta + 1)$
$= (2\theta + 2) \tanh^{-1} (\theta + 1) - 1$

29. $y = (1 - t) \coth^{-1} \sqrt{t} = (1 - t) \coth^{-1} \left(t^{1/2}\right) \Rightarrow \frac{dy}{dt} = (1 - t) \left[\frac{\left(\frac{1}{2}\right) t^{-1/2}}{1 - (t^{1/2})^2}\right] + (-1) \coth^{-1} \left(t^{1/2}\right) = \frac{1}{2\sqrt{t}} - \coth^{-1} \sqrt{t}$

30. $y = (1 - t^2) \coth^{-1} t \Rightarrow \frac{dy}{dt} = (1 - t^2) \left(\frac{1}{1-t^2}\right) + (-2t) \coth^{-1} t = 1 - 2t \coth^{-1} t$

31. $y = \cos^{-1} x - x \operatorname{sech}^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} - \left[x \left(\frac{-1}{x\sqrt{1-x^2}}\right) + (1) \operatorname{sech}^{-1} x\right] = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} - \operatorname{sech}^{-1} x = -\operatorname{sech}^{-1} x$

32. $y = \ln x + \sqrt{1 - x^2} \operatorname{sech}^{-1} x = \ln x + (1 - x^2)^{1/2} \operatorname{sech}^{-1} x$
$\Rightarrow \frac{dy}{dx} = \frac{1}{x} + (1 - x^2)^{1/2} \left(\frac{-1}{x\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right) (1 - x^2)^{-1/2}(-2x) \operatorname{sech}^{-1} x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1-x^2}} \operatorname{sech}^{-1} x = \frac{-x}{\sqrt{1-x^2}} \operatorname{sech}^{-1} x$

33. $y = \operatorname{csch}^{-1} \left(\frac{1}{2}\right)^\theta \Rightarrow \frac{dy}{d\theta} = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^\theta}{\left(\frac{1}{2}\right)^\theta \sqrt{1 + \left[\left(\frac{1}{2}\right)^\theta\right]^2}} = -\frac{\ln(1) - \ln(2)}{\sqrt{1 + \left(\frac{1}{2}\right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1 + \left(\frac{1}{2}\right)^{2\theta}}}$

34. $y = \operatorname{csch}^{-1} 2^\theta \Rightarrow \frac{dy}{d\theta} = -\frac{(\ln 2)\, 2^\theta}{2^\theta \sqrt{1+(2^\theta)^2}} = \frac{-\ln 2}{\sqrt{1+2^{2\theta}}}$

35. $y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x|\,|\sec x|}{|\sec x|} = |\sec x|$

36. $y = \cosh^{-1}(\sec x) \Rightarrow \frac{dy}{dx} = \frac{(\sec x)(\tan x)}{\sqrt{\sec^2 x - 1}} = \frac{(\sec x)(\tan x)}{\sqrt{\tan^2 x}} = \frac{(\sec x)(\tan x)}{|\tan x|} = \sec x,\ 0 < x < \frac{\pi}{2}$

37. (a) If $y = \tan^{-1}(\sinh x) + C$, then $\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$, which verifies the formula
(b) If $y = \sin^{-1}(\tanh x) + C$, then $\frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x$, which verifies the formula

38. If $y = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$, then $\frac{dy}{dx} = x \operatorname{sech}^{-1} x + \frac{x^2}{2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \frac{2x}{4\sqrt{1-x^2}} = x \operatorname{sech}^{-1} x$, which verifies the formula

39. If $y = \frac{x^2-1}{2} \coth^{-1} x + \frac{x}{2} + C$, then $\frac{dy}{dx} = x \coth^{-1} x + \left(\frac{x^2-1}{2}\right)\left(\frac{1}{1-x^2}\right) + \frac{1}{2} = x \coth^{-1} x$, which verifies the formula

40. If $y = x \tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$, then $\frac{dy}{dx} = \tanh^{-1} x + x\left(\frac{1}{1-x^2}\right) + \frac{1}{2}\left(\frac{-2x}{1-x^2}\right) = \tanh^{-1} x$, which verifies the formula

41. $\int \sinh 2x\, dx = \frac{1}{2}\int \sinh u\, du$, where $u = 2x$ and $du = 2\, dx$
$= \frac{\cosh u}{2} + C = \frac{\cosh 2x}{2} + C$

42. $\int \sinh \frac{x}{5}\, dx = 5\int \sinh u\, du$, where $u = \frac{x}{5}$ and $du = \frac{1}{5}\, dx$
$= 5\cosh u + C = 5\cosh \frac{x}{5} + C$

43. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right) dx = 12\int \cosh u\, du$, where $u = \frac{x}{2} - \ln 3$ and $du = \frac{1}{2}\, dx$
$= 12\sinh u + C = 12\sinh\left(\frac{x}{2} - \ln 3\right) + C$

44. $\int 4\cosh(3x - \ln 2)\, dx = \frac{4}{3}\int \cosh u\, du$, where $u = 3x - \ln 2$ and $du = 3\, dx$
$= \frac{4}{3}\sinh u + C = \frac{4}{3}\sinh(3x - \ln 2) + C$

45. $\int \tanh \frac{x}{7}\, dx = 7\int \frac{\sinh u}{\cosh u}\, du$, where $u = \frac{x}{7}$ and $du = \frac{1}{7}\, dx$
$= 7\ln|\cosh u| + C_1 = 7\ln\left|\cosh \frac{x}{7}\right| + C_1 = 7\ln\left|\frac{e^{x/7}+e^{-x/7}}{2}\right| + C_1 = 7\ln\left|e^{x/7}+e^{-x/7}\right| - 7\ln 2 + C_1$
$= 7\ln|e^{x/7}+e^{-x/7}| + C$

46. $\int \coth \frac{\theta}{\sqrt{3}}\, d\theta = \sqrt{3}\int \frac{\cosh u}{\sinh u}\, du$, where $u = \frac{\theta}{\sqrt{3}}$ and $du = \frac{d\theta}{\sqrt{3}}$
$= \sqrt{3}\ln|\sinh u| + C_1 = \sqrt{3}\ln\left|\sinh \frac{\theta}{\sqrt{3}}\right| + C_1 = \sqrt{3}\ln\left|\frac{e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}}{2}\right| + C_1$
$= \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| - \sqrt{3}\ln 2 + C_1 = \sqrt{3}\ln\left|e^{\theta/\sqrt{3}} - e^{-\theta/\sqrt{3}}\right| + C$

47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right) dx = \int \operatorname{sech}^2 u\, du$, where $u = \left(x - \frac{1}{2}\right)$ and $du = dx$
$= \tanh u + C = \tanh\left(x - \frac{1}{2}\right) + C$

48. $\int \operatorname{csch}^2(5-x)\,dx = -\int \operatorname{csch}^2 u\,du$, where $u = (5-x)$ and $du = -\,dx$
$= -(-\coth u) + C = \coth u + C = \coth(5-x) + C$

49. $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}}{\sqrt{t}}\,dt = 2\int \operatorname{sech} u \tanh u\,du$, where $u = \sqrt{t} = t^{1/2}$ and $du = \frac{dt}{2\sqrt{t}}$
$= 2(-\operatorname{sech} u) + C = -2\operatorname{sech}\sqrt{t} + C$

50. $\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)}{t}\,dt = \int \operatorname{csch} u \coth u\,du$, where $u = \ln t$ and $du = \frac{dt}{t}$
$= -\operatorname{csch} u + C = -\operatorname{csch}(\ln t) + C$

51. $\int_{\ln 2}^{\ln 4} \coth x\,dx = \int_{\ln 2}^{\ln 4} \frac{\cosh x}{\sinh x}\,dx = \int_{3/4}^{15/8} \frac{1}{u}\,du = [\ln|u|]_{3/4}^{15/8} = \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8}\cdot\frac{4}{3}\right| = \ln\frac{5}{2}$,
where $u = \sinh x$, $du = \cosh x\,dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2-\left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper limit is $\sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4-\left(\frac{1}{4}\right)}{2} = \frac{15}{8}$

52. $\int_0^{\ln 2} \tanh 2x\,dx = \int_0^{\ln 2} \frac{\sinh 2x}{\cosh 2x}\,dx = \frac{1}{2}\int_1^{17/8} \frac{1}{u}\,du = \frac{1}{2}[\ln|u|]_1^{17/8} = \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right] = \frac{1}{2}\ln\frac{17}{8}$, where
$u = \cosh 2x$, $du = 2\sinh(2x)\,dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is $\cosh(2\ln 2) = \cosh(\ln 4)$
$= \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4+\left(\frac{1}{4}\right)}{2} = \frac{17}{8}$

53. $\int_{-\ln 4}^{-\ln 2} 2e^\theta \cosh\theta\,d\theta = \int_{-\ln 4}^{-\ln 2} 2e^\theta\left(\frac{e^\theta + e^{-\theta}}{2}\right)d\theta = \int_{-\ln 4}^{-\ln 2} (e^{2\theta}+1)\,d\theta = \left[\frac{e^{2\theta}}{2} + \theta\right]_{-\ln 4}^{-\ln 2}$
$= \left(\frac{e^{-2\ln 2}}{2} - \ln 2\right) - \left(\frac{e^{-2\ln 4}}{2} - \ln 4\right) = \left(\frac{1}{8} - \ln 2\right) - \left(\frac{1}{32} - \ln 4\right) = \frac{3}{32} - \ln 2 + 2\ln 2 = \frac{3}{32} + \ln 2$

54. $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta = \int_0^{\ln 2} 4e^{-\theta}\left(\frac{e^\theta - e^{-\theta}}{2}\right)d\theta = 2\int_0^{\ln 2} (1 - e^{-2\theta})\,d\theta = 2\left[\theta + \frac{e^{-2\theta}}{2}\right]_0^{\ln 2}$
$= 2\left[\left(\ln 2 + \frac{e^{-2\ln 2}}{2}\right) - \left(0 + \frac{e^0}{2}\right)\right] = 2\left(\ln 2 + \frac{1}{8} - \frac{1}{2}\right) = 2\ln 2 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4}$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\,d\theta = \int_{-1}^{1} \cosh u\,du = [\sinh u]_{-1}^{1} = \sinh(1) - \sinh(-1) = \left(\frac{e^1 - e^{-1}}{2}\right) - \left(\frac{e^{-1} - e^1}{2}\right)$
$= \frac{e - e^{-1} - e^{-1} + e}{2} = e - e^{-1}$, where $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, the lower limit is $\tan\left(-\frac{\pi}{4}\right) = -1$ and the upper limit is $\tan\left(\frac{\pi}{4}\right) = 1$

56. $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\,d\theta = 2\int_0^1 \sinh u\,du = 2[\cosh u]_0^1 = 2(\cosh 1 - \cosh 0) = 2\left(\frac{e+e^{-1}}{2} - 1\right)$
$= e + e^{-1} - 2$, where $u = \sin\theta$, $du = \cos\theta\,d\theta$, the lower limit is $\sin 0 = 0$ and the upper limit is $\sin\left(\frac{\pi}{2}\right) = 1$

57. $\int_1^2 \frac{\cosh(\ln t)}{t}\,dt = \int_0^{\ln 2} \cosh u\,du = [\sinh u]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} - 0 = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$, where
$u = \ln t$, $du = \frac{1}{t}\,dt$, the lower limit is $\ln 1 = 0$ and the upper limit is $\ln 2$

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\,dx = 16\int_1^2 \cosh u\,du = 16[\sinh u]_1^2 = 16(\sinh 2 - \sinh 1) = 16\left[\left(\frac{e^2 - e^{-2}}{2}\right) - \left(\frac{e - e^{-1}}{2}\right)\right]$
$= 8(e^2 - e^{-2} - e + e^{-1})$, where $u = \sqrt{x} = x^{1/2}$, $du = \frac{1}{2}x^{-1/2}dx = \frac{dx}{2\sqrt{x}}$, the lower limit is $\sqrt{1} = 1$ and the upper limit is $\sqrt{4} = 2$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx = \int_{-\ln 2}^{0} \frac{\cosh x + 1}{2}\, dx = \frac{1}{2}\int_{-\ln 2}^{0} (\cosh x + 1)\, dx = \frac{1}{2}\left[\sinh x + x\right]_{-\ln 2}^{0}$
$= \frac{1}{2}\left[(\sinh 0 + 0) - (\sinh(-\ln 2) - \ln 2)\right] = \frac{1}{2}\left[(0+0) - \left(\frac{e^{-\ln 2} - e^{\ln 2}}{2} - \ln 2\right)\right] = \frac{1}{2}\left[-\frac{\left(\frac{1}{2}\right) - 2}{2} + \ln 2\right]$
$= \frac{1}{2}\left(1 - \frac{1}{4} + \ln 2\right) = \frac{3}{8} + \frac{1}{2}\ln 2 = \frac{3}{8} + \ln\sqrt{2}$

60. $\int_{0}^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx = \int_{0}^{\ln 10} 4\left(\frac{\cosh x - 1}{2}\right) dx = 2\int_{0}^{\ln 10} (\cosh x - 1)\, dx = 2\left[\sinh x - x\right]_{0}^{\ln 10}$
$= 2[(\sinh(\ln 10) - \ln 10) - (\sinh 0 - 0)] = e^{\ln 10} - e^{-\ln 10} - 2\ln 10 = 10 - \frac{1}{10} - 2\ln 10 = 9.9 - 2\ln 10$

61. $\sinh^{-1}\left(\frac{-5}{12}\right) = \ln\left(-\frac{5}{12} + \sqrt{\frac{25}{144} + 1}\right) = \ln\left(\frac{2}{3}\right)$

62. $\cosh^{-1}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right) = \ln 3$

63. $\tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1-(1/2)}{1+(1/2)}\right) = -\frac{\ln 3}{2}$

64. $\coth^{-1}\left(\frac{5}{4}\right) = \frac{1}{2}\ln\left(\frac{(9/4)}{(1/4)}\right) = \frac{1}{2}\ln 9 = \ln 3$

65. $\operatorname{sech}^{-1}\left(\frac{3}{5}\right) = \ln\left(\frac{1+\sqrt{1-(9/25)}}{(3/5)}\right) = \ln 3$

66. $\operatorname{csch}^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \ln\left(-\sqrt{3} + \frac{\sqrt{4/3}}{\left(1/\sqrt{3}\right)}\right) = \ln\left(-\sqrt{3} + 2\right)$

67. (a) $\int_{0}^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1}\frac{x}{2}\right]_{0}^{2\sqrt{3}} = \sinh^{-1}\sqrt{3} - \sinh 0 = \sinh^{-1}\sqrt{3}$
(b) $\sinh^{-1}\sqrt{3} = \ln\left(\sqrt{3} + \sqrt{3+1}\right) = \ln\left(\sqrt{3} + 2\right)$

68. (a) $\int_{0}^{1/3} \frac{6\, dx}{\sqrt{1+9x^2}} = 2\int_{0}^{1} \frac{dx}{\sqrt{a^2+u^2}}$, where $u = 3x$, $du = 3\, dx$, $a = 1$
$= \left[2\sinh^{-1} u\right]_{0}^{1} = 2\left(\sinh^{-1} 1 - \sinh^{-1} 0\right) = 2\sinh^{-1} 1$
(b) $2\sinh^{-1} 1 = 2\ln\left(1 + \sqrt{1^2+1}\right) = 2\ln\left(1+\sqrt{2}\right)$

69. (a) $\int_{5/4}^{2} \frac{1}{1-x^2}\, dx = \left[\coth^{-1} x\right]_{5/4}^{2} = \coth^{-1} 2 - \coth^{-1}\frac{5}{4}$
(b) $\coth^{-1} 2 - \coth^{-1}\frac{5}{4} = \frac{1}{2}\left[\ln 3 - \ln\left(\frac{9/4}{1/4}\right)\right] = \frac{1}{2}\ln\frac{1}{3}$

70. (a) $\int_{0}^{1/2} \frac{1}{1-x^2}\, dx = \left[\tanh^{-1} x\right]_{0}^{1/2} = \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1}\frac{1}{2}$
(b) $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln\left(\frac{1+(1/2)}{1-(1/2)}\right) = \frac{1}{2}\ln 3$

71. (a) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, where $u = 4x$, $du = 4\, dx$, $a = 1$
$= \left[-\operatorname{sech}^{-1} u\right]_{4/5}^{12/13} = -\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5}$
(b) $-\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5} = -\ln\left(\frac{1+\sqrt{1-(12/13)^2}}{(12/13)}\right) + \ln\left(\frac{1+\sqrt{1-(4/5)^2}}{(4/5)}\right)$
$= -\ln\left(\frac{13+\sqrt{169-144}}{12}\right) + \ln\left(\frac{5+\sqrt{25-16}}{4}\right) = \ln\left(\frac{5+3}{4}\right) - \ln\left(\frac{13+5}{12}\right) = \ln 2 - \ln\frac{3}{2} = \ln\left(2\cdot\frac{2}{3}\right) = \ln\frac{4}{3}$

72. (a) $\int_{1}^{2} \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2}\operatorname{csch}^{-1}\left|\frac{x}{2}\right|\right]_{1}^{2} = -\frac{1}{2}\left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1}\frac{1}{2}\right) = \frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right)$
(b) $\frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right) = \frac{1}{2}\left[\ln\left(2 + \frac{\sqrt{5/4}}{(1/2)}\right) - \ln\left(1+\sqrt{2}\right)\right] = \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$

73. (a) $\int_{0}^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\, dx = \int_{0}^{0} \frac{1}{\sqrt{1+u^2}}\, du = \left[\sinh^{-1} u\right]_{0}^{0} = \sinh^{-1} 0 - \sinh^{-1} 0 = 0$, where $u = \sin x$, $du = \cos x\, dx$

(b) $\sinh^{-1} 0 - \sinh^{-1} 0 = \ln\left(0 + \sqrt{0+1}\right) - \ln\left(0 + \sqrt{0+1}\right) = 0$

74. (a) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where $u = \ln x$, $du = \frac{1}{x}\,dx$, $a = 1$

$= [\sinh^{-1} u]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$

(b) $\sinh^{-1} 1 - \sinh^{-1} 0 = \ln\left(1 + \sqrt{1^2+1}\right) - \ln\left(0 + \sqrt{0^2+1}\right) = \ln\left(1+\sqrt{2}\right)$

75. Let $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$. Then $E(x) + O(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = \frac{2f(x)}{2} = f(x)$. Also, $E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E(x)$ is even, and $O(-x) = \frac{f(-x)-f(-(-x))}{2} = -\frac{f(x)-f(-x)}{2} = -O(x)$ $\Rightarrow O(x)$ is odd. Consequently, $f(x)$ can be written as a sum of an even and an odd function. $f(x) = \frac{f(x)+f(-x)}{2}$ because $\frac{f(x)-f(-x)}{2} = 0$ if f is even and $f(x) = \frac{f(x)-f(-x)}{2}$ because $\frac{f(x)+f(-x)}{2} = 0$ if f is odd. Thus, if f is even $f(x) = \frac{2f(x)}{2} + 0$ and if f is odd, $f(x) = 0 + \frac{2f(x)}{2}$

76. $y = \sinh^{-1} x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2} \Rightarrow 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = e^{2y} - 1 \Rightarrow e^{2y} - 2xe^y - 1 = 0$ $\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^y = x + \sqrt{x^2+1} \Rightarrow \sinh^{-1} x = y = \ln\left(x + \sqrt{x^2+1}\right)$. Since $e^y > 0$, we cannot choose $e^y = x - \sqrt{x^2+1}$ because $x - \sqrt{x^2+1} < 0$.

