

April 3, 2018
Time : 55 minutes
Spring 2017-18

MATHEMATICS 218
QUIZ II

NAME Key (Ver. 1)
ID# -----

Circle your section number :

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
8 M	11 M	2 M	12 M	3 M	4 M	8 F	11 M	11 F	2 T	3:30 T	5 T	1 M	3 M	4 M

PROBLEM GRADE

PART I

1 ----- / 16

2 ----- / 16

3 ----- / 16

4 ----- / 8

PART II

5	6	7	8	9	10	11	12
a	a	a	a	a	a	a	a
(b)	(b)	(b)	b	b	b	b	b
c	c	c	c	c	c	c	c
d	d	d	(d)	d	d	(d)	d
e	e	e	e	(e)	e	e	e

5-12 ----- / 32

PART III

13	14	15	16	17	18
(T)	T	T	(T)	(T)	T
F	(F)	(F)	F	F	(F)

13-18 ----- / 12

TOTAL ----- / 100

PART I. Answer each of the following problems in the space provided for each problem (Problem 1 to Problem 4).

1. Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 1 & 0 & 3 \\ 2 & 0 & 6 \end{pmatrix}$

(a) Find a basis for the null space $N(A)$ of A. (Do not prove it is a basis)

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 2 & 1 & 6 & 0 \\ 1 & 0 & 3 & 0 \\ 2 & 0 & 6 & 0 \end{array} \right] \xrightarrow{\begin{array}{l} -2R_1 + R_2 \\ -R_1 + R_3 \\ -2R_1 + R_4 \end{array}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad [8 \text{ points}]$$

$$x + 3z = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow x, y \text{ are leading variables} \\ y = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad z = t = \text{Parameter}$$

$$x = -3z = -3t \\ N(A) = \left\{ \begin{pmatrix} -3t \\ 0 \\ t \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$\text{Basis for } N(A) = \left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{Clearly } \begin{pmatrix} -3t \\ 0 \\ t \\ 1 \end{pmatrix} = t \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

(b) Find a basis of the column space $\text{Col}(A)$ of A. (Do not prove it is a basis)

[8 points]

Columns 1 & 2 in row-echelon form contain pivots
 \Rightarrow Corresponding columns 1 & 2 in the original matrix
 form a basis of the column space $\text{Col}(A)$
 \Rightarrow Basis of $\text{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

2. Prove or disprove the following statements:

(a) Let $W = \{p(x) \in P_2 \mid xp'(x) + 1 = p(x)\}$. Then W is a subspace of P_2

[8 points]

FALSE

The zero polynomial of P_2 , $p(x) = 0 \notin W$ since
 $x \cdot p'(x) + 1 = x \cdot 0 + 1 = 1 \neq 0$

(Can also give an example showing not closed
under addition)

TRUE (b) Let $V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \mid b = (a+c)/2 \right\}$. Then V is a subspace of \mathbb{R}^3 .

1. $V \neq \emptyset$ since $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V$

[8 points]

2. Closed under addition:

If $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \in V$, Then $b = \frac{a+c}{2}$, $b' = \frac{a'+c'}{2}$

$$\text{So } \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \\ c+c' \end{pmatrix} \text{ where } b+b' = \frac{a+c}{2} + \frac{a'+c'}{2} = \frac{(a+a')+(c+c')}{2} \in V$$

3. Closed under scalar multiplication

If $k \in \mathbb{R} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V$, Then

$$b = \frac{a+c}{2} \text{ and}$$

$$k \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix} \in V \text{ since } kb = k \left(\frac{a+c}{2} \right) = \frac{ka+kc}{2}$$

So V is a subspace

3. Consider the subspace $W = \{p(x) \in P_3 \mid p(1) = 0\}$ of P_3 .

Let $B = \{x-1, x^2-1, x^3-1\}$

(a) Prove that B is linearly independent

$$\begin{aligned} & c_1(x-1) + c_2(x^2-1) + c_3(x^3-1) = 0 && [8 \text{ points}] \\ & \Rightarrow c_3x^3 + c_2x^2 + c_1x - (c_1 - c_2 - c_3) = 0 \text{ identical to } 0 \\ & \Rightarrow c_3 = c_2 = c_1 = 0 \\ & \text{So } B \text{ is } \underline{\text{linearly independent}} \\ & \text{So } B \text{ forms a basis for } \text{Span}(B) \\ & \times \dim(\text{Span}(B)) = 3 \end{aligned}$$

(b) Prove that B is a basis for W

$B \subset W$ since $p(1) = 0$ for all $p(x) \in B \Rightarrow \text{Span } B \subseteq W$ [8 points]

Enough to show that $\text{Span } B = W$

$$\begin{aligned} & \text{If } p(x) = ax^3 + bx^2 + cx + d \in W, \text{ Then} \\ & p(1) = a + b + c + d = 0 \Rightarrow p(x) = ax^3 + bx^2 + cx - (a+b+c) \\ & \left. \begin{array}{l} a=1, b=0, c=0 \Rightarrow p(x) = x^3 - 1 \\ a=0, b=1, c=0 \Rightarrow p(x) = x^2 - 1 \\ a=0, b=0, c=1 \Rightarrow p(x) = x - 1 \end{array} \right\} \text{This shows that these vectors} \\ & \text{span (see below)} \\ & \text{and } p(x) = a(x^3-1) + b(x^2-1) + c(x-1) \in \text{Span}(B) \end{aligned}$$

