KEY

LEBANESE AMERICAN UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS

MTH 201 - Calculus III

Exam-I,Fall 2012

Duration: 60 minutes

INSTRUCTIONS: This exam consists of 8 pages and 6 problems. Check that none is missing. Answer the questions in the space provided for each problem; if more space is needed, you may use the back pages. To receive full credits, you have to justify your answers.

Student's Name: ____________________

Student's ID: ____________________

Select your instructor and section

(a) Dr. Hamdan: (b) Dr. Issa MWF

(c)Dr. Issa T-TR (d) Dr. Touma

Grading scheme
(Keep it empty)

Question 1	/10
Question 2	/14
Question 3	/25
Question 4	/14
Question 5	/30
Question 6	/7
Total	/100

Problem 1: (10%) Evaluate the following integrals

$$\int \frac{e^x}{e^{2x}-1}\,dx$$

let $u = e^x \qquad du = e^x\,dx$

$$\int \frac{e^x}{e^{2x}-1}\,dx = \int \frac{du}{u^2-1} = \int \frac{du}{(u-1)(u+1)} = \frac{1}{2}\left[\int \frac{1}{u-1}\,du - \int \frac{du}{u+1}\right]$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

6 pts

$$\int \sinh(2\ln t)\,dt$$

$$\int \sinh(2\ln t)\,dt = \int \frac{e^{2\ln t} - e^{-2\ln t}}{2}\,dt = \int \frac{e^{\ln t^2} - e^{\ln t^{-2}}}{2}$$

$$= \frac{1}{2}\int \left\{t^2 - \frac{1}{t^2}\right\} dt$$

$$= \frac{1}{2}\left\{\frac{t^3}{3} - \frac{t^{-2+1}}{-2+1}\right\} + C$$

4 pts

Problem 2: (14%) Evaluate the following improper integrals

$\int_2^\infty \frac{1}{\sqrt{t}(4+t)}\,dt$ Improper type ①, explicit

$u = \sqrt{t}$ $\quad du = \frac{1}{2\sqrt{t}}\,dt = \quad du = \frac{1}{2u}\,dt \Rightarrow dt = 2u\,du$

$\int \frac{1}{\sqrt{t}(4+t)}\,dt = \int \frac{2\cancel{u}\,du}{\cancel{u}(4+u^2)} = 2\left[\frac{1}{2}\tan^{-1} u/2\right] + C$

No limits $\qquad a=2 \qquad = \tan^{-1}\sqrt{t}/2$

$\lim_{b\to\infty} \int_2^b \frac{1}{\sqrt{t}(4+t)}\,dt = \lim_{b\to\infty} \tan^{-1}\sqrt{t}/2\Big]_2^b = \boxed{\pi/2 - \tan^{-1}\sqrt{2}/2}$

7pts

$\int_1^e \frac{1}{x\ln x}\,dx$ type ②, explicit

let $u = \ln x \quad du = 1/x\,dx$

$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln x| + C$

$\lim_{b\to 1^+} \int_b^e \frac{1}{x\ln x}\,dx = \lim_{b\to 1^+} \ln|\ln x|\Big]_b^e = -(-\infty) = +\infty$

7pts

Problem 3: (25%) Determine if each of the following improper integrals converges or diverges. Justify your answer and precise the test you used.

$$\int_0^\infty \frac{1}{x+e^x}dx$$

type ① comparison

let $g(x) = 1/e^x$

LCT with $g(x) = 1/e^x$ $\quad \lim f(x)/g(x) = \lim 1/x+e^x \cdot e^x$

$= \lim \frac{1}{1+xe^{-x}}$

$= 1. \quad \neq 0, \neq \infty$

Test applies

$\int_0^\infty 1/e^x dx = \int_0^\infty e^{-x} = -e^{-x}\Big|_0^{\infty \text{ limit}}$

$\rightarrow 0 - (-e^0)$

$\rightarrow 1$

$\Rightarrow$ Integral converges $\Rightarrow$ Original converges by LCT.