77. (a) $v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}\,t\right) \Rightarrow \frac{dv}{dt} = \sqrt{\frac{mg}{k}}\left[\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\,t\right)\right]\left(\sqrt{\frac{gk}{m}}\right) = g\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\,t\right)$.

Thus $m\frac{dv}{dt} = mg\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\,t\right) = mg\left(1 - \tanh^2\left(\sqrt{\frac{gk}{m}}\,t\right)\right) = mg - kv^2$. Also, since $\tanh x = 0$ when $x = 0$, $v = 0$ when $t = 0$.

(b) $\lim_{t\to\infty} v = \lim_{t\to\infty} \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right) = \sqrt{\frac{mg}{k}}\lim_{t\to\infty}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right) = \sqrt{\frac{mg}{k}}(1) = \sqrt{\frac{mg}{k}}$

(c) $\sqrt{\frac{160}{0.005}} = \sqrt{\frac{160{,}000}{5}} = \frac{400}{\sqrt{5}} = 80\sqrt{5} \approx 178.89$ ft/sec

78. (a) $s(t) = a\cos kt + b\sin kt \Rightarrow \frac{ds}{dt} = -ak\sin kt + bk\cos kt \Rightarrow \frac{d^2s}{dt^2} = -ak^2\cos kt - bk^2\sin kt$ $= -k^2(a\cos kt + b\sin kt) = -k^2 s(t) \Rightarrow$ acceleration is proportional to s. The negative constant $-k^2$ implies that the acceleration is directed toward the origin.

(b) $s(t) = a\cosh kt + b\sinh kt \Rightarrow \frac{ds}{dt} = ak\sinh kt + bk\cosh kt \Rightarrow \frac{d^2s}{dt^2} = ak^2\cosh kt + bk^2\sinh kt$ $= k^2(a\cosh kt + b\sinh kt) = k^2 s(t) \Rightarrow$ acceleration is proportional to s. The positive constant k^2 implies that the acceleration is directed away from the origin.

79. $V = \pi\int_0^2 (\cosh^2 x - \sinh^2 x)\,dx = \pi\int_0^2 1\,dx = 2\pi$

80. $V = 2\pi\int_0^{\ln\sqrt{3}} \operatorname{sech}^2 x\,dx = 2\pi\,[\tanh x]_0^{\ln\sqrt{3}} = 2\pi\left[\frac{\sqrt{3} - \left(1/\sqrt{3}\right)}{\sqrt{3} + \left(1/\sqrt{3}\right)}\right] = \pi$

81. $y = \frac{1}{2}\cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow L = \int_0^{\ln\sqrt{5}} \sqrt{1 + (\sinh 2x)^2}\,dx = \int_0^{\ln\sqrt{5}} \cosh 2x\,dx = \left[\frac{1}{2}\sinh 2x\right]_0^{\ln\sqrt{5}}$ $= \left[\frac{1}{2}\left(\frac{e^{2x} - e^{-2x}}{2}\right)\right]_0^{\ln\sqrt{5}} = \frac{1}{4}\left(5 - \frac{1}{5}\right) = \frac{6}{5}$

82. (a) $\lim\limits_{x\to\infty} \tanh x = \lim\limits_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{\left(e^x - \frac{1}{e^x}\right)}{\left(e^x + \frac{1}{e^x}\right)} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{1 - \frac{1}{e^{2x}}}{1 + \frac{1}{e^{2x}}} = \frac{1-0}{1+0} = 1$

(b) $\lim\limits_{x\to-\infty} \tanh x = \lim\limits_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x\to-\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim\limits_{x\to-\infty} \frac{\left(e^x - \frac{1}{e^x}\right)}{\left(e^x + \frac{1}{e^x}\right)} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

(c) $\lim\limits_{x\to\infty} \sinh x = \lim\limits_{x\to\infty} \frac{e^x - e^{-x}}{2} = \lim\limits_{x\to\infty} \frac{e^x - \frac{1}{e^x}}{2} = \lim\limits_{x\to\infty} \left(\frac{e^x}{2} - \frac{1}{2e^x}\right) = \infty - 0 = \infty$

(d) $\lim\limits_{x\to-\infty} \sinh x = \lim\limits_{x\to-\infty} \frac{e^x - e^{-x}}{2} = \lim\limits_{x\to-\infty} \left(\frac{e^x}{2} - \frac{e^{-x}}{2}\right) = 0 - \infty = -\infty$

(e) $\lim\limits_{x\to\infty} \operatorname{sech} x = \lim\limits_{x\to\infty} \frac{2}{e^x + e^{-x}} = \lim\limits_{x\to\infty} \frac{2}{e^x + \frac{1}{e^x}} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{\frac{2}{e^x}}{1 + \frac{1}{e^{2x}}} = \frac{0}{1+0} = 0$

(f) $\lim\limits_{x\to\infty} \coth x = \lim\limits_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x\to\infty} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{\left(e^x + \frac{1}{e^x}\right)}{\left(e^x - \frac{1}{e^x}\right)} \cdot \frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim\limits_{x\to\infty} \frac{1 + \frac{1}{e^{2x}}}{1 - \frac{1}{e^{2x}}} = \frac{1+0}{1-0} = 1$

(g) $\lim\limits_{x\to 0^+} \coth x = \lim\limits_{x\to 0^+} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x\to 0^+} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to 0^+} \frac{e^{2x} + 1}{e^{2x} - 1} = +\infty$

(h) $\lim\limits_{x\to 0^-} \coth x = \lim\limits_{x\to 0^-} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim\limits_{x\to 0^-} \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to 0^-} \frac{e^{2x} + 1}{e^{2x} - 1} = -\infty$

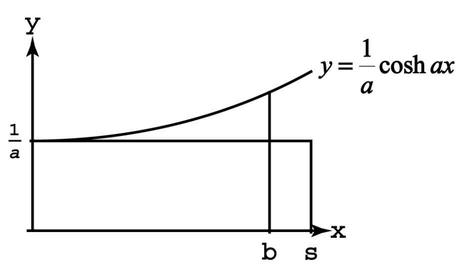
(i) $\lim\limits_{x\to-\infty} \operatorname{csch} x = \lim\limits_{x\to-\infty} \frac{2}{e^x - e^{-x}} = \lim\limits_{x\to-\infty} \frac{2}{e^x - \frac{1}{e^x}} \cdot \frac{e^x}{e^x} = \lim\limits_{x\to-\infty} \frac{2e^x}{e^{2x} - 1} = \frac{0}{0-1} = 0$

83. (a) $y = \frac{H}{w} \cosh\left(\frac{w}{H} x\right) \Rightarrow \tan\phi = \frac{dy}{dx} = \left(\frac{H}{w}\right)\left[\frac{w}{H} \sinh\left(\frac{w}{H} x\right)\right] = \sinh\left(\frac{w}{H} x\right)$

(b) The tension at P is given by $T \cos\phi = H \Rightarrow T = H \sec\phi = H\sqrt{1 + \tan^2\phi} = H\sqrt{1 + \left(\sinh \frac{w}{H} x\right)^2}$
$= H \cosh\left(\frac{w}{H} x\right) = w\left(\frac{H}{w}\right) \cosh\left(\frac{w}{H} x\right) = wy$

84. $s = \frac{1}{a} \sinh ax \Rightarrow \sinh ax = as \Rightarrow ax = \sinh^{-1} as \Rightarrow x = \frac{1}{a} \sinh^{-1} as$; $y = \frac{1}{a} \cosh ax = \frac{1}{a} \sqrt{\cosh^2 ax}$
$= \frac{1}{a} \sqrt{\sinh^2 ax + 1} = \frac{1}{a} \sqrt{a^2 s^2 + 1} = \sqrt{s^2 + \frac{1}{a^2}}$

85. To find the length of the curve: $y = \frac{1}{a} \cosh ax \Rightarrow y' = \sinh ax \Rightarrow L = \int_0^b \sqrt{1 + (\sinh ax)^2}\, dx$
$\Rightarrow L = \int_0^b \cosh ax\, dx = \left[\frac{1}{a} \sinh ax\right]_0^b = \frac{1}{a} \sinh ab$. The area under the curve is $A = \int_0^b \frac{1}{a} \cosh ax\, dx$
$= \left[\frac{1}{a^2} \sinh ax\right]_0^b = \frac{1}{a^2} \sinh ab = \left(\frac{1}{a}\right)\left(\frac{1}{a} \sinh ab\right)$ which is the area of the rectangle of height $\frac{1}{a}$ and length L as claimed, and which is illustrated below.



86. (a) Let the point located at $(\cosh u, 0)$ be called T. Then $A(u)$ = area of the triangle ΔOTP minus the area under the curve $y = \sqrt{x^2 - 1}$ from A to T $\Rightarrow A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx$.

(b) $A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx \Rightarrow A'(u) = \frac{1}{2}\left(\cosh^2 u + \sinh^2 u\right) - \left(\sqrt{\cosh^2 u - 1}\right)(\sinh u)$
$= \frac{1}{2} \cosh^2 u + \frac{1}{2} \sinh^2 u - \sinh^2 u = \frac{1}{2}\left(\cosh^2 u - \sinh^2 u\right) = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$

(c) $A'(u) = \frac{1}{2} \Rightarrow A(u) = \frac{u}{2} + C$, and from part (a) we have $A(0) = 0 \Rightarrow C = 0 \Rightarrow A(u) = \frac{u}{2} \Rightarrow u = 2A$

7.8 RELATIVE RATES OF GROWTH

1. (a) slower, $\lim\limits_{x\to\infty} \frac{x+3}{e^x} = \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$

(b) slower, $\lim\limits_{x\to\infty} \frac{x^3+\sin^2 x}{e^x} = \lim\limits_{x\to\infty} \frac{3x^2+2\sin x\cos x}{e^x} = \lim\limits_{x\to\infty} \frac{6x+2\cos 2x}{e^x} = \lim\limits_{x\to\infty} \frac{6-4\sin 2x}{e^x} = 0$ by the Sandwich Theorem because $\frac{2}{e^x} \le \frac{6-4\sin 2x}{e^x} \le \frac{10}{e^x}$ for all reals and $\lim\limits_{x\to\infty} \frac{2}{e^x} = 0 = \lim\limits_{x\to\infty} \frac{10}{e^x}$

(c) slower, $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim\limits_{x\to\infty} \frac{x^{1/2}}{e^x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{2}\right)x^{-1/2}}{e^x} = \lim\limits_{x\to\infty} \frac{1}{2\sqrt{x}\,e^x} = 0$

(d) faster, $\lim\limits_{x\to\infty} \frac{4^x}{e^x} = \lim\limits_{x\to\infty} \left(\frac{4}{e}\right)^x = \infty$ since $\frac{4}{e} > 1$

(e) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{3}{2}\right)^x}{e^x} = \lim\limits_{x\to\infty} \left(\frac{3}{2e}\right)^x = 0$ since $\frac{3}{2e} < 1$

(f) slower, $\lim\limits_{x\to\infty} \frac{e^{x/2}}{e^x} = \lim\limits_{x\to\infty} \frac{1}{e^{x/2}} = 0$

(g) same, $\lim\limits_{x\to\infty} \frac{\left(\frac{e^x}{2}\right)}{e^x} = \lim\limits_{x\to\infty} \frac{1}{2} = \frac{1}{2}$

(h) slower, $\lim\limits_{x\to\infty} \frac{\log_{10} x}{e^x} = \lim\limits_{x\to\infty} \frac{\ln x}{(\ln 10)\,e^x} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{(\ln 10)\,e^x} = \lim\limits_{x\to\infty} \frac{1}{(\ln 10)xe^x} = 0$

2. (a) slower, $\lim\limits_{x\to\infty} \frac{10x^4+30x+1}{e^x} = \lim\limits_{x\to\infty} \frac{40x^3+30}{e^x} = \lim\limits_{x\to\infty} \frac{120x^2}{e^x} = \lim\limits_{x\to\infty} \frac{240x}{e^x} = \lim\limits_{x\to\infty} \frac{240}{e^x} = 0$

(b) slower, $\lim\limits_{x\to\infty} \frac{x\ln x - x}{e^x} = \lim\limits_{x\to\infty} \frac{x(\ln x-1)}{e^x} = \lim\limits_{x\to\infty} \frac{\ln x - 1 + x\left(\frac{1}{x}\right)}{e^x} = \lim\limits_{x\to\infty} \frac{\ln x - 1 + 1}{e^x} = \lim\limits_{x\to\infty} \frac{\ln x}{e^x}$
$= \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{e^x} = \lim\limits_{x\to\infty} \frac{1}{xe^x} = 0$

(c) slower, $\lim\limits_{x\to\infty} \frac{\sqrt{1+x^4}}{e^x} = \sqrt{\lim\limits_{x\to\infty} \frac{1+x^4}{e^{2x}}} = \sqrt{\lim\limits_{x\to\infty} \frac{4x^3}{2e^{2x}}} = \sqrt{\lim\limits_{x\to\infty} \frac{12x^2}{4e^{2x}}} = \sqrt{\lim\limits_{x\to\infty} \frac{24x}{8e^{2x}}} = \sqrt{\lim\limits_{x\to\infty} \frac{24}{16e^{2x}}}$
$= \sqrt{0} = 0$

(d) slower, $\lim\limits_{x\to\infty} \frac{\left(\frac{5}{2}\right)^x}{e^x} = \lim\limits_{x\to\infty} \left(\frac{5}{2e}\right)^x = 0$ since $\frac{5}{2e} < 1$

(e) slower, $\lim\limits_{x\to\infty} \frac{e^{-x}}{e^x} = \lim\limits_{x\to\infty} \frac{1}{e^{2x}} = 0$

(f) faster, $\lim\limits_{x\to\infty} \frac{xe^x}{e^x} = \lim\limits_{x\to\infty} x = \infty$

(g) slower, since for all reals we have $-1 \le \cos x \le 1 \Rightarrow e^{-1} \le e^{\cos x} \le e^1 \Rightarrow \frac{e^{-1}}{e^x} \le \frac{e^{\cos x}}{e^x} \le \frac{e^1}{e^x}$ and also $\lim\limits_{x\to\infty} \frac{e^{-1}}{e^x} = 0 = \lim\limits_{x\to\infty} \frac{e^1}{e^x}$, so by the Sandwich Theorem we conclude that $\lim\limits_{x\to\infty} \frac{e^{\cos x}}{e^x} = 0$

(h) same, $\lim\limits_{x\to\infty} \frac{e^{x-1}}{e^x} = \lim\limits_{x\to\infty} \frac{1}{e^{(x-x+1)}} = \lim\limits_{x\to\infty} \frac{1}{e} = \frac{1}{e}$

3. (a) same, $\lim\limits_{x\to\infty} \frac{x^2+4x}{x^2} = \lim\limits_{x\to\infty} \frac{2x+4}{2x} = \lim\limits_{x\to\infty} \frac{2}{2} = 1$

(b) faster, $\lim\limits_{x\to\infty} \frac{x^5-x^2}{x^2} = \lim\limits_{x\to\infty} (x^3-1) = \infty$

(c) same, $\lim\limits_{x\to\infty} \frac{\sqrt{x^4+x^3}}{x^2} = \sqrt{\lim\limits_{x\to\infty} \frac{x^4+x^3}{x^4}} = \sqrt{\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)} = \sqrt{1} = 1$

