MATH 201

EXAM II REVIEW

(1) Testing convergence or divergence of $\sum a_n$

We do, in order;

1) Always simplify expression or split (for ex, rationalize, write $n^{1/n} \to e^{\frac{1}{n}\ln n}$ ...)
2) Check if it's geometric or telescoping
3) N^{th} term test [Remark: for series that look like $\sum(-1)^n u_n$, $u_n > 0$ If $u_n \not\to 0$, the limit doesn't exist]

4) Here we can have

- $a_n > 0$

(a) Integral test

- check $a_n \searrow$

typical example $\sum \frac{1}{n(\ln n)^p}$

(b) LCT, DCT

When we can think of dominating terms at ∞

When we can compare rates of growth of functions

(Remember things from LCT of integrals, $\ln x < x^{\text{any power}}$, $|\sin x| < 1$, $\sin x < x$, ⋮)

(c) Root & Ratio test. When we see $(\ )^n$ with $n!$

Remark: $\rho = 1$ inconclusive

- $\sum(-1)^n u_n$, $u_n > 0$

$(-1)^n$ could be $\cos n\pi$, $\sin(2n+1)\frac{\pi}{2}$

$\to u_n$ decreasing

$\to \lim_{n\to\infty} u_n = 0$

for u_n decreasing:

(a) $u_n = \frac{1}{b_n}$ b_n increasing ex $\frac{1}{n \cdot \ln n}$

(b) $\frac{u_{n+1}}{u_n} < 1$

(c) $u_{n+1} - u_n < 0$

(d) $f(x) = \ldots$
$f'(x) < 0$

Use Leibniz to say it converges

- $\sum a_n$ a_n can be of any sign/type

ex: $\sum \frac{\sin n}{n^2+1}$

Here we check absolute convergence directly

(2) Testing absolute and conditional convergence

(a) alternating series

$\sum (-1)^n u_n$, $u_n > 0$

Test absolute convergence first by testing $\sum u_n$ as described previously

~~If test~~

If $\sum u_n$ doesn't converge, go back to $\sum (-1)^n u_n$ and see if Leibniz works

(b) series of +ve terms

$\sum a_n$, $a_n > 0$

$\rightsquigarrow$ absolute convergence is same as convergence

$\rightsquigarrow$ do same as before on item (1)

(c) general

$\sum a_n$, a_n can be anything

$\to$ test $\sum |a_n|$ using previous tests

③ Find the sum $\sum a_n$

case (A) Geometric Series.

$$\sum_{n=1}^{\infty} a r^{n-1} = \sum_{n=0}^{\infty} a r^n = \lim_{n\to\infty} \sum_{k=0}^{n-1} a r^k$$

$$= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} \quad \text{when } |r|<1,$$

diverges $|r|\geq 1$

⇝ Sometimes we get $\sum_{n=k}^{\infty} a r^n$

↳ doesn't always start at 0 (imagine $k=3,4,\ldots$)

Here we can do:

(1) Reindexing $\sum_{n=k}^{\infty} a r^n = \sum_{n=0}^{\infty} a r^{n+k} = r^k \sum_{n=0}^{\infty} a r^n$

$$= r^k \frac{a}{1-r}, \ |r|<1$$

(2) $$\sum_{n=k}^{\infty} a r^n = \left(\sum_{n=0}^{\infty} a r^n\right) - \left\{a + ar + \cdots + ar^{k-1}\right\}$$

$$= \frac{a}{1-r} - \left\{a + ar + \cdots + ar^{k-1}\right\}$$

for $|r|<1$

Case B) Telescoping series

⇝ Either we see it directly: Ex $\sum \frac{1}{n} - \frac{1}{n+1}$

$\sum \ln\left(\frac{n}{n+1}\right)$

⇝ or we use partial fractions to see it

$\sum \frac{1}{n} - \frac{1}{n+2}$

In any case, we carefully write

$S_n = \sum_{k=1}^{n} a_k$ and see the cancelations

then we compute $\lim_{n\to\infty} S_n = L = \sum_{k=1}^{\infty} a_k$

(4) Using geometric series, write a number with repeating decimals as a fraction.
See Exercise 4 page 547 in Book or Class notes.