So $\text{Span}(B) = W$

$\Rightarrow B = \{x-1, x^2-1, x^3-1\}$ is a basis for W

4. If $T: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a linear transformation with $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Prove that T is the zero linear transformation. (That is show $T(v)=0$ for every $v \in \mathbb{R}^2$)

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = T\begin{pmatrix} 2 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \textcircled{1} \quad [8 \text{ points}]$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = T\begin{pmatrix} 1 \\ 2 \end{pmatrix} - T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \textcircled{0}$$

For any $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$$\begin{aligned} T(v) &= T\begin{pmatrix} x \\ y \end{pmatrix} \\ &= T(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\ &\stackrel{\perp}{=} x \cdot T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= x \cdot \textcircled{1} + y \cdot \textcircled{0} \\ &= \textcircled{0} \end{aligned}$$

PART II. Circle the correct answer for each of the following problems
(Problem 5 to Problem 12) IN THE TABLE OF THE FRONT PAGE.
[4 points for each correct answer (No penalty)].

5. Let A be $m \times n$ matrix and let R be a row echelon form of A .

Which one of the following is FALSE?

- (a) $N(A) = N(R)$
- (b) $\text{col}(A) = \text{col}(R)$
- (c) If columns 1 and 2 of R are linearly dependent then columns 1 and 2 of A are linearly dependent.
- (d) The nonzero rows of R form a basis of $\text{row}(A)$

[4 points]

6. The subspace W of \mathbb{R}^4 spanned by $S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \\ 0 \end{pmatrix} \right\}$ has a dimension equal to

- (a) 1
- (b) 2
- (c) 3
- (d) 4
- (e) None of the above

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ -2 & 3 & 4 & 4 \\ 3 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[3R_1 + R_3]{2R_1 + R_2} \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 3 & 6 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{Row reduction}} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- [4 points]
7. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation such that $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Then, $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$$= T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ $= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

- (e) None of the above

[4 points]

8. Let $S = \{x+1, x^2+x, x^3+x^2\} \subset P_3$. Which one of the following is TRUE:

- (a) S is a basis of P_3
- (b) S is linearly dependent
- (c) the constant polynomial 1 belongs to $\text{Span}(S)$
- (d) $\text{Span}(S)$ is a subspace of P_3 of dimension 3
- (e) None of the above

$S \Rightarrow$ linearly independent

[4 points]

9. Let $S = \{u, v, w, z\}$ be a linearly independent subset of a vector space V . Which one of the following statements is FALSE:

- (a) The dimension of V is at least 4
- (b) The set $\{u, v, z\}$ is linearly independent
- (c) $\text{Span}(S)$ has dimension 3
- (d) S is a basis of $\text{Span}(S)$

[4 points]

10. Let A be $m \times n$ matrix with $m < n$. Which one of the following statements is FALSE?

- (a) The columns of A are linearly dependent.
- (b) $\text{col}(A)$ is a subspace of \mathbb{R}^m .
- (c) $\text{row}(A)$ is a subspace of \mathbb{R}^n .
- (d) $\dim(\text{N}(A)) \neq 0$
- (e) The rows of A are linearly independent.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad 2 \times 3$$

[4 points]

11. Let $W = \{p(x) \in P_2 \mid p(1)x^2 + p(-1)x + p(0) = 0\}$, then $\dim W =$

- (a) 3
- (b) 2
- (c) 1
- (d) 0
- (e) None of the above

$$\begin{aligned} p(x) &= ax^2 + bx + c \\ p(1) &= a + b + c, \quad p(-1) = a - b + c \\ (a+b+c)x^2 + (a-b+c)x + c &= 0 \\ a+b+c=0 &\Rightarrow a=-b \\ a-b+c=0 &\Rightarrow a=b \\ c=0 & \end{aligned}$$

[4 points]

12. Let U be the subset of P_3 given by

$$U = \{p(x) = ax^3 + bx^2 + cx + d \in P_3 \mid p''(x) = 0\}.$$

Then $\dim U =$

- (a) 0
- (b) 1
- (c) 2
- (d) 3
- (e) 4

$$\begin{aligned} p'(x) &= 3ax^2 + 2bx + c \\ p''(x) &= 6ax + 2b = 0 \\ \Rightarrow a=0, b=0 & \end{aligned}$$

[4 points]

PART III. Answer TRUE or FALSE only: (Questions 13 to 18) IN THE TABLE OF THE FRONT PAGE [2 points for each correct answer, NO PENALTY]

T

13. If A and B are two $n \times n$ matrices such that $AB = 3I$, then $\text{Col}(A) = \mathbb{R}^n$. *A is invertible*

F

14. If $S = \{v\}$, then S is linearly independent. *If $v=0$, then S is dependent*

F

15. A subspace W of a vector space V is linearly independent. *$\oplus c_i W$*

T

16. The set of polynomials $\{1+x, x^2+3, x^2+x\}$ is a basis for P_2 .

T

17. If A is a 3×5 matrix, then the columns of A are linearly dependent. *5 vectors in \mathbb{R}^3*

F

18. $T: P_2 \rightarrow \mathbb{R}$ defined by $T(ax^2+bx+c) = |a|$ is a linear transformation

$$\begin{aligned} T(x^2 + (-x^2)) &= T(0) = 0 \\ T(x^2) + T(-x^2) &= 1 + 1 = 2 \end{aligned}$$

[12 points]