(d) same, $\lim\limits_{x\to\infty} \frac{(x+3)^2}{x^2} = \lim\limits_{x\to\infty} \frac{2(x+3)}{2x} = \lim\limits_{x\to\infty} \frac{2}{2} = 1$

(e) slower, $\lim\limits_{x\to\infty} \frac{x\ln x}{x^2} = \lim\limits_{x\to\infty} \frac{\ln x}{x} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$

(f) faster, $\lim\limits_{x\to\infty} \frac{2^x}{x^2} = \lim\limits_{x\to\infty} \frac{(\ln 2)\,2^x}{2x} = \lim\limits_{x\to\infty} \frac{(\ln 2)^2\,2^x}{2} = \infty$

(g) slower, $\lim\limits_{x\to\infty} \frac{x^3e^{-x}}{x^2} = \lim\limits_{x\to\infty} \frac{x}{e^x} = \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$

(h) same, $\lim\limits_{x\to\infty} \frac{8x^2}{x^2} = \lim\limits_{x\to\infty} 8 = 8$

4. (a) same, $\lim\limits_{x\to\infty} \frac{x^2+\sqrt{x}}{x^2} = \lim\limits_{x\to\infty} \left(1+\frac{1}{x^{3/2}}\right) = 1$

(b) same, $\lim\limits_{x\to\infty} \frac{10x^2}{x^2} = \lim\limits_{x\to\infty} 10 = 10$

(c) slower, $\lim\limits_{x\to\infty} \frac{x^2e^{-x}}{x^2} = \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$

(d) slower, $\lim_{x \to \infty} \frac{\log_{10} x^2}{x^2} = \lim_{x \to \infty} \frac{\left(\frac{\ln x^2}{\ln 10}\right)}{x^2} = \frac{1}{\ln 10} \lim_{x \to \infty} \frac{2 \ln x}{x^2} = \frac{2}{\ln 10} \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{2x} = \frac{1}{\ln 10} \lim_{x \to \infty} \frac{1}{x^2} = 0$

(e) faster, $\lim_{x \to \infty} \frac{x^3 - x^2}{x^2} = \lim_{x \to \infty} (x - 1) = \infty$

(f) slower, $\lim_{x \to \infty} \frac{\left(\frac{1}{10}\right)^x}{x^2} = \lim_{x \to \infty} \frac{1}{10^x x^2} = 0$

(g) faster, $\lim_{x \to \infty} \frac{(1.1)^x}{x^2} = \lim_{x \to \infty} \frac{(\ln 1.1)(1.1)^x}{2x} = \lim_{x \to \infty} \frac{(\ln 1.1)^2 (1.1)^x}{2} = \infty$

(h) same, $\lim_{x \to \infty} \frac{x^2 + 100x}{x^2} = \lim_{x \to \infty} \left(1 + \frac{100}{x}\right) = 1$

5. (a) same, $\lim_{x \to \infty} \frac{\log_3 x}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{\ln x}{\ln 3}\right)}{\ln x} = \lim_{x \to \infty} \frac{1}{\ln 3} = \frac{1}{\ln 3}$

(b) same, $\lim_{x \to \infty} \frac{\ln 2x}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{2}{2x}\right)}{\left(\frac{1}{x}\right)} = 1$

(c) same, $\lim_{x \to \infty} \frac{\ln \sqrt{x}}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{1}{2}\right) \ln x}{\ln x} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2}$

(d) faster, $\lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \to \infty} \frac{x^{1/2}}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{1}{2}\right) x^{-1/2}}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{x}{2\sqrt{x}} = \lim_{x \to \infty} \frac{\sqrt{x}}{2} = \infty$

(e) faster, $\lim_{x \to \infty} \frac{x}{\ln x} = \lim_{x \to \infty} \frac{1}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} x = \infty$

(f) same, $\lim_{x \to \infty} \frac{5 \ln x}{\ln x} = \lim_{x \to \infty} 5 = 5$

(g) slower, $\lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\ln x} = \lim_{x \to \infty} \frac{1}{x \ln x} = 0$

(h) faster, $\lim_{x \to \infty} \frac{e^x}{\ln x} = \lim_{x \to \infty} \frac{e^x}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} x e^x = \infty$

6. (a) same, $\lim_{x \to \infty} \frac{\log_2 x^2}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{\ln x^2}{\ln 2}\right)}{\ln x} = \frac{1}{\ln 2} \lim_{x \to \infty} \frac{\ln x^2}{\ln x} = \frac{1}{\ln 2} \lim_{x \to \infty} \frac{2 \ln x}{\ln x} = \frac{1}{\ln 2} \lim_{x \to \infty} 2 = \frac{2}{\ln 2}$

(b) same, $\lim_{x \to \infty} \frac{\log_{10} 10x}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{\ln 10x}{\ln 10}\right)}{\ln x} = \frac{1}{\ln 10} \lim_{x \to \infty} \frac{\ln 10x}{\ln x} = \frac{1}{\ln 10} \lim_{x \to \infty} \frac{\left(\frac{10}{10x}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{\ln 10} \lim_{x \to \infty} 1 = \frac{1}{\ln 10}$

(c) slower, $\lim_{x \to \infty} \frac{\left(\frac{1}{\sqrt{x}}\right)}{\ln x} = \lim_{x \to \infty} \frac{1}{\left(\sqrt{x}\right)(\ln x)} = 0$

(d) slower, $\lim_{x \to \infty} \frac{\left(\frac{1}{x^2}\right)}{\ln x} = \lim_{x \to \infty} \frac{1}{x^2 \ln x} = 0$

(e) faster, $\lim_{x \to \infty} \frac{x - 2 \ln x}{\ln x} = \lim_{x \to \infty} \left(\frac{x}{\ln x} - 2\right) = \left(\lim_{x \to \infty} \frac{x}{\ln x}\right) - 2 = \left(\lim_{x \to \infty} \frac{1}{\left(\frac{1}{x}\right)}\right) - 2 = \left(\lim_{x \to \infty} x\right) - 2 = \infty$

(f) slower, $\lim_{x \to \infty} \frac{e^{-x}}{\ln x} = \lim_{x \to \infty} \frac{1}{e^x \ln x} = 0$

(g) slower, $\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{1/x}{\ln x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{1}{\ln x} = 0$

(h) same, $\lim_{x \to \infty} \frac{\ln(2x+5)}{\ln x} = \lim_{x \to \infty} \frac{\left(\frac{2}{2x+5}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{2x}{2x+5} = \lim_{x \to \infty} \frac{2}{2} = \lim_{x \to \infty} 1 = 1$

7. $\lim_{x \to \infty} \frac{e^x}{e^{x/2}} = \lim_{x \to \infty} e^{x/2} = \infty \Rightarrow e^x$ grows faster than $e^{x/2}$; since for $x > e^e$ we have $\ln x > e$ and $\lim_{x \to \infty} \frac{(\ln x)^x}{e^x} = \lim_{x \to \infty} \left(\frac{\ln x}{e}\right)^x = \infty \Rightarrow (\ln x)^x$ grows faster than e^x; since $x > \ln x$ for all $x > 0$ and $\lim_{x \to \infty} \frac{x^x}{(\ln x)^x} = \lim_{x \to \infty} \left(\frac{x}{\ln x}\right)^x = \infty \Rightarrow x^x$ grows faster than $(\ln x)^x$. Therefore, slowest to fastest are: $e^{x/2}$, e^x, $(\ln x)^x$, x^x so the order is d, a, c, b

8. $\lim_{x \to \infty} \frac{(\ln 2)^x}{x^2} = \lim_{x \to \infty} \frac{(\ln(\ln 2))(\ln 2)^x}{2x} = \lim_{x \to \infty} \frac{(\ln(\ln 2))^2 (\ln 2)^x}{2} = \frac{(\ln(\ln 2))^2}{2} \lim_{x \to \infty} (\ln 2)^x = 0$ $\Rightarrow (\ln 2)^x$ grows slower than x^2; $\lim_{x \to \infty} \frac{x^2}{2^x} = \lim_{x \to \infty} \frac{2x}{(\ln 2) 2^x} = \lim_{x \to \infty} \frac{2}{(\ln 2)^2 2^x} = 0 \Rightarrow x^2$ grows slower than 2^x; $\lim_{x \to \infty} \frac{2^x}{e^x} = \lim_{x \to \infty} \left(\frac{2}{e}\right)^x = 0 \Rightarrow 2^x$ grows slower than e^x. Therefore, the slowest to the fastest is: $(\ln 2)^x$, x^2, 2^x and e^x so the order is c, b, a, d

9. (a) false; $\lim\limits_{x \to \infty} \frac{x}{x} = 1$

   (b) false; $\lim\limits_{x \to \infty} \frac{x}{x+5} = \frac{1}{1} = 1$

   (c) true; $x < x + 5 \Rightarrow \frac{x}{x+5} < 1$ if $x > 1$ (or sufficiently large)

   (d) true; $x < 2x \Rightarrow \frac{x}{2x} < 1$ if $x > 1$ (or sufficiently large)

   (e) true; $\lim\limits_{x \to \infty} \frac{e^x}{e^{2x}} = \lim\limits_{x \to 0} \frac{1}{e^x} = 0$

   (f) true; $\frac{x + \ln x}{x} = 1 + \frac{\ln x}{x} < 1 + \frac{\sqrt{x}}{x} = 1 + \frac{1}{\sqrt{x}} < 2$ if $x > 1$ (or sufficiently large)

   (g) false; $\lim\limits_{x \to \infty} \frac{\ln x}{\ln 2x} = \lim\limits_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2}{2x}\right)} = \lim\limits_{x \to \infty} 1 = 1$

   (h) true; $\frac{\sqrt{x^2+5}}{x} < \frac{\sqrt{(x+5)^2}}{x} < \frac{x+5}{x} = 1 + \frac{5}{x} < 6$ if $x > 1$ (or sufficiently large)

10. (a) true; $\frac{\left(\frac{1}{x+3}\right)}{\left(\frac{1}{x}\right)} = \frac{x}{x+3} < 1$ if $x > 1$ (or sufficiently large)

   (b) true; $\frac{\left(\frac{1}{x} + \frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = 1 + \frac{1}{x} < 2$ if $x > 1$ (or sufficiently large)

   (c) false; $\lim\limits_{x \to \infty} \frac{\left(\frac{1}{x} - \frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \left(1 - \frac{1}{x}\right) = 1$

   (d) true; $2 + \cos x \le 3 \Rightarrow \frac{2 + \cos x}{2} \le \frac{3}{2}$ if x is sufficiently large

   (e) true; $\frac{e^x + x}{e^x} = 1 + \frac{x}{e^x}$ and $\frac{x}{e^x} \to 0$ as $x \to \infty \Rightarrow 1 + \frac{x}{e^x} < 2$ if x is sufficiently large

   (f) true; $\lim\limits_{x \to \infty} \frac{x \ln x}{x^2} = \lim\limits_{x \to \infty} \frac{\ln x}{x} = \lim\limits_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$

   (g) true; $\frac{\ln(\ln x)}{\ln x} < \frac{\ln x}{\ln x} = 1$ if x is sufficiently large

   (h) false; $\lim\limits_{x \to \infty} \frac{\ln x}{\ln(x^2+1)} = \lim\limits_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{2x}{x^2+1}\right)} = \lim\limits_{x \to \infty} \frac{x^2+1}{2x^2} = \lim\limits_{x \to \infty} \left(\frac{1}{2} + \frac{1}{2x^2}\right) = \frac{1}{2}$

11. If f(x) and g(x) grow at the same rate, then $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0 \Rightarrow \lim\limits_{x \to \infty} \frac{g(x)}{f(x)} = \frac{1}{L} \neq 0$. Then $\left|\frac{f(x)}{g(x)} - L\right| < 1$ if x is sufficiently large $\Rightarrow L - 1 < \frac{f(x)}{g(x)} < L + 1 \Rightarrow \frac{f(x)}{g(x)} \le |L| + 1$ if x is sufficiently large $\Rightarrow f = O(g)$. Similarly, $\frac{g(x)}{f(x)} \le \left|\frac{1}{L}\right| + 1 \Rightarrow g = O(f)$.

12. When the degree of f is less than the degree of g since in that case $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

13. When the degree of f is less than or equal to the degree of g since $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 0$ when the degree of f is smaller than the degree of g, and $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ (the ratio of the leading coefficients) when the degrees are the same.

14. Polynomials of a greater degree grow at a greater rate than polynomials of a lesser degree. Polynomials of the same degree grow at the same rate.

15. $\lim\limits_{x \to \infty} \frac{\ln(x+1)}{\ln x} = \lim\limits_{x \to \infty} \frac{\left(\frac{1}{x+1}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \frac{x}{x+1} = \lim\limits_{x \to \infty} \frac{1}{1} = 1$ and $\lim\limits_{x \to \infty} \frac{\ln(x+999)}{\ln x} = \lim\limits_{x \to \infty} \frac{\left(\frac{1}{x+999}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \frac{x}{x+999} = 1$

16. $\lim\limits_{x \to \infty} \frac{\ln(x+a)}{\ln x} = \lim\limits_{x \to \infty} \frac{\left(\frac{1}{x+a}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \frac{x}{x+a} = \lim\limits_{x \to \infty} \frac{1}{1} = 1$. Therefore, the relative rates are the same.

17. $\lim\limits_{x \to \infty} \frac{\sqrt{10x+1}}{\sqrt{x}} = \sqrt{\lim\limits_{x \to \infty} \frac{10x+1}{x}} = \sqrt{10}$ and $\lim\limits_{x \to \infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \sqrt{\lim\limits_{x \to \infty} \frac{x+1}{x}} = \sqrt{1} = 1$. Since the growth rate is transitive, we conclude that $\sqrt{10x+1}$ and $\sqrt{x+1}$ have the same growth rate $\left(\text{that of } \sqrt{x}\right)$.

18. $\lim\limits_{x \to \infty} \frac{\sqrt{x^4+x}}{x^2} = \sqrt{\lim\limits_{x \to \infty} \frac{x^4+x}{x^4}} = 1$ and $\lim\limits_{x \to \infty} \frac{\sqrt{x^4-x^3}}{x^2} = \sqrt{\lim\limits_{x \to \infty} \frac{x^4-x^3}{x^4}} = 1$. Since the growth rate is transitive, we conclude that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ have the same growth rate (that of x^2).

19. $\lim\limits_{x \to \infty} \frac{x^n}{e^x} = \lim\limits_{x \to \infty} \frac{nx^{n-1}}{e^x} = \ldots = \lim\limits_{x \to \infty} \frac{n!}{e^x} = 0 \Rightarrow x^n = o\left(e^x\right)$ for any non-negative integer n

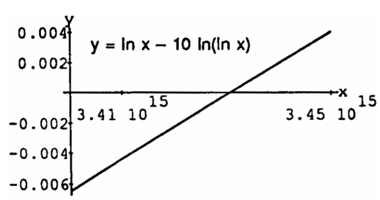
20. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then $\lim\limits_{x \to \infty} \frac{p(x)}{e^x} = a_n \lim\limits_{x \to \infty} \frac{x^n}{e^x} + a_{n-1} \lim\limits_{x \to \infty} \frac{x^{n-1}}{e^x} + \ldots$ $+ a_1 \lim\limits_{x \to \infty} \frac{x}{e^x} + a_0 \lim\limits_{x \to \infty} \frac{1}{e^x}$ where each limit is zero (from Exercise 19). Therefore, $\lim\limits_{x \to \infty} \frac{p(x)}{e^x} = 0$ $\Rightarrow$ e^x grows faster than any polynomial.

21. (a) $\lim\limits_{x \to \infty} \frac{x^{1/n}}{\ln x} = \lim\limits_{x \to \infty} \frac{x^{(1-n)/n}}{n\left(\frac{1}{x}\right)} = \left(\frac{1}{n}\right) \lim\limits_{x \to \infty} x^{1/n} = \infty \Rightarrow \ln x = o\left(x^{1/n}\right)$ for any positive integer n

(b) $\ln\left(e^{17{,}000{,}000}\right) = 17{,}000{,}000 < \left(e^{17\times 10^6}\right)^{1/10^6} = e^{17} \approx 24{,}154{,}952.75$

(c) $x \approx 3.430631121 \times 10^{15}$

(d) In the interval $[3.41 \times 10^{15}, 3.45 \times 10^{15}]$ we have $\ln x = 10 \ln(\ln x)$. The graphs cross at about 3.4306311×10^{15}.