(5) Error Estimation for alternating series

Question: <u>Estimate</u> the <u>error</u> if we approximate

$$\sum_{k=1}^{\infty} a_k \quad \text{by} \quad \sum_{k=1}^{N} a_k = S_{\underline{N}} \leftarrow \text{some fixed number like } 5, 10\ldots$$

For alternating series, $\sum_{k=1}^{\infty} (-1)^{k-1} u_k$, $u_k > 0$ that are convergent, there's a theorem which says:

$$|\text{error}| = \Bigg| \underbrace{\overbrace{\underset{\substack{\downarrow \\ L \\ \text{unknown in most cases}}}{\sum_{k=1}^{\infty} (-1)^{k-1} u_k} - \underbrace{\sum_{k=1}^{N} (-1)^{k-1} u_k}_{\substack{S_N \\ \text{known}}}}^{\text{difference between the infinite sum and the finite sum}}}_{} \Bigg| < \underbrace{u_{N+1}}_{\substack{\text{upper bound for} \\ \text{error is} \\ 1^{st} \text{ unused} \\ \text{term.} \\ (u_{N+1} > 0)}}$$

$$|\text{error}| \overset{\text{or}}{=} |L - S_N| < u_{N+1}$$

In some particular cases, like geometric series, we can explicitly compute $L = \sum_{k=1}^{\infty} a_k$, so we know the error $|L - S_N|$ excatly! (above we have an upper bound for it)

→ See example from previous

(6) Power Series

Series of the form $\sum_{k=0}^{\infty} C_k (x-a)^k$ ↳ Real variable.

- Types of Q: 1) for what values of x does series converge
  2) Find the interval of convergence
  3) Find the radius of "
  4) Find the interval of absolute convergence
  5) Find the points where series is conditionally convergent.

<u>Method:</u> Let $u_n = C_n (x-a)^n$

Let $\rho = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$ will always ~~look~~ like $= \lim_{n\to\infty} b_n \cdot |x-a|$ → $\left|\frac{C_{n+1}}{C_n}\right|$

We can have <u>3 cases here</u>, depending on C_n:

case(1)

$\lim_{n\to\infty} b_n = \infty$

Then $\rho = \infty$ unless $x-a=0$ or $x=a$

So $\begin{cases} \rho > 1 & \text{if } x \neq a \text{. Series diverge} \\ \rho = 0 < 1 & \text{if } x = a \text{. Series is [redacted] convergent at a point } x=a \end{cases}$

So $R=0$
Radius is zero

case(2)

$\lim_{n\to\infty} b_n = 0$

$\Rightarrow \rho = 0 < 1$ no matter what $x \in \mathbb{R}$ is!

$\Rightarrow R = \infty$ and series is <u>absolutely convergent</u> <u>everywhere</u> on the real line.

case (3)

$\lim_{n\to\infty} b_n =$ some finite number $= c$

We then need the condition

$\rho = \boxed{c \cdot |x-a| < 1}$

to get abs. convergence

- From that condition we solve for x and get the radius R.
- Then WE <u>MANUALLY</u> <u>TEST the ENDpoints</u>

i.e. we plug in $x = a \pm R$ into series and see what we get...

(7) Other various questions that need some analysis.

But ALWAYS, if you claim something is true in all cases, PROVE IT.

OTHERWISE, give COUNTEREXAMPLES.

- Remember $\sum a_n$ converges $\Rightarrow a_n \to 0$
  but the converse is NOT always true

- Review theorems in section 10.2 on combining series
- Review Problem 1 in Hamdan's previous.

Good luck!