Lebanese American University — Fall 2013

Byblos

Calculus III
Test #1

Date: Nov/13
Duration: 2h

Name: N. El Khatib

ID:

**The quality and cleanliness of the copy will be considered during the correction!**

I. **(25 Points)** Find the following integrals

a) $\int_0^3 x^2\sqrt{9-x^2}\,dx$, b) $\int \frac{\tan^{-1}(x)}{(x+1)^2}\,dx$, c) $\int_{\pi/4}^{\pi/2} \sin^3(x)\sin(2x)\,dx$, d) $\int \frac{3x^3+7x^2+5x+2}{(x^2+x)(x^2+2x)}\,dx$,

e) $\int \frac{1+x}{1+\sqrt{x}}\,dx$.

a) $\int_0^3 x^2\sqrt{9-x^2}\,dx$ let $x = 3\sin\theta \longrightarrow dx = 3\cos\theta\,d\theta$

$x=0 \longrightarrow \theta = 0$, $x = 3 \longrightarrow \theta = \frac{\pi}{2}$

$$\int^{\frac{\pi}{2}} 9\sin^2\theta\, 3\cos\theta\, 3\cos\theta\, d\theta = 81\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta = \frac{81}{4}\int_0^{\frac{\pi}{2}}\sin^2(2\theta)\,d\theta$$

$$= \frac{81}{4}\int_0^{\pi/2}\frac{1-\cos(4\theta)}{2}\,d\theta = \frac{81}{8}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} = \frac{81\pi}{16}.$$

b) $\int \frac{\tan^{-1}(x)}{(x+1)^2}\,dx$ let $u = \tan^{-1}(x) \longrightarrow du = \frac{dx}{1+x^2}$

$dv = \frac{dx}{(1+x)^2} \longrightarrow v = -\frac{1}{1+x}$.

$$-\frac{\tan^{-1}(x)}{1+x} + \int \frac{dx}{(1+x)(1+x^2)}$$

$$-\frac{\tan^{-1}(x)}{1+x} + \frac{1}{2}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{x-1}{x^2+1}\,dx$$

$$-\frac{\tan^{-1}(x)}{1+x} + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}(x) + C$$

$$\frac{1}{(1+x)(1+x^2)} = \frac{a}{1+x} + \frac{cx+d}{1+x^2}$$

$$= \frac{ax^2+a+cx^2+(c+d)x+d}{(1+x)(1+x^2)}$$

$$= \frac{(a+c)x^2+(c+d)x+a+d}{(1+x)(1+x^2)}$$

$\begin{cases} a+c=0 \\ c+d=0 \end{cases} \to \begin{matrix} a+c=0 \\ a-c=1 \end{matrix}$ $\boxed{a=\frac{1}{2} \quad c=-\frac{1}{2} \quad d=\frac{1}{2}}$

c) $\int_{\pi/4} \sin^3(x)\sin(2x)\,dx = 2\int_{\pi/4} \sin^4(x)\cos(x)\,dx = \left[\frac{2}{5}\sin^5(x)\right]_{\pi/4}$

$= \frac{2}{5}\left(\left(\frac{\sqrt{3}}{2}\right)^5 - \left(\frac{\sqrt{2}}{2}\right)^5\right) =$

d) $\int \frac{3x^3+7x^2+5x+2}{(x^2+x)(x^2+2x)}\,dx \qquad \frac{3x^3+7x^2+5x+2}{(x^2+x)(x^2+2x)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} + \frac{d}{x+2}$

let's use Heaviside to find c & d then the classic way for a & b

we get $a=b=c=d=1$.

hence the integral is $\int \frac{dx}{x} + \int \frac{dx}{x^2} + \int \frac{dx}{x+1} + \int \frac{dx}{x+2}$

$= \ln|x| - \frac{1}{x} + \ln|x+1| + \ln|x+2| + C.$

e) $\int \frac{1+x}{1+\sqrt{x}}\,dx \qquad$ let $u = \sqrt{x} \to x = u^2 \Rightarrow dx = 2u\,du$

$= \int \frac{1+u^2}{1+u}\,2u\,du = 2\int \frac{u^3+u}{u+1}\,du.$

$$\begin{array}{r|l} u^3+u & u+1 \\ \underline{u^3+u^2} & u^2-u+2 \\ -u^2+u & \\ \underline{-u^2-u} & \\ 2u & \\ \underline{2u+2} & \\ -2 & \end{array}$$