22. $\lim\limits_{x \to \infty} \frac{\ln x}{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0} = \frac{\lim\limits_{x\to\infty}\left(\frac{\ln x}{x^n}\right)}{\lim\limits_{x\to\infty}\left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)} = \frac{\lim\limits_{x\to\infty}\left[\frac{1/x}{nx^{n-1}}\right]}{a_n} = \lim\limits_{x \to \infty} \frac{1}{(a_n)(nx^n)} = 0$

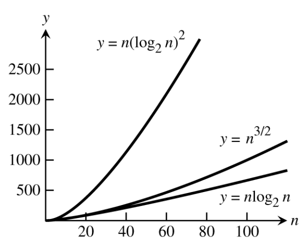
$\Rightarrow \ln x$ grows slower than any non-constant polynomial $(n \geq 1)$

23. (a) $\lim\limits_{n \to \infty} \frac{n \log_2 n}{n(\log_2 n)^2} = \lim\limits_{n \to \infty} \frac{1}{\log_2 n} = 0 \Rightarrow n \log_2 n$ grows slower than $n(\log_2 n)^2$; $\lim\limits_{n \to \infty} \frac{n \log_2 n}{n^{3/2}} = \lim\limits_{n \to \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}}$

$= \frac{1}{\ln 2} \lim\limits_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2}\right)n^{-1/2}} = \frac{2}{\ln 2} \lim\limits_{n \to \infty} \frac{1}{n^{1/2}} = 0$

$\Rightarrow n \log_2 n$ grows slower than $n^{3/2}$. Therefore, $n \log_2 n$ grows at the slowest rate $\Rightarrow$ the algorithm that takes $O(n \log_2 n)$ steps is the most efficient in the long run.

(b)



24. (a) $\lim\limits_{n \to \infty} \frac{(\log_2 n)^2}{n} = \lim\limits_{n \to \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)^2}{n} = \lim\limits_{n \to \infty} \frac{(\ln n)^2}{n(\ln 2)^2}$

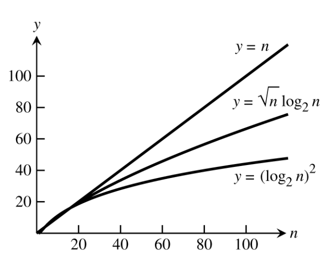
$= \lim\limits_{n \to \infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{(\ln 2)^2} = \frac{2}{(\ln 2)^2} \lim\limits_{n \to \infty} \frac{\ln n}{n}$

$= \frac{2}{(\ln 2)^2} \lim\limits_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 \Rightarrow (\log_2 n)^2$ grows slower than n; $\lim\limits_{n \to \infty} \frac{(\log_2 n)^2}{\sqrt{n}\log_2 n} = \lim\limits_{n \to \infty} \frac{\log_2 n}{\sqrt{n}}$

$= \lim\limits_{n \to \infty} \frac{\left(\frac{\ln n}{\ln 2}\right)}{n^{1/2}} = \frac{1}{\ln 2} \lim\limits_{n \to \infty} \frac{\ln n}{n^{1/2}}$

$= \frac{1}{\ln 2} \lim\limits_{x \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2}\right)n^{-1/2}} = \frac{2}{\ln 2} \lim\limits_{n \to \infty} \frac{1}{n^{1/2}} = 0 \Rightarrow (\log_2 n)^2$ grows slower than $\sqrt{n}\log_2 n$. Therefore $(\log_2 n)^2$ grows at the slowest rate $\Rightarrow$ the algorithm that takes $O\left((\log_2 n)^2\right)$ steps is the most efficient in the long run.

(b)



25. It could take one million steps for a sequential search, but at most 20 steps for a binary search because $2^{19} = 524{,}288 < 1{,}000{,}000 < 1{,}048{,}576 = 2^{20}$.

26. It could take 450,000 steps for a sequential search, but at most 19 steps for a binary search because $2^{18} = 262{,}144 < 450{,}000 < 524{,}288 = 2^{19}$.

CHAPTER 7 PRACTICE EXERCISES

1. $y = 10e^{-x/5} \Rightarrow \frac{dy}{dx} = (10)\left(-\frac{1}{5}\right)e^{-x/5} = -2e^{-x/5}$

2. $y = \sqrt{2}\,e^{\sqrt{2}x} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(\sqrt{2}\right)e^{\sqrt{2}x} = 2e^{\sqrt{2}x}$

3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left[x\left(4e^{4x}\right) + e^{4x}(1)\right] - \frac{1}{16}\left(4e^{4x}\right) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$

4. $y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2\left[\left(2x^{-2}\right)e^{-2x^{-1}}\right] + e^{-2x^{-1}}(2x) = (2+2x)e^{-2x^{-1}} = 2e^{-2/x}(1+x)$

5. $y = \ln\left(\sin^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin\theta)(\cos\theta)}{\sin^2\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

6. $y = \ln\left(\sec^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sec\theta)(\sec\theta\tan\theta)}{\sec^2\theta} = 2\tan\theta$

7. $y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$

8. $y = \log_5(3x-7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$

9. $y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8)$

10. $y = 9^{2t} \Rightarrow \frac{dy}{dt} = 9^{2t}(\ln 9)(2) = 9^{2t}(2\ln 9)$

11. $y = 5x^{3.6} \Rightarrow \frac{dy}{dx} = 5(3.6)x^{2.6} = 18x^{2.6}$

12. $y = \sqrt{2}\,x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(-\sqrt{2}\right)x^{\left(-\sqrt{2}-1\right)} = -2x^{\left(-\sqrt{2}-1\right)}$

13. $y = (x+2)^{x+2} \Rightarrow \ln y = \ln(x+2)^{x+2} = (x+2)\ln(x+2) \Rightarrow \frac{y'}{y} = (x+2)\left(\frac{1}{x+2}\right) + (1)\ln(x+2)$
$\Rightarrow \frac{dy}{dx} = (x+2)^{x+2}\left[\ln(x+2)+1\right]$

14. $y = 2(\ln x)^{x/2} \Rightarrow \ln y = \ln\left[2(\ln x)^{x/2}\right] = \ln(2) + \left(\frac{x}{2}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = 0 + \left(\frac{x}{2}\right)\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right] + (\ln(\ln x))\left(\frac{1}{2}\right)$
$\Rightarrow y' = \left[\frac{1}{2\ln x} + \left(\frac{1}{2}\right)\ln(\ln x)\right]2(\ln x)^{x/2} = (\ln x)^{x/2}\left[\ln(\ln x) + \frac{1}{\ln x}\right]$

15. $y = \sin^{-1}\sqrt{1-u^2} = \sin^{-1}\left(1-u^2\right)^{1/2} \Rightarrow \frac{dy}{du} = \frac{\frac{1}{2}\left(1-u^2\right)^{-1/2}(-2u)}{\sqrt{1-\left[\left(1-u^2\right)^{1/2}\right]^2}} = \frac{-u}{\sqrt{1-u^2}\sqrt{1-\left(1-u^2\right)}} = \frac{-u}{|u|\sqrt{1-u^2}}$
$= \frac{-u}{u\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-u^2}},\ 0 < u < 1$

16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right) = \sin^{-1}v^{-1/2} \Rightarrow \frac{dy}{dv} = \frac{-\frac{1}{2}v^{-3/2}}{\sqrt{1-\left(v^{-1/2}\right)^2}} = \frac{-1}{2v^{3/2}\sqrt{1-v^{-1}}} = \frac{-1}{2v^{3/2}\sqrt{\frac{v-1}{v}}} = \frac{-\sqrt{v}}{2v^{3/2}\sqrt{v-1}} = \frac{-1}{2v\sqrt{v-1}}$

17. $y = \ln\left(\cos^{-1}x\right) \Rightarrow y' = \frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\cos^{-1}x} = \frac{-1}{\sqrt{1-x^2}\cos^{-1}x}$

18. $y = z\cos^{-1} z - \sqrt{1-z^2} = z\cos^{-1} z - (1-z^2)^{1/2} \Rightarrow \frac{dy}{dz} = \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} - \left(\frac{1}{2}\right)(1-z^2)^{-1/2}(-2z)$
$= \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}} = \cos^{-1} z$

19. $y = t\tan^{-1} t - \left(\frac{1}{2}\right)\ln t \Rightarrow \frac{dy}{dt} = \tan^{-1} t + t\left(\frac{1}{1+t^2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{t}\right) = \tan^{-1} t + \frac{t}{1+t^2} - \frac{1}{2t}$

20. $y = (1+t^2)\cot^{-1} 2t \Rightarrow \frac{dy}{dt} = 2t\cot^{-1} 2t + (1+t^2)\left(\frac{-2}{1+4t^2}\right)$

21. $y = z\sec^{-1} z - \sqrt{z^2-1} = z\sec^{-1} z - (z^2-1)^{1/2} \Rightarrow \frac{dy}{dz} = z\left(\frac{1}{|z|\sqrt{z^2-1}}\right) + (\sec^{-1} z)(1) - \frac{1}{2}(z^2-1)^{-1/2}(2z)$
$= \frac{z}{|z|\sqrt{z^2-1}} - \frac{z}{\sqrt{z^2-1}} + \sec^{-1} z = \frac{1-z}{\sqrt{z^2-1}} + \sec^{-1} z,\ z > 1$

22. $y = 2\sqrt{x-1}\sec^{-1}\sqrt{x} = 2(x-1)^{1/2}\sec^{-1}\left(x^{1/2}\right)$
$\Rightarrow \frac{dy}{dx} = 2\left[\left(\frac{1}{2}\right)(x-1)^{-1/2}\sec^{-1}\left(x^{1/2}\right) + (x-1)^{1/2}\left(\frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{x}\sqrt{x-1}}\right)\right] = 2\left(\frac{\sec^{-1}\sqrt{x}}{2\sqrt{x-1}} + \frac{1}{2x}\right) = \frac{\sec^{-1}\sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$

23. $y = \csc^{-1}(\sec\theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec\theta\tan\theta}{|\sec\theta|\sqrt{\sec^2\theta - 1}} = -\frac{\tan\theta}{|\tan\theta|} = -1,\ 0 < \theta < \frac{\pi}{2}$

24. $y = (1+x^2)e^{\tan^{-1} x} \Rightarrow y' = 2xe^{\tan^{-1} x} + (1+x^2)\left(\frac{e^{\tan^{-1} x}}{1+x^2}\right) = 2xe^{\tan^{-1} x} + e^{\tan^{-1} x}$

25. $y = \frac{2(x^2+1)}{\sqrt{\cos 2x}} \Rightarrow \ln y = \ln\left(\frac{2(x^2+1)}{\sqrt{\cos 2x}}\right) = \ln(2) + \ln(x^2+1) - \frac{1}{2}\ln(\cos 2x) \Rightarrow \frac{y'}{y} = 0 + \frac{2x}{x^2+1} - \left(\frac{1}{2}\right)\frac{(-2\sin 2x)}{\cos 2x}$
$\Rightarrow y' = \left(\frac{2x}{x^2+1} + \tan 2x\right)y = \frac{2(x^2+1)}{\sqrt{\cos 2x}}\left(\frac{2x}{x^2+1} + \tan 2x\right)$

26. $y = \sqrt[10]{\frac{3x+4}{2x-4}} \Rightarrow \ln y = \ln\sqrt[10]{\frac{3x+4}{2x-4}} = \frac{1}{10}\left[\ln(3x+4) - \ln(2x-4)\right] \Rightarrow \frac{y'}{y} = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{2}{2x-4}\right)$
$\Rightarrow y' = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)y = \sqrt[10]{\frac{3x+4}{2x-4}}\left(\frac{1}{10}\right)\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)$

27. $y = \left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5 \Rightarrow \ln y = 5\left[\ln(t+1) + \ln(t-1) - \ln(t-2) - \ln(t+3)\right] \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{dt}\right)$
$= 5\left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3}\right) \Rightarrow \frac{dy}{dt} = 5\left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5\left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3}\right)$

28. $y = \frac{2u2^u}{\sqrt{u^2+1}} \Rightarrow \ln y = \ln 2 + \ln u + u\ln 2 - \frac{1}{2}\ln(u^2+1) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{du}\right) = \frac{1}{u} + \ln 2 - \frac{1}{2}\left(\frac{2u}{u^2+1}\right)$
$\Rightarrow \frac{dy}{du} = \frac{2u2^u}{\sqrt{u^2+1}}\left(\frac{1}{u} + \ln 2 - \frac{u}{u^2+1}\right)$

29. $y = (\sin\theta)^{\sqrt{\theta}} \Rightarrow \ln y = \sqrt{\theta}\ln(\sin\theta) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{d\theta}\right) = \sqrt{\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{1}{2}\theta^{-1/2}\ln(\sin\theta)$
$\Rightarrow \frac{dy}{d\theta} = (\sin\theta)^{\sqrt{\theta}}\left(\sqrt{\theta}\cot\theta + \frac{\ln(\sin\theta)}{2\sqrt{\theta}}\right)$

30. $y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x}\right)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) + \ln(\ln x)\left[\frac{-1}{(\ln x)^2}\right]\left(\frac{1}{x}\right)$
$\Rightarrow y' = (\ln x)^{1/\ln x}\left[\frac{1-\ln(\ln x)}{x(\ln x)^2}\right]$

31. $\int e^x\sin(e^x)\,dx = \int \sin u\,du$, where $u = e^x$ and $du = e^x\,dx$
$= -\cos u + C = -\cos(e^x) + C$

32. $\int e^t \cos(3e^t - 2)\,dt = \frac{1}{3}\int \cos u\,du$, where $u = 3e^t - 2$ and $du = 3e^t\,dt$
$= \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3e^t - 2) + C$

33. $\int e^x \sec^2(e^x - 7)\,dx = \int \sec^2 u\,du$, where $u = e^x - 7$ and $du = e^x\,dx$
$= \tan u + C = \tan(e^x - 7) + C$

34. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\,dy = \int \csc u \cot u\,du$, where $u = e^y + 1$ and $du = e^y\,dy$
$= -\csc u + C = -\csc(e^y + 1) + C$

35. $\int (\sec^2 x)\,e^{\tan x}\,dx = \int e^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$
$= e^u + C = e^{\tan x} + C$

36. $\int (\csc^2 x)\,e^{\cot x}\,dx = -\int e^u\,du$, where $u = \cot x$ and $du = -\csc^2 x\,dx$
$= -e^u + C = -e^{\cot x} + C$

37. $\int_{-1}^{1} \frac{1}{3x-4}\,dx = \frac{1}{3}\int_{-7}^{-1} \frac{1}{u}\,du$, where $u = 3x - 4$, $du = 3\,dx$; $x = -1 \Rightarrow u = -7$, $x = 1 \Rightarrow u = -1$
$= \frac{1}{3}\left[\ln|u|\right]_{-7}^{-1} = \frac{1}{3}\left[\ln|-1| - \ln|-7|\right] = \frac{1}{3}\left[0 - \ln 7\right] = -\frac{\ln 7}{3}$

38. $\int_1^e \frac{\sqrt{\ln x}}{x}\,dx = \int_0^1 u^{1/2}\,du$, where $u = \ln x$, $du = \frac{1}{x}\,dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$
$= \left[\frac{2}{3}u^{3/2}\right]_0^1 = \left[\frac{2}{3}1^{3/2} - \frac{2}{3}0^{3/2}\right] = \frac{2}{3}$

39. $\int_0^{\pi} \tan\left(\frac{x}{3}\right)dx = \int_0^{\pi} \frac{\sin\left(\frac{x}{3}\right)}{\cos\left(\frac{x}{3}\right)}\,dx = -3\int_1^{1/2} \frac{1}{u}\,du$, where $u = \cos\left(\frac{x}{3}\right)$, $du = -\frac{1}{3}\sin\left(\frac{x}{3}\right)dx$; $x = 0 \Rightarrow u = 1$, $x = \pi$
$\Rightarrow u = \frac{1}{2}$
$= -3\left[\ln|u|\right]_1^{1/2} = -3\left[\ln\left|\frac{1}{2}\right| - \ln|1|\right] = -3\ln\frac{1}{2} = \ln 2^3 = \ln 8$

40. $\int_{1/6}^{1/4} 2\cot \pi x\,dx = 2\int_{1/6}^{1/4} \frac{\cos \pi x}{\sin \pi x}\,dx = \frac{2}{\pi}\int_{1/2}^{1/\sqrt{2}} \frac{1}{u}\,du$, where $u = \sin \pi x$, $du = \pi \cos \pi x\,dx$; $x = \frac{1}{6} \Rightarrow u = \frac{1}{2}$, $x = \frac{1}{4}$
$\Rightarrow u = \frac{1}{\sqrt{2}}$
$= \frac{2}{\pi}\left[\ln|u|\right]_{1/2}^{1/\sqrt{2}} = \frac{2}{\pi}\left[\ln\left|\frac{1}{\sqrt{2}}\right| - \ln\left|\frac{1}{2}\right|\right] = \frac{2}{\pi}\left[\ln 1 - \frac{1}{2}\ln 2 - \ln 1 + \ln 2\right] = \frac{2}{\pi}\left[\frac{1}{2}\ln 2\right] = \frac{\ln 2}{\pi}$

41. $\int_0^4 \frac{2t}{t^2 - 25}\,dt = \int_{-25}^{-9} \frac{1}{u}\,du$, where $u = t^2 - 25$, $du = 2t\,dt$; $t = 0 \Rightarrow u = -25$, $t = 4 \Rightarrow u = -9$
$= \left[\ln|u|\right]_{-25}^{-9} = \ln|-9| - \ln|-25| = \ln 9 - \ln 25 = \ln\frac{9}{25}$

42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\,dt = -\int_2^{1/2} \frac{1}{u}\,du$, where $u = 1 - \sin t$, $du = -\cos t\,dt$; $t = -\frac{\pi}{2} \Rightarrow u = 2$, $t = \frac{\pi}{6} \Rightarrow u = \frac{1}{2}$
$= -\left[\ln|u|\right]_2^{1/2} = -\left[\ln\left|\frac{1}{2}\right| - \ln|2|\right] = -\ln 1 + \ln 2 + \ln 2 = 2\ln 2 = \ln 4$

