Extra Sheet: (p: 602 study.)

1) a- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots = \sum_0^\infty \frac{x^k}{k!}$ all $x \in \mathbb{R}$

$xe^x = x + x^2 + \frac{x^3}{2!} \dots = x \sum_0^\infty \frac{x^k}{k!} = \sum \frac{x^{k+1}}{k!}$

* 2 ways to prove convergence:

Ratio test

1) $\lim_{k \to \infty} \left| \frac{x^{k+2}}{(k+1)!} \cdot \frac{k!}{x^{k+1}} \right|$

$= 0 < 1 \ \forall x \in \mathbb{R}$

Series Multiplication theorem

2) Proved $1 + x + \frac{x^2}{2} \dots = e^x$ all $x \in \mathbb{R}$

$x = x \ \forall x$ all $x \in \mathbb{R}$.

Series multiplication theorem. (p:580)

$\Rightarrow$ the product of the 2 series converge to $e^x . x$

b- $\sum_0^\infty \frac{n+1}{n!} = \sum \frac{n}{n!} + \sum \frac{1}{n!} = \sum_{n=1} \frac{1}{(n-1)!} + \sum_1^\infty \frac{1}{n!}$

We know $xe^x = \sum_0^\infty \frac{x^{k+1}}{k!} = \sum_m^\infty \frac{x^m}{k-1!}$

$\Rightarrow$ for $x = 1$, we get $1 . e^1 = \sum_{k=0}^\infty \frac{1}{k!}$

& $1 . e^1 = \sum_{k=1}^\infty \frac{1}{(k-1)!}$

$\Rightarrow \sum_0^\infty \frac{n+1}{n!} = 2e.$

5> a) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{1 + x - e^x}$

$= \lim\limits_{x \to 0} \dfrac{1 - \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right]}{1 + x - \left\{1 + x + \frac{x^2}{2!} \dots\right]}$

$= \lim\limits_{x \to 0} \dfrac{x^2/2! + x^4/4! \dots}{-x^2/2! - x^3/3!}$

$= \lim\limits_{x \to 0} \dfrac{\cancel{x^2/2!} \left[1 \cdot \overbrace{\frac{x^2 \cdot 2!}{4!}}^{0} \dots\right]}{\cancel{x^2/2!} \left[-1 - \underbrace{\frac{x \cdot 2!}{3!}}_{0} \dots\right]} = \dfrac{1}{-1} = -1$

b> Solved in class.

4> $f(x) = \dfrac{1}{7 - 2x} = \dfrac{1}{7 - 2(x - 3 + 3)}$

$= \dfrac{1}{7 - 2(x-3) + 6}$

$= \dfrac{1}{1 - 2(x-3)} = \dfrac{1}{1-r}$

$$= \sum_{K=0}^{\infty} [2(x-3)]^K$$

$$r = 2(x-3)$$

$$|r| < 1 \Rightarrow \text{ if } |2x-3| < 1$$

5) a) $f(x) = \frac{1}{1-x} = \sum_0^\infty x^k$ for $|x| < 1$

We need to get from $\frac{1}{1-x}$ to $\tan^{-1} x^2$

First note $\int \frac{2x}{1+x^4}\,dx = \tan^{-1} x^2 + C$

and that $2x \cdot \frac{1}{1-(-x^4)} = \frac{2x}{1+x^4}$.

So here's how the solution goes:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x| < 1$$

Substitute $-x^4$

$$\frac{1}{1+x^4} = \sum_{k=0}^{\infty} (-x^4)^k \qquad |x^4| < 1$$

multiply by $2x$

$$\frac{2x}{1+x^4} = \sum_0^{\infty} (-x^4)^k 2x \qquad |x^4| < 1$$

$$= 2\sum (-1)^k x^{4k+1}$$

Now term by term integration:

$$\tan^{-1} x^2 = \int \frac{2x}{1+x^4}\,dx = \int 2\sum (-1)^k x^{4k+1}\,dx$$

$$= 2\sum (-1)^k \int x^{4k+1}\,dx$$

$$\boxed{\tan^{-1} x^2 = 2\sum (-1)^k \frac{x^{4k+2}}{4k+2}} \quad \text{for } |x| < 1$$

We can also prove that at $x = \pm 1$, the series converge by Leibniz. $\Rightarrow$ it converges for $|x| \le 1$

this point is essential for part b).

b) $\tan^{-1} x^2 = 2 \sum (-1)^k \cdot \frac{x^{4k+2}}{4k+2}$ $\quad |x| \leq 1$

at $x = 1$.

$\Rightarrow \tan^{-1} 1 = 2 \sum (-1)^k \frac{1}{4k+2}$

$\Rightarrow \frac{\tan^{-1} 1}{2} = \sum \frac{(-1)^k}{4k+2}$

7) a) e^{-x^2}

$$e^x = 1 + x + \frac{x^2}{2} \dots = \sum_{n=0}^{\infty} \frac{x^k}{k!} \quad x \in \mathbb{R}$$

Substitute $-x^2$ for x $\Rightarrow$ $$e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2!} \dots = \sum_{n=0}^{\infty} \frac{(-x^2)^k}{k!} \quad |x^2| \in \mathbb{R} \Rightarrow x \in \mathbb{R}$$

$$= 1 - x^2 - \frac{x^4}{2!} \dots - \sum_{n=0}^{\infty} - \frac{x^{2k}}{k!}$$

$$\boxed{e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} \quad \text{all } x \in \mathbb{R}}$$

plug in

$x = 1$ $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \left\{1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} \dots \right\}.$$

$$= e^{-1}$$

Now, $$\sum_{1}^{\infty} \frac{(-1)^n (2n)}{n!} = +2 \sum_{1}^{\infty} \frac{(-1)^n}{(n-1)!}$$

$$= 2 \left\{-1 + \frac{1}{1} - \frac{1}{2!} + \frac{1}{3!} \dots \right\}$$

$$= -2 \left\{1 - 1 + \frac{1}{2!} + \dots \right\}$$

$$= -2 e^{(-1)^2} = -\frac{2}{e}$$