Problem 1: (8%)

(a) Evaluate the following improper integral

$$\int_0^4 \frac{x}{\sqrt{16 - x^2}} \, dx$$

(b) Determine the convergence or divergence of

$$\int_2^\infty \frac{1}{\sqrt[4]{x^4 - x}} \, dx$$

Problem 2: (12%) Consider the following series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

(a) Find S_{20} and simplify your answer.

(b) Find the infinite sum
$$\sum_{n=1}^{\infty} \ln \left(\frac{n}{n+1} \right)$$
.

Problem 3: (20%) Which series converge? Converge conditionally? Diverge? Justify your answer.

(a)
$$\sum_{n=1}^{\infty} \frac{(\sin n) \ln n}{n^2}$$

(b)
$$\sum_{n=1}^{\infty} \left(1 - \frac{49}{n^2}\right)^n$$

(c)
$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$$

(d)
$$\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)^2}$$

Problem 4: (8%) Find the interval of convergence of:

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x-2)^n}{4^n \ln n}$$

Problem 5: (12%)

(a) Express the indefinite integral $\int e^{-x^3} dx$ as an (alternating) power series.

(b) Deduce the definite integral $\int_0^{0.1} e^{-x^3} dx$ as a series.

(c) Find the appropriate value of n for which Sn approximates the definite integral with an error of magnitude less than or equal to 10^{-11} .

Problem 6: (8%) Consider $f(x) = x^2 e^x$.

(a) Express f(x) as a power series.

(b) Deduce a value for the sum

$$\sum_{n=1}^{\infty} \frac{n+2}{n!}$$

(You may use term by term integration or differentiation on your result in part (a)).

Problem 7: (4%) Find the following limit using Maclaurin series

$$\lim_{n \to \infty} \left(n^2 \ln\left(1 + \frac{1}{n}\right) - n \right)$$

Problem 8: (8%) Evaluate the following or show that the limit does not exist

(a)
$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^6}{x^3}$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

Problem 9: (6%) Consider the following function

$$f(x,y) = e^{10y} \cos\left(10x\right)$$

Verify whether f(x, y) satisfies

$$f_{xx} + f_{yy} = 0$$

(This is called Laplace's equation)

Problem 10: (6%) Consider the following function

$$f(x,y) = \sqrt{49 - x^2 - y^2}$$

(a) Find the domain and range of this function.

(b) Find the equation of the level curve of f(x, y) that passes through the point (2, 5).

Problem 11: (8%) Evaluate the integral

$$\int_0^1 \int_0^1 e^{(2x+3y)} \, dx \, dy$$