43. $\int \frac{\tan(\ln v)}{v}\,dv = \int \tan u\,du = \int \frac{\sin u}{\cos u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$
$= -\ln|\cos u| + C = -\ln|\cos(\ln v)| + C$

44. $\int \frac{1}{v \ln v}\,dv = \int \frac{1}{u}\,du$, where $u = \ln v$ and $du = \frac{1}{v}\,dv$
$= \ln|u| + C = \ln|\ln v| + C$

45. $\int \frac{(\ln x)^{-3}}{x}\,dx = \int u^{-3}\,du$, where $u = \ln x$ and $du = \frac{1}{x}\,dx$
$= \frac{u^{-2}}{-2} + C = -\frac{1}{2}(\ln x)^{-2} + C$

46. $\int \frac{\ln(x-5)}{x-5}\,dx = \int u\,du$, where $u = \ln(x-5)$ and $du = \frac{1}{x-5}\,dx$
$= \frac{u^2}{2} + C = \frac{[\ln(x-5)]^2}{2} + C$

47. $\int \frac{1}{r}\csc^2(1+\ln r)\,dr = \int \csc^2 u\,du$, where $u = 1+\ln r$ and $du = \frac{1}{r}\,dr$
$= -\cot u + C = -\cot(1+\ln r) + C$

48. $\int \frac{\cos(1-\ln v)}{v}\,dv = -\int \cos u\,du$, where $u = 1-\ln v$ and $du = -\frac{1}{v}\,dv$
$= -\sin u + C = -\sin(1-\ln v) + C$

49. $\int x3^{x^2}\,dx = \frac{1}{2}\int 3^u\,du$, where $u = x^2$ and $du = 2x\,dx$
$= \frac{1}{2\ln 3}(3^u) + C = \frac{1}{2\ln 3}\left(3^{x^2}\right) + C$

50. $\int 2^{\tan x}\sec^2 x\,dx = \int 2^u\,du$, where $u = \tan x$ and $du = \sec^2 x\,dx$
$= \frac{1}{\ln 2}(2^u) + C = \frac{2^{\tan x}}{\ln 2} + C$

51. $\int_1^7 \frac{3}{x}\,dx = 3\int_1^7 \frac{1}{x}\,dx = 3[\ln|x|]_1^7 = 3(\ln 7 - \ln 1) = 3\ln 7$

52. $\int_1^{32} \frac{1}{5x}\,dx = \frac{1}{5}\int_1^{32}\frac{1}{x}\,dx = \frac{1}{5}[\ln|x|]_1^{32} = \frac{1}{5}(\ln 32 - \ln 1) = \frac{1}{5}\ln 32 = \ln\left(\sqrt[5]{32}\right) = \ln 2$

53. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x}\right)dx = \frac{1}{2}\int_1^4\left(\frac{1}{4}x + \frac{1}{x}\right)dx = \frac{1}{2}\left[\frac{1}{8}x^2 + \ln|x|\right]_1^4 = \frac{1}{2}\left[\left(\frac{16}{8} + \ln 4\right) - \left(\frac{1}{8} + \ln 1\right)\right] = \frac{15}{16} + \frac{1}{2}\ln 4$
$= \frac{15}{16} + \ln\sqrt{4} = \frac{15}{16} + \ln 2$

54. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2}\right)dx = \frac{2}{3}\int_1^8\left(\frac{1}{x} - 12x^{-2}\right)dx = \frac{2}{3}\left[\ln|x| + 12x^{-1}\right]_1^8 = \frac{2}{3}\left[\left(\ln 8 + \frac{12}{8}\right) - (\ln 1 + 12)\right]$
$= \frac{2}{3}\left(\ln 8 + \frac{3}{2} - 12\right) = \frac{2}{3}\left(\ln 8 - \frac{21}{2}\right) = \frac{2}{3}(\ln 8) - 7 = \ln\left(8^{2/3}\right) - 7 = \ln 4 - 7$

55. $\int_{-2}^{-1} e^{-(x+1)}\,dx = -\int_1^0 e^u\,du$, where $u = -(x+1)$, $du = -dx$; $x = -2 \Rightarrow u = 1$, $x = -1 \Rightarrow u = 0$
$= -[e^u]_1^0 = -(e^0 - e^1) = e - 1$

56. $\int_{-\ln 2}^{0} e^{2w}\,dw = \frac{1}{2}\int_{\ln(1/4)}^{0} e^u\,du$, where $u = 2w$, $du = 2\,dw$; $w = -\ln 2 \Rightarrow u = \ln\frac{1}{4}$, $w = 0 \Rightarrow u = 0$
$= \frac{1}{2}[e^u]_{\ln(1/4)}^{0} = \frac{1}{2}\left[e^0 - e^{\ln(1/4)}\right] = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$

57. $\int_1^{\ln 5} e^r(3e^r+1)^{-3/2}\,dr = \frac{1}{3}\int_4^{16} u^{-3/2}\,du$, where $u = 3e^r + 1$, $du = 3e^r dr$; $r = 0 \Rightarrow u = 4$, $r = \ln 5 \Rightarrow u = 16$
$= -\frac{2}{3}\left[u^{-1/2}\right]_4^{16} = -\frac{2}{3}\left(16^{-1/2} - 4^{-1/2}\right) = \left(-\frac{2}{3}\right)\left(\frac{1}{4} - \frac{1}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{1}{4}\right) = \frac{1}{6}$

58. $\int_0^{\ln 9} e^\theta(e^\theta - 1)^{1/2}\,d\theta = \int_0^8 u^{1/2}\,du$, where $u = e^\theta - 1$, $du = e^\theta\,d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \ln 9 \Rightarrow u = 8$
$= \frac{2}{3}\left[u^{3/2}\right]_0^8 = \frac{2}{3}\left(8^{3/2} - 0^{3/2}\right) = \frac{2}{3}\left(2^{9/2} - 0\right) = \frac{2^{11/2}}{3} = \frac{32\sqrt{2}}{3}$

59. $\int_1^e \frac{1}{x}(1+7\ln x)^{-1/3}\,dx = \frac{1}{7}\int_1^8 u^{-1/3}\,du$, where $u = 1+7\ln x$, $du = \frac{7}{x}\,dx$, $x = 1 \Rightarrow u = 1$, $x = e \Rightarrow u = 8$
$= \frac{3}{14}\left[u^{2/3}\right]_1^8 = \frac{3}{14}\left(8^{2/3} - 1^{2/3}\right) = \left(\frac{3}{14}\right)(4-1) = \frac{9}{14}$

60. $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx = \int_e^{e^2} (\ln x)^{-1/2}\,\frac{1}{x}\,dx = \int_1^2 u^{-1/2}\,du$, where $u = \ln x$, $du = \frac{1}{x}\,dx$; $x = e \Rightarrow u = 1$, $x = e^2 \Rightarrow u = 2$
$= 2\left[u^{1/2}\right]_1^2 = 2\left(\sqrt{2}-1\right) = 2\sqrt{2} - 2$

61. $\int_1^3 \frac{[\ln(v+1)]^2}{v+1}\,dv = \int_1^3 [\ln(v+1)]^2\,\frac{1}{v+1}\,dv = \int_{\ln 2}^{\ln 4} u^2\,du$, where $u = \ln(v+1)$, $du = \frac{1}{v+1}\,dv$;
$v = 1 \Rightarrow u = \ln 2$, $v = 3 \Rightarrow u = \ln 4$;
$= \frac{1}{3}\left[u^3\right]_{\ln 2}^{\ln 4} = \frac{1}{3}\left[(\ln 4)^3 - (\ln 2)^3\right] = \frac{1}{3}\left[(2\ln 2)^3 - (\ln 2)^3\right] = \frac{(\ln 2)^3}{3}(8-1) = \frac{7}{3}(\ln 2)^3$

62. $\int_2^4 (1+\ln t)(t\ln t)\,dt = \int_2^4 (t\ln t)(1+\ln t)\,dt = \int_{2\ln 2}^{4\ln 4} u\,du$, where $u = t\ln t$, $du = \left((t)\left(\frac{1}{t}\right) + (\ln t)(1)\right)dt$
$= (1+\ln t)\,dt$; $t = 2 \Rightarrow u = 2\ln 2$, $t = 4$
$\Rightarrow u = 4\ln 4$
$= \frac{1}{2}\left[u^2\right]_{2\ln 2}^{4\ln 4} = \frac{1}{2}\left[(4\ln 4)^2 - (2\ln 2)^2\right] = \frac{1}{2}\left[(8\ln 2)^2 - (2\ln 2)^2\right] = \frac{(2\ln 2)^2}{2}(16-1) = 30(\ln 2)^2$

63. $\int_1^8 \frac{\log_4 \theta}{\theta}\,d\theta = \frac{1}{\ln 4}\int_1^8 (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = \frac{1}{\ln 4}\int_0^{\ln 8} u\,du$, where $u = \ln\theta$, $du = \frac{1}{\theta}\,d\theta$, $\theta = 1 \Rightarrow u = 0$, $\theta = 8 \Rightarrow u = \ln 8$
$= \frac{1}{2\ln 4}\left[u^2\right]_0^{\ln 8} = \frac{1}{\ln 16}\left[(\ln 8)^2 - 0^2\right] = \frac{(3\ln 2)^2}{4\ln 2} = \frac{9\ln 2}{4}$

64. $\int_1^e \frac{8(\ln 3)(\log_3\theta)}{\theta}\,d\theta = \int_1^e \frac{8(\ln 3)(\ln\theta)}{\theta(\ln 3)}\,d\theta = 8\int_1^e (\ln\theta)\left(\frac{1}{\theta}\right)d\theta = 8\int_0^1 u\,du$, where $u = \ln\theta$, $du = \frac{1}{\theta}\,d\theta$;
$\theta = 1 \Rightarrow u = 0$, $\theta = e \Rightarrow u = 1$
$= 4\left[u^2\right]_0^1 = 4\left(1^2 - 0^2\right) = 4$

65. $\int_{-3/4}^{3/4} \frac{6}{\sqrt{9-4x^2}}\,dx = 3\int_{-3/4}^{3/4} \frac{2}{\sqrt{3^2-(2x)^2}}\,dx = 3\int_{-3/2}^{3/2} \frac{1}{\sqrt{3^2-u^2}}\,du$, where $u = 2x$, $du = 2\,dx$;
$x = -\frac{3}{4} \Rightarrow u = -\frac{3}{2}$, $x = \frac{3}{4} \Rightarrow u = \frac{3}{2}$
$= 3\left[\sin^{-1}\left(\frac{u}{3}\right)\right]_{-3/2}^{3/2} = 3\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = 3\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = 3\left(\frac{\pi}{3}\right) = \pi$

66. $\int_{-1/5}^{1/5} \frac{6}{\sqrt{4-25x^2}}\,dx = \frac{6}{5}\int_{-1/5}^{1/5} \frac{5}{\sqrt{2^2-(5x)^2}}\,dx = \frac{6}{5}\int_{-1}^{1} \frac{1}{\sqrt{2^2-u^2}}\,du$, where $u = 5x$, $du = 5\,dx$;
$x = -\frac{1}{5} \Rightarrow u = -1$, $x = \frac{1}{5} \Rightarrow u = 1$
$= \frac{6}{5}\left[\sin^{-1}\left(\frac{u}{2}\right)\right]_{-1}^{1} = \frac{6}{5}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \frac{6}{5}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{6}{5}\left(\frac{\pi}{3}\right) = \frac{2\pi}{5}$

67. $\int_{-2}^{2} \frac{3}{4+3t^2}\,dt = \sqrt{3}\int_{-2}^{2} \frac{\sqrt{3}}{2^2+\left(\sqrt{3}t\right)^2}\,dt = \sqrt{3}\int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{2^2+u^2}\,du$, where $u = \sqrt{3}t$, $du = \sqrt{3}\,dt$;
$t = -2 \Rightarrow u = -2\sqrt{3}$, $t = 2 \Rightarrow u = 2\sqrt{3}$
$= \sqrt{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_{-2\sqrt{3}}^{2\sqrt{3}} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}\left(-\sqrt{3}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \frac{\pi}{\sqrt{3}}$

68. $\int_{\sqrt{3}}^{3} \frac{1}{3+t^2}\,dt = \int_{\sqrt{3}}^{3} \frac{1}{\left(\sqrt{3}\right)^2+t^2}\,dt = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_{\sqrt{3}}^{3} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 1\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{3}\pi}{36}$

69. $\int \frac{1}{y\sqrt{4y^2-1}}\,dy = \int \frac{2}{(2y)\sqrt{(2y)^2-1}}\,dy = \int \frac{1}{u\sqrt{u^2-1}}\,du$, where $u = 2y$ and $du = 2\,dy$
$= \sec^{-1}|u| + C = \sec^{-1}|2y| + C$

70. $\int \frac{24}{y\sqrt{y^2-16}}\,dy = 24\int \frac{1}{y\sqrt{y^2-4^2}}\,dy = 24\left(\frac{1}{4}\sec^{-1}\left|\frac{y}{4}\right|\right) + C = 6\sec^{-1}\left|\frac{y}{4}\right| + C$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{1}{|y|\sqrt{9y^2-1}}\,dy = \int_{\sqrt{2}/3}^{2/3} \frac{3}{|3y|\sqrt{(3y)^2-1}}\,dy = \int_{\sqrt{2}}^{2} \frac{1}{|u|\sqrt{u^2-1}}\,du$, where $u = 3y$, $du = 3\,dy$;

$y = \frac{\sqrt{2}}{3} \Rightarrow u = \sqrt{2}$, $y = \frac{2}{3} \Rightarrow u = 2$

$= \left[\sec^{-1} u\right]_{\sqrt{2}}^{2} = \left[\sec^{-1} 2 - \sec^{-1}\sqrt{2}\right] = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{1}{|y|\sqrt{5y^2-3}}\,dy = \int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{\sqrt{5}}{-\sqrt{5}y\sqrt{\left(\sqrt{5}y\right)^2-\left(\sqrt{3}\right)^2}}\,dy = \int_{-2}^{-\sqrt{6}} \frac{1}{-u\sqrt{u^2-\left(\sqrt{3}\right)^2}}\,du$,

where $u = \sqrt{5}y$, $du = \sqrt{5}\,dy$; $y = -\frac{2}{\sqrt{5}} \Rightarrow u = -2$, $y = -\frac{\sqrt{6}}{\sqrt{5}} \Rightarrow u = -\sqrt{6}$

$= \left[-\frac{1}{\sqrt{3}}\sec^{-1}\left|\frac{u}{\sqrt{3}}\right|\right]_{-2}^{-\sqrt{6}} = \frac{-1}{\sqrt{3}}\left[\sec^{-1}\sqrt{2} - \sec^{-1}\frac{2}{\sqrt{3}}\right] = \frac{-1}{\sqrt{3}}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{-1}{\sqrt{3}}\left[\frac{3\pi}{12} - \frac{2\pi}{12}\right] = \frac{-\pi}{12\sqrt{3}} = \frac{-\sqrt{3}\pi}{36}$

73. $\int \frac{1}{\sqrt{-2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x^2+2x+1)}}\,dx = \int \frac{1}{\sqrt{1-(x+1)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$, where $u = x+1$ and

$du = dx$

$= \sin^{-1} u + C = \sin^{-1}(x+1) + C$

74. $\int \frac{1}{\sqrt{-x^2+4x-1}}\,dx = \int \frac{1}{\sqrt{3-(x^2-4x+4)}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-(x-2)^2}}\,dx = \int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-u^2}}\,du$

where $u = x-2$ and $du = dx$

$= \sin^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + C$

75. $\int_{-2}^{-1} \frac{2}{v^2+4v+5}\,dv = 2\int_{-2}^{-1} \frac{1}{1+(v^2+4v+4)}\,dv = 2\int_{-2}^{-1} \frac{1}{1+(v+2)^2}\,dv = 2\int_{0}^{1} \frac{1}{1+u^2}\,du$,

where $u = v+2$, $du = dv$; $v = -2 \Rightarrow u = 0$, $v = -1 \Rightarrow u = 1$

$= 2\left[\tan^{-1} u\right]_0^1 = 2\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 2\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2}$

76. $\int_{-1}^{1} \frac{3}{4v^2+4v+4}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\frac{3}{4}+\left(v^2+v+\frac{1}{4}\right)}\,dv = \frac{3}{4}\int_{-1}^{1} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+\left(v+\frac{1}{2}\right)^2}\,dv = \frac{3}{4}\int_{-1/2}^{3/2} \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+u^2}\,du$

where $u = v+\frac{1}{2}$, $du = dv$; $v = -1 \Rightarrow u = -\frac{1}{2}$, $v = 1 \Rightarrow u = \frac{3}{2}$

$= \frac{3}{4}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right]_{-1/2}^{3/2} = \frac{\sqrt{3}}{2}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \frac{\sqrt{3}}{2}\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{2}\left(\frac{2\pi}{6} + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\pi}{2}$

$= \frac{\sqrt{3}\pi}{4}$

77. $\int \frac{1}{(t+1)\sqrt{t^2+2t-8}}\,dt = \int \frac{1}{(t+1)\sqrt{(t^2+2t+1)-9}}\,dt = \int \frac{1}{(t+1)\sqrt{(t+1)^2-3^2}}\,dt = \int \frac{1}{u\sqrt{u^2-3^2}}\,du$

where $u = t+1$ and $du = dt$

$= \frac{1}{3}\sec^{-1}\left|\frac{u}{3}\right| + C = \frac{1}{3}\sec^{-1}\left|\frac{t+1}{3}\right| + C$