7pts

$$\int_1^\infty \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx$$

type① explicit

Let $u = \sqrt{x}$ $\quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$ $\quad \Rightarrow \frac{1}{\sqrt{x}}dx = du \cdot 2$

$\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx = \int \frac{2du}{e^u} = 2\int e^{-u}du = -2e^{-u}$

$= -2e^{-\sqrt{x}}$

$\int_1^\infty \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx = \lim_{b\to\infty} -2e^{-\sqrt{x}}\Big|_1^b = 0 + 2e^{-1}$

Converges

7pts

$\int_2^\infty \frac{x}{\ln x} dx$ Improper type (1)

Divergence test

$x/\ln x \to \infty$ as $x \to \infty$

6 pts

$\Rightarrow$ Integral diverges

$\int_1^\infty \sin\left(\frac{1}{x}\right) dx$. Hint: First find $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$

Since $\lim_{x\to\infty} \frac{\overbrace{\sin 1/x}^{=f}}{\underbrace{\boxed{1/x}}_{=g}} = 1$ $\quad \lim_{y\to 0} \frac{\sin y}{y} \to 1$

then $\int f(x)\,dx$ & $\int g(x)\,dx$ behave alike

by LCT $\Rightarrow$ Since $\int_1^\infty \frac{1}{x} dx$ diverges p-int $p=1$

then $\int_1^\infty \sin\left(\frac{1}{x}\right) dx$ diverges

5pts

Problem 4: (14%) Find the limit of each of the following sequences.

$$a_n = \left(\frac{n+4}{n+5}\right)^n$$

1^∞ is undetermined

$$a_n = \left[\frac{\cancel{n}(1+4/n)}{\cancel{n}(1+5/n)}\right]^n = \frac{(1+4/n)^n \to e^4}{(1+5/n)^n \to e^5} \quad \text{by Rule \#--}$$

$$\Rightarrow a_n \to e^4/e^5 = 1/e$$

10pts

$$b_n = \frac{n2^n}{3^n}$$

$n\,(2/3)^n$

form $\infty \cdot 0$ undetermined

~~Cont'd~~ ~~do~~

$$\cancel{n\,(2/3)^n = n\,e^{\ln(2/3)^n} = n\,e^{n\ln(2/3)} = n\,e^{n\ln(2/3)<0}}$$

$$\cancel{\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{e^{n\ln(2/3)}}{1/n} = \frac{\ln(2/3)\,e^{n\ln(2/3)}}{-1/n^2}}$$

~~$e^{-\infty} \to 0$~~ ~~$\infty \cdot 0$ undetermined~~

4pts

$$b_n = e^{\ln(n(2/3)^n)} = e^{\ln(n^{1/n}\cdot 2/3)^n} = e^{n\ln(n^{1/n}\cdot 2/3)}, \text{ as } n\to\infty$$

$$\infty(-\text{ve}\ \#)$$

$$\Rightarrow e \longrightarrow 0$$

Rember $\ln 2/3 < 0$

$$\Rightarrow n\ln(2/3) \to -\infty$$

Problem 5: (30%) Find the sum (if it exists) of each of the following infinite series

$$\sum_{n=1}^{\infty} \sqrt[n]{10n+5}$$

$a_n \to 1$ as $n \to \infty$ Since $(10n+5)^{\frac{1}{n}}$

$= e^{\ln (10n+5)^{\frac{1}{n}}}$

This means

$= e^{\left(\frac{\ln(10n+5)}{n}\right)} \to 0$

by l'Hop

Series diverge by n^{th} term test $\Rightarrow$ Infinite sum is infinite (does not exist)

10 pts

$$\sum_{n=2}^{\infty} \frac{2^n+5}{3^n}$$

2 Geometric series $|r|<1$ in each [careful does NOT start at 0]

$$\sum_{2}^{\infty} (2/3)^n + 5\sum_{2}^{\infty} (1/3)^n$$

$\downarrow$

10 pts

$$\frac{(2/3)^2}{1-2/3} + \frac{5(1/3)^2}{1-1/3}$$

$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ Telescoping series

$$S_n = \ln\left(\tfrac{1}{2}\right) + \ln\left(\tfrac{2}{3}\right) + \cdots + \ln\left(\tfrac{n}{n+1}\right)$$

$$= \ln 1 - \cancel{\ln 2} + \cancel{\ln 2} - \cancel{\ln 3} + \cdots$$

$$+ \cdots \cancel{\ln n} - \ln(n+1)$$

$$= 0 - \ln(n+1)$$

$$\lim_{n\to\infty} S_n = -\infty \Rightarrow \text{Series diverge}$$

Problem 6: (7%) Show that $0.\bar{9} = 1$

$$0.\bar{9} = 0.999 \cdots \qquad 7pts$$

$$= 0.9 + 0.99 + 0.999 + \cdots$$

$$= 0.9\,(1 + 0.1 + 0.01 + \cdots)$$

$$= 0.9 \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k \qquad r = 1/10 < 1, \; a = 1$$

$$= 0.9 \, \frac{1}{1 - 1/10} = \frac{0.9}{0.9} = 1$$