78. $\int \frac{1}{(3t+1)\sqrt{9t^2+6t}}\,dt = \int \frac{1}{(3t+1)\sqrt{(9t^2+6t+1)-1}}\,dt = \int \frac{1}{(3t+1)\sqrt{(3t+1)^2-1^2}}\,dt = \frac{1}{3}\int \frac{1}{u\sqrt{u^2-1}}\,du$

where $u = 3t+1$ and $du = 3\,dt$

$= \frac{1}{3}\sec^{-1}|u| + C = \frac{1}{3}\sec^{-1}|3t+1| + C$

79. $3^y = 2^{y+1} \Rightarrow \ln 3^y = \ln 2^{y+1} \Rightarrow y(\ln 3) = (y+1)\ln 2 \Rightarrow (\ln 3 - \ln 2)y = \ln 2 \Rightarrow \left(\ln\frac{3}{2}\right)y = \ln 2 \Rightarrow y = \frac{\ln 2}{\ln\left(\frac{3}{2}\right)}$

80. $4^{-y} = 3^{y+2} \Rightarrow \ln 4^{-y} = \ln 3^{y+2} \Rightarrow -y \ln 4 = (y+2)\ln 3 \Rightarrow -2\ln 3 = (\ln 3 + \ln 4)y \Rightarrow (\ln 12)y = -2\ln 3$
$\Rightarrow y = -\frac{\ln 9}{\ln 12}$

81. $9e^{2y} = x^2 \Rightarrow e^{2y} = \frac{x^2}{9} \Rightarrow \ln e^{2y} = \ln\left(\frac{x^2}{9}\right) \Rightarrow 2y(\ln e) = \ln\left(\frac{x^2}{9}\right) \Rightarrow y = \frac{1}{2}\ln\left(\frac{x^2}{9}\right) = \ln\sqrt{\frac{x^2}{9}} = \ln\left|\frac{x}{3}\right| = \ln|x| - \ln 3$

82. $3^y = 3\ln x \Rightarrow \ln 3^y = \ln(3\ln x) \Rightarrow y\ln 3 = \ln(3\ln x) \Rightarrow y = \frac{\ln(3\ln x)}{\ln 3} = \frac{\ln 3 + \ln(\ln x)}{\ln 3}$

83. $\ln(y-1) = x + \ln y \Rightarrow e^{\ln(y-1)} = e^{(x+\ln y)} = e^x e^{\ln y} \Rightarrow y - 1 = ye^x \Rightarrow y - ye^x = 1 \Rightarrow y(1-e^x) = 1 \Rightarrow y = \frac{1}{1-e^x}$

84. $\ln(10\ln y) = \ln 5x \Rightarrow e^{\ln(10\ln y)} = e^{\ln 5x} \Rightarrow 10\ln y = 5x \Rightarrow \ln y = \frac{x}{2} \Rightarrow e^{\ln y} = e^{x/2} \Rightarrow y = e^{x/2}$

85. $\lim_{x\to 1} \frac{x^2+3x-4}{x-1} = \lim_{x\to 1} \frac{2x+3}{1} = 5$

86. $\lim_{x\to 1} \frac{x^a-1}{x^b-1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

87. $\lim_{x\to \pi} \frac{\tan x}{x} = \frac{\tan \pi}{\pi} = 0$

88. $\lim_{x\to 0} \frac{\tan x}{x+\sin x} = \lim_{x\to 0} \frac{\sec^2 x}{1+\cos x} = \frac{1}{1+1} = \frac{1}{2}$

89. $\lim_{x\to 0} \frac{\sin^2 x}{\tan(x^2)} = \lim_{x\to 0} \frac{2\sin x\cdot\cos x}{2x\sec^2(x^2)} = \lim_{x\to 0} \frac{\sin(2x)}{2x\sec^2(x^2)} = \lim_{x\to 0} \frac{2\cos(2x)}{2x(2\sec^2(x^2)\tan(x^2)\cdot 2x) + 2\sec^2(x^2)} = \frac{2}{0+2\cdot 1} = 1$

90. $\lim_{x\to 0} \frac{\sin(mx)}{\sin(nx)} = \lim_{x\to 0} \frac{m\cos(mx)}{n\cos(nx)} = \frac{m}{n}$

91. $\lim_{x\to \pi/2^-} \sec(7x)\cos(3x) = \lim_{x\to \pi/2^-} \frac{\cos(3x)}{\cos(7x)} = \lim_{x\to \pi/2^-} \frac{-3\sin(3x)}{-7\sin(7x)} = \frac{3}{7}$

92. $\lim_{x\to 0^+} \sqrt{x}\sec x = \lim_{x\to 0^+} \frac{\sqrt{x}}{\cos x} = \frac{0}{1} = 0$

93. $\lim_{x\to 0} (\csc x - \cot x) = \lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$

94. $\lim_{x\to 0} \left(\frac{1}{x^4} - \frac{1}{x^2}\right) = \lim_{x\to 0} \left(\frac{1-x^2}{x^4}\right) = \lim_{x\to 0} (1-x^2)\cdot\frac{1}{x^4} = \lim_{x\to 0} (1-x^2) = \lim_{x\to 0} \frac{1}{x^4} = 1\cdot\infty = \infty$

95. $\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right) = \lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right)\cdot\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$
$= \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$
Notice that $x = \sqrt{x^2}$ for $x > 0$ so this is equivalent to
$= \lim_{x\to\infty} \frac{\frac{2x+1}{x}}{\sqrt{\frac{x^2+x+1}{x^2}}+\sqrt{\frac{x^2-x}{x^2}}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1$

96. $\lim_{x\to\infty} \left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right) = \lim_{x\to\infty} \frac{x^3(x^2+1) - x^3(x^2-1)}{(x^2-1)(x^2+1)} = \lim_{x\to\infty} \frac{2x^3}{x^4-1} = \lim_{x\to\infty} \frac{6x^2}{4x^3} = \lim_{x\to\infty} \frac{12x}{12x^2}$
$= \lim_{x\to\infty} \frac{12}{24x} = \lim_{x\to\infty} \frac{1}{2x} = 0$

97. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 0} \frac{10^x-1}{x} = \lim_{x\to 0} \frac{(\ln 10)10^x}{1} = \ln 10$

98. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{\theta\to 0} \frac{3^\theta-1}{\theta} = \lim_{\theta\to 0} \frac{(\ln 3)3^\theta}{1} = \ln 3$

99. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim_{x\to 0} \frac{2^{\sin x}-1}{e^x-1} = \lim_{x\to 0} \frac{2^{\sin x}(\ln 2)(\cos x)}{e^x} = \ln 2$

100. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 0} \frac{2^{-\sin x} - 1}{e^x - 1} = \lim\limits_{x \to 0} \frac{2^{-\sin x}(\ln 2)(-\cos x)}{e^x} = -\ln 2$

101. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 0} \frac{5 - 5\cos x}{e^x - x - 1} = \lim\limits_{x \to 0} \frac{5\sin x}{e^x - 1} = \lim\limits_{x \to 0} \frac{5\cos x}{e^x} = 5$

102. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 0} \frac{x\sin x^2}{\tan^3 x} = \lim\limits_{x \to 0} \frac{2x^2\cos x^2 + \sin x^2}{3\tan^2 x\sec^2 x} = \lim\limits_{x \to 0} \frac{2x^2\cos x^2 + \sin x^2}{3\tan^4 x + 3\tan^2 x}$
$= \lim\limits_{x \to 0} \frac{6x\cos x^2 - 4x^3\sin x^2}{12\tan^3 x\sec^2 x + 6\tan x\sec^2 x} = \lim\limits_{x \to 0} \frac{6x\cos x^2 - 4x^3\sin x^2}{12\tan^5 x + 18\tan^3 x + 6\tan x} = \lim\limits_{x \to 0} \frac{(6 - 8x^4)\cos x^2 - 24x^2\sin x^2}{60\tan^4 x\sec^2 x + 54\tan^2 x\sec^2 x + 6\sec^2 x} = \frac{6}{6} = 1$

103. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{t \to 0^+} \frac{t - \ln(1 + 2t)}{t^2} = \lim\limits_{t \to 0^+} \frac{\left(1 - \frac{2}{1+2t}\right)}{2t} = -\infty$

104. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x} = \lim\limits_{x \to 4} \frac{2\pi(\sin \pi x)(\cos \pi x)}{e^{x-4} - 1}$
$= \lim\limits_{x \to 4} \frac{\pi\sin(2\pi x)}{e^{x-4} - 1} = \lim\limits_{x \to 4} \frac{2\pi^2\cos(2\pi x)}{e^{x-4}} = 2\pi^2$

105. The limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{t \to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right) = \lim\limits_{t \to 0^+} \left(\frac{e^t - 1}{t}\right) = \lim\limits_{t \to 0^+} \frac{e^t}{1} = 1$

106. The limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim\limits_{y \to 0^+} e^{-1/y}\ln y = \lim\limits_{y \to 0^+} \frac{\ln y}{e^{y^{-1}}} = \lim\limits_{y \to 0^+} \frac{y^{-1}}{-e^{y^{-1}}(y^{-2})}$
$= \lim\limits_{y \to 0^+} \left(-\frac{y}{e^{y^{-1}}}\right) = 0$

107. Let $f(x) = \left(\frac{e^x + 1}{e^x - 1}\right)^{\ln x} \Rightarrow \ln f(x) = \ln x \ln\left(\frac{e^x + 1}{e^x - 1}\right) \Rightarrow \lim\limits_{x \to \infty} \ln f(x) = \lim\limits_{x \to \infty} \ln x \ln\left(\frac{e^x + 1}{e^x - 1}\right)$; this is limit is currently of the form $0 \cdot \infty$. Before we put in one of the indeterminate forms, we rewrite $\frac{e^x + 1}{e^x - 1} = \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} = \coth\left(\frac{x}{2}\right)$; the limit is $\lim\limits_{x \to \infty} \ln x \ln\coth\left(\frac{x}{2}\right) = \lim\limits_{x \to \infty} \frac{\ln\coth\left(\frac{x}{2}\right)}{\frac{1}{\ln x}}$; the limit leads to the indeterminate form $\frac{0}{0}$: $\lim\limits_{x \to \infty} \frac{\ln\coth\left(\frac{x}{2}\right)}{\frac{1}{\ln x}}$
$= \lim\limits_{x \to \infty} \left(\frac{\frac{\operatorname{csch}^2\left(\frac{x}{2}\right)}{\coth\left(\frac{x}{2}\right)}\left(-\frac{1}{2}\right)}{-\frac{1}{(\ln x)^2}\left(\frac{1}{x}\right)}\right) = \lim\limits_{x \to \infty} \left(\frac{x(\ln x)^2}{2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)}\right) = \lim\limits_{x \to \infty} \left(\frac{x(\ln x)^2}{\sinh x}\right) = \lim\limits_{x \to \infty} \left(\frac{2x(\ln x)\left(\frac{1}{x}\right) + (\ln x)^2}{\cosh x}\right)$
$= \lim\limits_{x \to \infty} \left(\frac{2\ln x + (\ln x)^2}{\cosh x}\right) = \lim\limits_{x \to \infty} \left(\frac{2\left(\frac{1}{x}\right) + 2(\ln x)\left(\frac{1}{x}\right)}{\sinh x}\right) = \lim\limits_{x \to \infty} \left(\frac{2 + 2\ln x}{x\sinh x}\right) = \lim\limits_{x \to \infty} \left(\frac{\frac{2}{x}}{x\cosh x + \sinh x}\right)$
$= \lim\limits_{x \to \infty} \left(\frac{2}{x^2\cosh x + x\sinh x}\right) = 0 \Rightarrow \lim\limits_{x \to \infty} \left(\frac{e^x + 1}{e^x - 1}\right)^{\ln x} = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^0 = 1$

108. Let $f(x) = \left(1 + \frac{3}{x}\right)^x \Rightarrow \ln f(x) = x\ln\left(1 + \frac{3}{x}\right) \Rightarrow \lim\limits_{x \to 0^+} \ln f(x) = \lim\limits_{x \to 0^+} \frac{\ln(1 + 3x^{-1})}{x^{-1}}$; the limit leads to the indeterminate form $\frac{\infty}{\infty}$: $\lim\limits_{x \to 0^+} \frac{\left(\frac{-3x^{-2}}{1 + 3x^{-1}}\right)}{-x^{-2}} = \lim\limits_{x \to 0^+} \frac{3x}{x + 3} = 0 \Rightarrow \lim\limits_{x \to 0^+} \left(1 + \frac{3}{x}\right)^x = \lim\limits_{x \to 0^+} e^{\ln f(x)} = e^0 = 1$

109. (a) $\lim\limits_{x \to \infty} \frac{\log_2 x}{\log_3 x} = \lim\limits_{x \to \infty} \frac{\left(\frac{\ln x}{\ln 2}\right)}{\left(\frac{\ln x}{\ln 3}\right)} = \lim\limits_{x \to \infty} \frac{\ln 3}{\ln 2} = \frac{\ln 3}{\ln 2} \Rightarrow$ same rate

(b) $\lim\limits_{x \to \infty} \frac{x}{x + \left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \frac{x^2}{x^2 + 1} = \lim\limits_{x \to \infty} \frac{2x}{2x} = \lim\limits_{x \to \infty} 1 = 1 \Rightarrow$ same rate

(c) $\lim\limits_{x \to \infty} \frac{\left(\frac{x}{100}\right)}{xe^{-x}} = \lim\limits_{x \to \infty} \frac{xe^x}{100x} = \lim\limits_{x \to \infty} \frac{e^x}{100} = \infty \Rightarrow$ faster

(d) $\lim\limits_{x \to \infty} \frac{x}{\tan^{-1} x} = \infty \Rightarrow$ faster

(e) $\lim\limits_{x \to \infty} \frac{\csc^{-1} x}{\left(\frac{1}{x}\right)} = \lim\limits_{x \to \infty} \frac{\sin^{-1}(x^{-1})}{x^{-1}} = \lim\limits_{x \to \infty} \frac{\frac{(-x^{-2})}{\sqrt{1 - (x^{-1})^2}}}{-x^{-2}} = \lim\limits_{x \to \infty} \frac{1}{\sqrt{1 - \left(\frac{1}{x^2}\right)}} = 1 \Rightarrow$ same rate

(f) $\lim\limits_{x\to\infty} \frac{\sinh x}{e^x} = \lim\limits_{x\to\infty} \frac{(e^x-e^{-x})}{2e^x} = \lim\limits_{x\to\infty} \frac{1-e^{-2x}}{2} = \frac{1}{2} \Rightarrow$ same rate

110. (a) $\lim\limits_{x\to\infty} \frac{3^{-x}}{2^{-x}} = \lim\limits_{x\to\infty} \left(\frac{2}{3}\right)^x = 0 \Rightarrow$ slower

(b) $\lim\limits_{x\to\infty} \frac{\ln 2x}{\ln x^2} = \lim\limits_{x\to\infty} \frac{\ln 2 + \ln x}{2(\ln x)} = \lim\limits_{x\to\infty} \left(\frac{\ln 2}{2\ln x} + \frac{1}{2}\right) = \frac{1}{2} \Rightarrow$ same rate

(c) $\lim\limits_{x\to\infty} \frac{10x^3+2x^2}{e^x} = \lim\limits_{x\to\infty} \frac{30x^2+4x}{e^x} = \lim\limits_{x\to\infty} \frac{60x+4}{e^x} = \lim\limits_{x\to\infty} \frac{60}{e^x} = 0 \Rightarrow$ slower

(d) $\lim\limits_{x\to\infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{\tan^{-1}(x^{-1})}{x^{-1}} = \lim\limits_{x\to\infty} \frac{\left(\frac{-x^{-2}}{1+x^{-2}}\right)}{-x^{-2}} = \lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1 \Rightarrow$ same rate

(e) $\lim\limits_{x\to\infty} \frac{\sin^{-1}\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{\sin^{-1}(x^{-1})}{x^{-2}} = \lim\limits_{x\to\infty} \frac{\left(\frac{-x^{-2}}{\sqrt{1-(x^{-1})^2}}\right)}{-2x^{-3}} = \lim\limits_{x\to\infty} \frac{x}{2\sqrt{1-\frac{1}{x^2}}} = \infty \Rightarrow$ faster

(f) $\lim\limits_{x\to\infty} \frac{\operatorname{sech} x}{e^{-x}} = \lim\limits_{x\to\infty} \frac{\left(\frac{2}{e^x+e^{-x}}\right)}{e^{-x}} = \lim\limits_{x\to\infty} \frac{2}{e^{-x}(e^x+e^{-x})} = \lim\limits_{x\to\infty} \left(\frac{2}{1+e^{-2x}}\right) = 2 \Rightarrow$ same rate

111. (a) $\frac{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}\right)} = 1 + \frac{1}{x^2} \le 2$ for x sufficiently large $\Rightarrow$ true

(b) $\frac{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = x^2 + 1 > M$ for any positive integer M whenever $x > \sqrt{M} \Rightarrow$ false

(c) $\lim\limits_{x\to\infty} \frac{x}{x+\ln x} = \lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \Rightarrow$ the same growth rate $\Rightarrow$ false

(d) $\lim\limits_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim\limits_{x\to\infty} \frac{\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right]}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{1}{\ln x} = 0 \Rightarrow$ grows slower $\Rightarrow$ true

(e) $\frac{\tan^{-1} x}{1} \le \frac{\pi}{2}$ for all x $\Rightarrow$ true

(f) $\frac{\cosh x}{e^x} = \frac{1}{2}(1+e^{-2x}) \le \frac{1}{2}(1+1) = 1$ if $x > 0 \Rightarrow$ true

112. (a) $\frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)} = \frac{1}{x^2+1} \le 1$ if $x > 0 \Rightarrow$ true

(b) $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^2}+\frac{1}{x^4}\right)} = \lim\limits_{x\to\infty} \left(\frac{1}{x^2+1}\right) = 0 \Rightarrow$ true

(c) $\lim\limits_{x\to\infty} \frac{\ln x}{x+1} = \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0 \Rightarrow$ true

(d) $\frac{\ln 2x}{\ln x} = \frac{\ln 2}{\ln x} + 1 \le 1 + 1 = 2$ if $x \ge 2 \Rightarrow$ true

(e) $\frac{\sec^{-1} x}{1} = \frac{\cos^{-1}\left(\frac{1}{x}\right)}{1} \le \frac{\left(\frac{\pi}{2}\right)}{1} = \frac{\pi}{2}$ if $x > 1 \Rightarrow$ true

(f) $\frac{\sinh x}{e^x} = \frac{1}{2}(1-e^{-2x}) \le \frac{1}{2}$ if $x > 0 \Rightarrow$ true

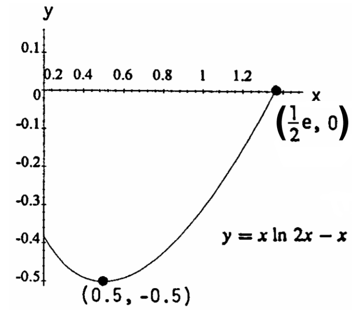
113. $\frac{df}{dx} = e^x + 1 \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=\ln 2}} \Rightarrow \left(\frac{df^{-1}}{dx}\right)_{x=f(\ln 2)} = \frac{1}{(e^x+1)_{x=\ln 2}} = \frac{1}{2+1} = \frac{1}{3}$

114. $y = f(x) \Rightarrow y = 1 + \frac{1}{x} \Rightarrow \frac{1}{x} = y - 1 \Rightarrow x = \frac{1}{y-1} \Rightarrow f^{-1}(x) = \frac{1}{x-1}$; $f^{-1}(f(x)) = \frac{1}{\left(1+\frac{1}{x}\right)-1} = \frac{1}{\left(\frac{1}{x}\right)} = x$ and

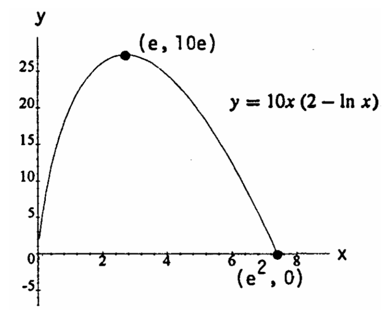
$f(f^{-1}(x)) = 1 + \frac{1}{\left(\frac{1}{x-1}\right)} = 1 + (x-1) = x$; $\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \left.\frac{-1}{(x-1)^2}\right|_{f(x)} = \frac{-1}{\left[\left(1+\frac{1}{x}\right)-1\right]^2} = -x^2$;

$f'(x) = -\frac{1}{x^2} \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}$

115. $y = x \ln 2x - x \Rightarrow y' = x\left(\frac{2}{2x}\right) + \ln(2x) - 1 = \ln 2x$; solving $y' = 0 \Rightarrow x = \frac{1}{2}$; $y' > 0$ for $x > \frac{1}{2}$ and $y' < 0$ for $x < \frac{1}{2} \Rightarrow$ relative minimum of $-\frac{1}{2}$ at $x = \frac{1}{2}$; $f\left(\frac{1}{2e}\right) = -\frac{1}{e}$ and $f\left(\frac{e}{2}\right) = 0 \Rightarrow$ absolute minimum is $-\frac{1}{2}$ at $x = \frac{1}{2}$ and the absolute maximum is 0 at $x = \frac{e}{2}$



116. $y = 10x(2 - \ln x) \Rightarrow y' = 10(2 - \ln x) - 10x\left(\frac{1}{x}\right)$
$= 20 - 10 \ln x - 10 = 10(1 - \ln x)$; solving $y' = 0$
$\Rightarrow x = e$; $y' < 0$ for $x > e$ and $y' > 0$ for $x < e$
$\Rightarrow$ relative maximum at $x = e$ of $10e$; $y \geq 0$ on $(0, e^2]$ and $y(e^2) = 10e^2(2 - 2\ln e) = 0 \Rightarrow$ absolute minimum is 0 at $x = e^2$ and the absolute maximum is $10e$ at $x = e$



117. $A = \int_1^e \frac{2\ln x}{x}\,dx = \int_0^1 2u\,du = \left[u^2\right]_0^1 = 1$, where $u = \ln x$ and $du = \frac{1}{x}\,dx$; $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$

118. (a) $A_1 = \int_{10}^{20} \frac{1}{x}\,dx = \left[\ln|x|\right]_{10}^{20} = \ln 20 - \ln 10 = \ln \frac{20}{10} = \ln 2$, and $A_2 = \int_1^2 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2$
(b) $A_1 = \int_{ka}^{kb} \frac{1}{x}\,dx = \left[\ln|x|\right]_{ka}^{kb} = \ln kb - \ln ka = \ln \frac{kb}{ka} = \ln \frac{b}{a} = \ln b - \ln a$, and $A_2 = \int_a^b \frac{1}{x}\,dx = \left[\ln|x|\right]_a^b = \ln b - \ln a$

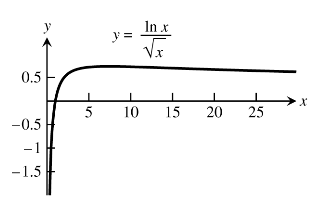
119. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$; $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \left(\frac{1}{x}\right)\sqrt{x} = \frac{1}{\sqrt{x}} \Rightarrow \left.\frac{dy}{dt}\right|_{e^2} = \frac{1}{e}$ m/sec

120. $y = 9e^{-x/3} \Rightarrow \frac{dy}{dx} = -3e^{-x/3}$; $\frac{dx}{dt} = \frac{(dy/dt)}{(dy/dx)} \Rightarrow \frac{dx}{dt} = \frac{\left(-\frac{1}{4}\right)\sqrt{9-y}}{-3e^{-x/3}}$; $x = 9 \Rightarrow y = 9e^{-3}$
$\Rightarrow \left.\frac{dx}{dt}\right|_{x=9} = \frac{\left(-\frac{1}{4}\right)\sqrt{9 - \frac{9}{e^3}}}{\left(-\frac{3}{e^3}\right)} = \frac{1}{4}\sqrt{e^3}\sqrt{e^3 - 1} \approx 5$ ft/sec

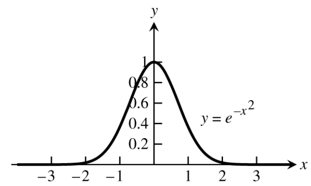
121. $A = xy = xe^{-x^2} \Rightarrow \frac{dA}{dx} = e^{-x^2} + (x)(-2x)e^{-x^2} = e^{-x^2}\left(1 - 2x^2\right)$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - 2x^2 = 0$
$\Rightarrow x = \frac{1}{\sqrt{2}}$; $\frac{dA}{dx} < 0$ for $x > \frac{1}{\sqrt{2}}$ and $\frac{dA}{dx} > 0$ for $0 < x < \frac{1}{\sqrt{2}} \Rightarrow$ absolute maximum of $\frac{1}{\sqrt{2}}e^{-1/2} = \frac{1}{\sqrt{2e}}$ at $x = \frac{1}{\sqrt{2}}$ units long by $y = e^{-1/2} = \frac{1}{\sqrt{e}}$ units high.

122. $A = xy = x\left(\frac{\ln x}{x^2}\right) = \frac{\ln x}{x} \Rightarrow \frac{dA}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1 - \ln x}{x^2}$. Solving $\frac{dA}{dx} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow x = e$; $\frac{dA}{dx} < 0$ for $x > e$ and $\frac{dA}{dx} > 0$ for $x < e \Rightarrow$ absolute maximum of $\frac{\ln e}{e} = \frac{1}{e}$ at $x = e$ units long and $y = \frac{1}{e^2}$ units high.

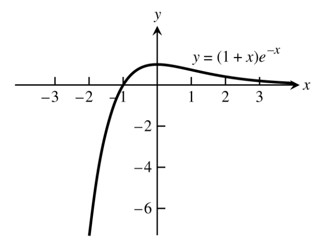
123. (a) $y = \frac{\ln x}{\sqrt{x}} \Rightarrow y' = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x^{3/2}} = \frac{2 - \ln x}{2x\sqrt{x}}$
$\Rightarrow y'' = -\frac{3}{4}x^{-5/2}(2 - \ln x) - \frac{1}{2}x^{-5/2} = x^{-5/2}\left(\frac{3}{4}\ln x - 2\right)$;
solving $y' = 0 \Rightarrow \ln x = 2 \Rightarrow x = e^2$; $y' < 0$ for $x > e^2$ and and $y' > 0$ for $x < e^2 \Rightarrow$ a maximum of $\frac{2}{e}$; $y'' = 0$
$\Rightarrow \ln x = \frac{8}{3} \Rightarrow x = e^{8/3}$; the curve is concave down on $\left(0, e^{8/3}\right)$ and concave up on $\left(e^{8/3}, \infty\right)$; so there is an inflection point at $\left(e^{8/3}, \frac{8}{3e^{4/3}}\right)$.



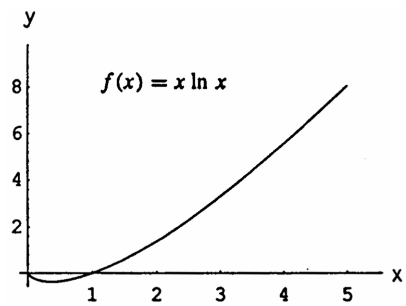
(b) $y = e^{-x^2} \Rightarrow y' = -2xe^{-x^2} \Rightarrow y'' = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2 - 2)e^{-x^2}$; solving $y' = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $e^0 = 1$; there are points of inflection at $x = \pm\frac{1}{\sqrt{2}}$; the curve is concave down for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and concave up otherwise.



(c) $y = (1 + x)e^{-x} \Rightarrow y' = e^{-x} - (1 + x)e^{-x} = -xe^{-x} \Rightarrow y'' = -e^{-x} + xe^{-x} = (x - 1)e^{-x}$; solving $y' = 0 \Rightarrow -xe^{-x} = 0 \Rightarrow x = 0$; $y' < 0$ for $x > 0$ and $y' > 0$ for $x < 0 \Rightarrow$ a maximum at $x = 0$ of $(1 + 0)e^0 = 1$; there is a point of inflection at $x = 1$ and the curve is concave up for $x > 1$ and concave down for $x < 1$.



124. $y = x \ln x \Rightarrow y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$; solving $y' = 0 \Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$; $y' > 0$ for $x > e^{-1}$ and $y' < 0$ for $x < e^{-1} \Rightarrow$ a minimum of $e^{-1} \ln e^{-1} = -\frac{1}{e}$ at $x = e^{-1}$. This minimum is an absolute minimum since $y'' = \frac{1}{x}$ is positive for all $x > 0$.



125. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}\cos^2\sqrt{y}} = dx \Rightarrow 2\tan\sqrt{y} = x + C \Rightarrow y = \left(\tan^{-1}\left(\frac{x+C}{2}\right)\right)^2$

126. $y' = \frac{3y(x+1)^2}{y-1} \Rightarrow \frac{(y-1)}{y}dy = 3(x+1)^2dx \Rightarrow y - \ln y = (x+1)^3 + C$

127. $yy' = \sec(y^2)\sec^2 x \Rightarrow \frac{y\,dy}{\sec(y^2)} = \sec^2 x\,dx \Rightarrow \frac{\sin(y^2)}{2} = \tan x + C \Rightarrow \sin(y^2) = 2\tan x + C_1$

128. $y\cos^2(x)\,dy + \sin x\,dx = 0 \Rightarrow y\,dy = -\frac{\sin x}{\cos^2(x)}dx \Rightarrow \frac{y^2}{2} = -\frac{1}{\cos(x)} + C \Rightarrow y = \pm\sqrt{\frac{-2}{\cos(x)} + C_1}$

129. $\frac{dy}{dx} = e^{-x-y-2} \Rightarrow e^y dy = e^{-(x+2)}dx \Rightarrow e^y = -e^{-(x+2)} + C$. We have $y(0) = -2$, so $e^{-2} = -e^{-2} + C \Rightarrow C = 2e^{-2}$ and $e^y = -e^{-(x+2)} + 2e^{-2} \Rightarrow y = \ln\left(-e^{-(x+2)} + 2e^{-2}\right)$

130. $\frac{dy}{dx} = \frac{y \ln y}{1+x^2} \Rightarrow \frac{dy}{y \ln y} = \frac{dx}{1+x^2} \Rightarrow \ln(\ln y) = \tan^{-1}(x) + C \Rightarrow y = e^{e^{\tan^{-1}(x)+C}}$. We have $y(0) = e^2 \Rightarrow e^2 = e^{e^{\tan^{-1}(0)+C}}$ $\Rightarrow e^{\tan^{-1}(0)+C} = 2 \Rightarrow \tan^{-1}(0) + C = \ln 2 \Rightarrow 0 + C = \ln 2 \Rightarrow C = \ln 2 \Rightarrow y = e^{e^{\tan^{-1}(x)+\ln 2}}$

131. $x\,dy - \left(y + \sqrt{y}\right)dx = 0 \Rightarrow \frac{dy}{(y+\sqrt{y})} = \frac{dx}{x} \Rightarrow 2\ln\left(\sqrt{y} + 1\right) = \ln x + C$. We have $y(1) = 1 \Rightarrow 2\ln\left(\sqrt{1} + 1\right) = \ln 1 + C$ $\Rightarrow 2\ln 2 = C = \ln 2^2 = \ln 4$. So $2\ln\left(\sqrt{y} + 1\right) = \ln x + \ln 4 = \ln(4x) \Rightarrow \ln\left(\sqrt{y} + 1\right) = \frac{1}{2}\ln(4x) = \ln(4x)^{1/2}$ $\Rightarrow e^{\ln\left(\sqrt{y}+1\right)} = e^{\ln(4x)^{1/2}} \Rightarrow \sqrt{y} + 1 = 2\sqrt{x} \Rightarrow y = \left(2\sqrt{x} - 1\right)^2$

132. $y^{-2}\frac{dx}{dy} = \frac{e^x}{e^{2x}+1} \Rightarrow \frac{e^{2x}+1}{e^x}dx = \frac{dy}{y^{-2}} \Rightarrow \frac{y^3}{3} = e^x - e^{-x} + C$. We have $y(0) = 1 \Rightarrow \frac{(1)^3}{3} = e^0 - e^0 + C \Rightarrow C = \frac{1}{3}$. So $\frac{y^3}{3} = e^x - e^{-x} + \frac{1}{3} \Rightarrow y^3 = 3(e^x - e^{-x}) + 1 \Rightarrow y = [3(e^x - e^{-x}) + 1\,]^{1/3}$

133. Since the half life is 5700 years and $A(t) = A_0 e^{kt}$ we have $\frac{A_0}{2} = A_0 e^{5700k} \Rightarrow \frac{1}{2} = e^{5700k} \Rightarrow \ln(0.5) = 5700k$ $\Rightarrow k = \frac{\ln(0.5)}{5700}$. With 10% of the original carbon-14 remaining we have $0.1A_0 = A_0 e^{\frac{\ln(0.5)}{5700}t} \Rightarrow 0.1 = e^{\frac{\ln(0.5)}{5700}t}$ $\Rightarrow \ln(0.1) = \frac{\ln(0.5)}{5700}t \Rightarrow t = \frac{(5700)\ln(0.1)}{\ln(0.5)} \approx 18{,}935$ years (rounded to the nearest year).

134. $T - T_s = (T_o - T_s)e^{-kt} \Rightarrow 180 - 40 = (220 - 40)e^{-k/4}$, time in hours, $\Rightarrow k = -4\ln\left(\frac{7}{9}\right) = 4\ln\left(\frac{9}{7}\right) \Rightarrow 70 - 40$ $= (220 - 40)e^{-4\ln(9/7)t} \Rightarrow t = \frac{\ln 6}{4\ln\left(\frac{9}{7}\right)} \approx 1.78$ hr ≈ 107 min, the total time $\Rightarrow$ the time it took to cool from 180° F to 70° F was $107 - 15 = 92$ min

135. $\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{5}{3} - \frac{x}{30}\right), 0 < x < 50 \Rightarrow \frac{d\theta}{dx} = \frac{\left(\frac{1}{60}\right)}{1+\left(\frac{x}{60}\right)^2} + \frac{\left(-\frac{1}{30}\right)}{1+\left(\frac{50-x}{30}\right)^2}$ $= 30\left[\frac{2}{60^2+x^2} - \frac{1}{30^2+(50-x)^2}\right]$; solving $\frac{d\theta}{dx} = 0 \Rightarrow x^2 - 200x + 3200 = 0 \Rightarrow x = 100 \pm 20\sqrt{17}$, but $100 + 20\sqrt{17}$ is not in the domain; $\frac{d\theta}{dx} > 0$ for $x < 20\left(5 - \sqrt{17}\right)$ and $\frac{d\theta}{dx} < 0$ for $20\left(5 - \sqrt{17}\right) < x < 50$ $\Rightarrow x = 20\left(5 - \sqrt{17}\right) \approx 17.54$ m maximizes θ

136. $v = x^2\ln\left(\frac{1}{x}\right) = x^2(\ln 1 - \ln x) = -x^2\ln x \Rightarrow \frac{dv}{dx} = -2x\ln x - x^2\left(\frac{1}{x}\right) = -x(2\ln x + 1)$; solving $\frac{dv}{dx} = 0$ $\Rightarrow 2\ln x + 1 = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-1/2}$; $\frac{dv}{dx} < 0$ for $x > e^{-1/2}$ and $\frac{dv}{dx} > 0$ for $x < e^{-1/2} \Rightarrow$ a relative maximum at $x = e^{-1/2}$; $\frac{r}{h} = x$ and $r = 1 \Rightarrow h = e^{1/2} = \sqrt{e} \approx 1.65$ cm

CHAPTER 7 ADDITIONAL AND ADVANCED EXERCISES

1. $\lim\limits_{b \to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim\limits_{b \to 1^-} \left[\sin^{-1} x\right]_0^b = \lim\limits_{b \to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \lim\limits_{b \to 1^-} \left(\sin^{-1} b - 0\right) = \lim\limits_{b \to 1^-} \sin^{-1} b = \frac{\pi}{2}$

2. $\lim\limits_{x \to \infty} \frac{1}{x}\int_0^x \tan^{-1} t\,dt = \lim\limits_{x \to \infty} \frac{\int_0^x \tan^{-1} t\,dt}{x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$
$= \lim\limits_{x \to \infty} \frac{\tan^{-1} x}{1} = \frac{\pi}{2}$

3. $y = \left(\cos\sqrt{x}\right)^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln\left(\cos\sqrt{x}\right)$ and $\lim\limits_{x \to 0^+} \frac{\ln\left(\cos\sqrt{x}\right)}{x} = \lim\limits_{x \to 0^+} \frac{-\sin\sqrt{x}}{2\sqrt{x}\cos\sqrt{x}} = \frac{-1}{2}\lim\limits_{x \to 0^+} \frac{\tan\sqrt{x}}{\sqrt{x}}$ $= -\frac{1}{2}\lim\limits_{x \to 0^+} \frac{\frac{1}{2}x^{-1/2}\sec^2\sqrt{x}}{\frac{1}{2}x^{-1/2}} = -\frac{1}{2} \Rightarrow \lim\limits_{x \to 0^+} \left(\cos\sqrt{x}\right)^{1/x} = e^{-1/2} = \frac{1}{\sqrt{e}}$

4. $y = (x + e^x)^{2/x} \Rightarrow \ln y = \frac{2\ln(x+e^x)}{x} \Rightarrow \lim\limits_{x \to \infty} \ln y = \lim\limits_{x \to \infty} \frac{2(1+e^x)}{x+e^x} = \lim\limits_{x \to \infty} \frac{2e^x}{1+e^x} = \lim\limits_{x \to \infty} \frac{2e^x}{e^x} = 2$ $\Rightarrow \lim\limits_{x \to \infty} (x+e^x)^{2/x} = \lim\limits_{x \to \infty} e^y = e^2$

5. $\lim\limits_{x \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right) = \lim\limits_{x \to \infty} \left(\left(\frac{1}{n}\right)\left[\frac{1}{1+\left(\frac{1}{n}\right)}\right] + \left(\frac{1}{n}\right)\left[\frac{1}{1+2\left(\frac{1}{n}\right)}\right] + \ldots + \left(\frac{1}{n}\right)\left[\frac{1}{1+n\left(\frac{1}{n}\right)}\right]\right)$ which can be interpreted as a Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim\limits_{x \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$ $= \int_0^1 \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^1 = \ln 2$

6. $\lim\limits_{x \to \infty} \frac{1}{n}\left[e^{1/n} + e^{2/n} + \ldots + e\right] = \lim\limits_{x \to \infty} \left[\left(\frac{1}{n}\right)e^{(1/n)} + \left(\frac{1}{n}\right)e^{2(1/n)} + \ldots + \left(\frac{1}{n}\right)e^{n(1/n)}\right]$ which can be interpreted as a Riemann sum with partitioning $\Delta x = \frac{1}{n} \Rightarrow \lim\limits_{x \to \infty} \frac{1}{n}\left[e^{1/n} + e^{2/n} + \ldots + e\right] = \int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$

7. $A(t) = \int_0^t e^{-x}\,dx = \left[-e^{-x}\right]_0^t = 1 - e^{-t}$, $V(t) = \pi\int_0^t e^{-2x}\,dx = \left[-\frac{\pi}{2}\,e^{-2x}\right]_0^t = \frac{\pi}{2}\left(1 - e^{-2t}\right)$

(a) $\lim\limits_{t \to \infty} A(t) = \lim\limits_{t \to \infty} \left(1 - e^{-t}\right) = 1$

(b) $\lim\limits_{t \to \infty} \frac{V(t)}{A(t)} = \lim\limits_{t \to \infty} \frac{\frac{\pi}{2}\left(1-e^{-2t}\right)}{1-e^{-t}} = \frac{\pi}{2}$

(c) $\lim\limits_{t \to 0^+} \frac{V(t)}{A(t)} = \lim\limits_{t \to 0^+} \frac{\frac{\pi}{2}\left(1-e^{-2t}\right)}{1-e^{-t}} = \lim\limits_{t \to 0^+} \frac{\frac{\pi}{2}\left(1-e^{-t}\right)\left(1+e^{-t}\right)}{\left(1-e^{-t}\right)} = \lim\limits_{t \to 0^+} \frac{\pi}{2}\left(1+e^{-t}\right) = \pi$

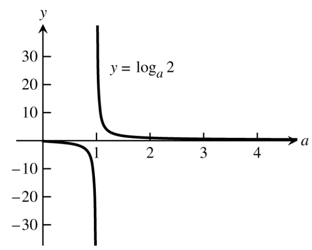
8. (a) $\lim\limits_{a \to 0^+} \log_a 2 = \lim\limits_{a \to 0^+} \frac{\ln 2}{\ln a} = 0$;

$\lim\limits_{a \to 1^-} \log_a 2 = \lim\limits_{a \to 1^-} \frac{\ln 2}{\ln a} = -\infty$;

$\lim\limits_{a \to 1^+} \log_a 2 = \lim\limits_{a \to 1^+} \frac{\ln 2}{\ln 1} = \infty$;

$\lim\limits_{a \to \infty} \log_a 2 = \lim\limits_{a \to \infty} \frac{\ln 2}{\ln a} = 0$

(b)



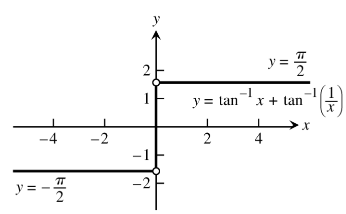
9. $A_1 = \int_1^e \frac{2\log_2 x}{x}\,dx = \frac{2}{\ln 2}\int_1^e \frac{\ln x}{x}\,dx = \left[\frac{(\ln x)^2}{\ln 2}\right]_1^e = \frac{1}{\ln 2}$; $A_2 = \int_1^e \frac{2\log_4 x}{4}\,dx = \frac{2}{\ln 4}\int_1^e \frac{\ln x}{x}\,dx$

$= \left[\frac{(\ln x)^2}{2\ln 2}\right]_1^e = \frac{1}{2\ln 2} \Rightarrow A_1 : A_2 = 2 : 1$

10. $y = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) \Rightarrow y' = \frac{1}{1+x^2} + \frac{\left(-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x^2}\right)}$

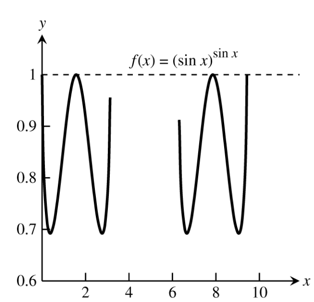
$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \Rightarrow \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ is a constant and the constant is $\frac{\pi}{2}$ for $x > 0$; it is $-\frac{\pi}{2}$ for $x < 0$ since $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ is odd. Next the $\lim\limits_{x \to 0^+} \left[\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right] = 0 + \frac{\pi}{2} = \frac{\pi}{2}$

and $\lim\limits_{x \to 0^-} \left(\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right) = 0 + \left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$



11. $\ln x^{(x^x)} = x^x \ln x$ and $\ln\left(x^x\right)^x = x \ln x^x = x^2 \ln x$; then, $x^x \ln x = x^2 \ln x \Rightarrow (x^x - x^2)\ln x = 0 \Rightarrow x^x = x^2$ or $\ln x = 0$.
$\ln x = 0 \Rightarrow x = 1$; $x^x = x^2 \Rightarrow x \ln x = 2 \ln x \Rightarrow x = 2$. Therefore, $x^{(x^x)} = \left(x^x\right)^x$ when $x = 2$ or $x = 1$.

12. In the interval $\pi < x < 2\pi$ the function $\sin x < 0$ $\Rightarrow (\sin x)^{\sin x}$ is not defined for all values in that interval or its translation by 2π.



13. $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}\,g'(x)$, where $g'(x) = \frac{x}{1+x^4} \Rightarrow f'(2) = e^0\left(\frac{2}{1+16}\right) = \frac{2}{17}$

14. (a) $\frac{df}{dx} = \frac{2\ln e^x}{e^x} \cdot e^x = 2x$

(b) $f(0) = \int_1^1 \frac{2\ln t}{t}\,dt = 0$

(c) $\frac{df}{dx} = 2x \Rightarrow f(x) = x^2 + C$; $f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = x^2 \Rightarrow$ the graph of $f(x)$ is a parabola

15. (a) $g(x) + h(x) = 0 \Rightarrow g(x) = -h(x)$; also $g(x) + h(x) = 0 \Rightarrow g(-x) + h(-x) = 0 \Rightarrow g(x) - h(x) = 0$
$\Rightarrow g(x) = h(x)$; therefore $-h(x) = h(x) \Rightarrow h(x) = 0 \Rightarrow g(x) = 0$

(b) $\frac{f(x)+f(-x)}{2} = \frac{[f_E(x)+f_O(x)]+[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)+f_E(x)-f_O(x)}{2} = f_E(x)$;
$\frac{f(x)-f(-x)}{2} = \frac{[f_E(x)+f_O(x)]-[f_E(-x)+f_O(-x)]}{2} = \frac{f_E(x)+f_O(x)-f_E(x)+f_O(x)}{2} = f_O(x)$

(c) Part b $\Rightarrow$ such a decomposition is unique.

16. (a) $g(0+0) = \frac{g(0)+g(0)}{1-g(0)\,g(0)} \Rightarrow [1-g^2(0)]\,g(0) = 2g(0) \Rightarrow g(0) - g^3(0) = 2g(0) \Rightarrow g^3(0) + g(0) = 0$
$\Rightarrow g(0)\,[g^2(0)+1] = 0 \Rightarrow g(0) = 0$

(b) $g'(x) = \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim\limits_{h\to 0} \frac{\left[\frac{g(x)+g(h)}{1-g(x)\,g(h)}\right]-g(x)}{h} = \lim\limits_{h\to 0} \frac{g(x)+g(h)-g(x)+g^2(x)\,g(h)}{h\,[1-g(x)\,g(h)]}$
$= \lim\limits_{h\to 0} \left[\frac{g(h)}{h}\right]\left[\frac{1+g^2(x)}{1-g(x)\,g(h)}\right] = 1\cdot[1+g^2(x)] = 1+g^2(x) = 1+[g(x)]^2$

(c) $\frac{dy}{dx} = 1+y^2 \Rightarrow \frac{dy}{1+y^2} = dx \Rightarrow \tan^{-1} y = x + C \Rightarrow \tan^{-1}(g(x)) = x + C$; $g(0) = 0 \Rightarrow \tan^{-1} 0 = 0 + C$
$\Rightarrow C = 0 \Rightarrow \tan^{-1}(g(x)) = x \Rightarrow g(x) = \tan x$

17. $M = \int_0^1 \frac{2}{1+x^2}\,dx = 2\left[\tan^{-1} x\right]_0^1 = \frac{\pi}{2}$ and $M_y = \int_0^1 \frac{2x}{1+x^2}\,dx = \left[\ln(1+x^2)\right]_0^1 = \ln 2 \Rightarrow \bar{x} = \frac{M_y}{M}$
$= \frac{\ln 2}{\left(\frac{\pi}{2}\right)} = \frac{\ln 4}{\pi}$; $\bar{y} = 0$ by symmetry

18. (a) $V = \pi \int_{1/4}^4 \left(\frac{1}{2\sqrt{x}}\right)^2 dx = \frac{\pi}{4}\int_{1/4}^4 \frac{1}{x}\,dx = \frac{\pi}{4}\left[\ln|x|\right]_{1/4}^4 = \frac{\pi}{4}\left(\ln 4 - \ln\frac{1}{4}\right) = \frac{\pi}{4}\ln 16 = \frac{\pi}{4}\ln(2^4) = \pi \ln 2$

(b) $M_y = \int_{1/4}^4 x\left(\frac{1}{2\sqrt{x}}\right) dx = \frac{1}{2}\int_{1/4}^4 x^{1/2}\,dx = \left[\frac{1}{3}x^{3/2}\right]_{1/4}^4 = \left(\frac{8}{3} - \frac{1}{24}\right) = \frac{64-1}{24} = \frac{63}{24}$;
$M_x = \int_{1/4}^4 \frac{1}{2}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right) dx = \frac{1}{8}\int_{1/4}^4 \frac{1}{x}\,dx = \left[\frac{1}{8}\ln|x|\right]_{1/4}^4 = \frac{1}{8}\ln 16 = \frac{1}{2}\ln 2$;
$M = \int_{1/4}^4 \frac{1}{2\sqrt{x}}\,dx = \int_{1/4}^4 \frac{1}{2}x^{-1/2}\,dx = \left[x^{1/2}\right]_{1/4}^4 = 2 - \frac{1}{2} = \frac{3}{2}$; therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{63}{24}\right)\left(\frac{2}{3}\right) = \frac{21}{12} = \frac{7}{4}$ and
$\bar{y} = \frac{M_x}{M} = \left(\frac{1}{2}\ln 2\right)\left(\frac{2}{3}\right) = \frac{\ln 2}{3}$

19. (a) $L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right) \Rightarrow \frac{dL}{d\theta} = k\left(\frac{b\csc^2\theta}{R^4} - \frac{b\csc\theta\cot\theta}{r^4}\right)$; solving $\frac{dL}{d\theta} = 0$
$\Rightarrow r^4 b\csc^2\theta - bR^4\csc\theta\cot\theta = 0 \Rightarrow (b\csc\theta)(r^4\csc\theta - R^4\cot\theta) = 0$; but $b\csc\theta \neq 0$ since
$\theta \neq \frac{\pi}{2} \Rightarrow r^4\csc\theta - R^4\cot\theta = 0 \Rightarrow \cos\theta = \frac{r^4}{R^4} \Rightarrow \theta = \cos^{-1}\left(\frac{r^4}{R^4}\right)$, the critical value of θ

(b) $\theta = \cos^{-1}\left(\frac{5}{6}\right)^4 \approx \cos^{-1}(0.48225) \approx 61^\circ$

20. In order to maximize the amount of sunlight, we need to maximize the angle θ formed by extending the two red line segments to their vertex. The angle between the two lines is given by $\theta = \pi - (\theta_1 + (\pi - \theta_2))$. From trig we have $\tan\theta_1 = \frac{350}{450-x} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{350}{450-x}\right)$ and $\tan(\pi - \theta_2) = \frac{200}{x} \Rightarrow (\pi - \theta_2) = \tan^{-1}\left(\frac{200}{x}\right)$
$\Rightarrow \theta = \pi - (\theta_1 + (\pi - \theta_2)) = \pi - \tan^{-1}\left(\frac{350}{450-x}\right) - \tan^{-1}\left(\frac{200}{x}\right)$
$\Rightarrow \frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{350}{450-x}\right)^2}\cdot\frac{350}{(450-x)^2} - \frac{1}{1+\left(\frac{200}{x}\right)^2}\cdot\left(-\frac{200}{x^2}\right) = \frac{-350}{(450-x)^2+122500} + \frac{200}{x^2+40000}$
$\frac{d\theta}{dx} = 0 \Rightarrow \frac{-350}{(450-x)^2+122500} + \frac{200}{x^2+40000} = 0 \Rightarrow 200\Big((450-x)^2 + 122500\Big) = 350(x^2 + 40000)$
$\Rightarrow 3x^2 + 3600x - 1020000 = 0 \Rightarrow x = -600 \pm 100\sqrt{70}$. Since $x > 0$, consider only $x = -600 + 100\sqrt{70}$.
Using the first derivative test, $\left.\frac{d\theta}{dx}\right|_{x=100} = \frac{9}{3500} > 0$ and $\left.\frac{d\theta}{dx}\right|_{x=400} = \frac{-9}{5000} < 0 \Rightarrow$ local max when
$x = -600 + 100\sqrt{70} \approx 236.67$ ft.