$\frac{u^3+u}{u+1} = u^2-u+2 - \frac{2}{u+1}$

$= 2\int (u^2-u+2)\,du - 4\int \frac{du}{u+1}$

$= \frac{2}{3}u^3 - u^2 + 4u - 4\ln|u+1| + C$

$= \frac{2}{3}x^{\frac{3}{2}} - x + 4\sqrt{x} - 4\ln(\sqrt{x}+1) + C.$

II. **(20 Points)** Study the convergence of the following improper integrals

a) $\int_1^\infty \frac{x^3}{x^4+\ln(x)}\,dx$, b) $\int_0^\infty \frac{dx}{\sqrt[3]{x^2+x^5}}$, c) $\int_0^\infty \frac{1}{\sqrt{x^4+x^3}-x^2}\,dx$, d) $\int_0^\infty \frac{\sqrt{x}}{e^{2x}+x}\,dx$.

a) $\int_1^\infty \frac{x^3}{x^4+\ln(x)}\,dx$

$$\frac{x^3}{x^4+\ln(x)} = \frac{x^3}{x^4\left(1+\frac{\ln(x)}{x^4}\right)} \underset{\text{at } \infty}{\sim} \frac{x^3}{x^4} = \frac{1}{x}$$

hence by LCT the integral $\boxed{\text{Diverges}}$

b) $$\int_0^\infty \frac{dx}{\sqrt[3]{x^2+x^5}} = \int_0^\infty \frac{dx}{\left(x^2(1+x^3)\right)^{\frac{1}{3}}} = \int_0^\infty \frac{dx}{x^{\frac{2}{3}}(1+x^3)^{\frac{1}{3}}}$$

$$= \int_0^1 \frac{dx}{x^{\frac{2}{3}}(1+x^3)^{\frac{1}{3}}} + \int_1^\infty \frac{dx}{x^{\frac{2}{3}}(1+x^3)^{\frac{1}{3}}}$$

$$\frac{1}{x^{\frac{2}{3}}(1+x^3)^{\frac{1}{3}}} \underset{\text{at } 0}{\sim} \frac{1}{x^{\frac{2}{3}}} \Rightarrow \int_0^1 \frac{dx}{x^{\frac{2}{3}}(1+x^3)^{\frac{1}{3}}} \text{ Converges}$$

$$\frac{1}{x^{\frac{2}{3}}(1+x^3)^{\frac{1}{3}}} \underset{\text{at } \infty}{\sim} \frac{1}{x^{\frac{2}{3}}\cdot x} = \frac{1}{x^{\frac{5}{3}}} \Rightarrow \int_1^\infty \frac{dx}{x^{2/3}(1+x^3)^{\frac{1}{3}}} \text{ Converges}$$

hence $\int_0^\infty \frac{dx}{\sqrt[3]{x^2+x^5}}$ $\boxed{\text{Converges.}}$

c) $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x^4+x^3} - x^2} = \int_0^{\infty} \frac{1}{\sqrt{x^4+x^3} - x^2} \cdot \frac{\sqrt{x^4+x^3}+x^2}{\sqrt{x^4+x^3}+x^2}\, dx$

$$= \int_0^{\infty} \frac{\sqrt{x^4+x^3}+x^2}{x^3}\, dx = \int_0^{1} \frac{\sqrt{x^4+x^3}+x^2}{x^3}\, dx + \int_1^{\infty} \frac{\sqrt{x^4+x^3}+x^2}{x^3}\, dx$$

$$\frac{\sqrt{x^4+x^3}+x^2}{x^3} \underset{\text{at } \infty}{\sim} \frac{2x^2}{x^3} = \frac{2}{x}$$

hence $\displaystyle\int_1^{\infty} \frac{\sqrt{x^4+x^3}+x^2}{x^3}\, dx$ diverges

then $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x^4+x^3} - x^2}$ $\boxed{\text{diverges}}$

d) $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{e^{2x}+x} = \underbrace{\int_0^{1} \frac{\sqrt{x}}{e^{2x}+x}\, dx}_{\text{finite}} + \int_1^{\infty} \frac{\sqrt{x}}{e^{2x}+x}\, dx$

in $[1, \infty)$

$$\frac{\sqrt{x}}{e^{2x}+x} < \frac{\sqrt{x}}{e^{2x}} < x e^{-2x}$$

but $\displaystyle\int_1^{\infty} x e^{-2x}\, dx$ converges (see below).

hence by DCT $\displaystyle\int_1^{\infty} \frac{\sqrt{x}}{e^{2x}+x}\, dx$ converges.

then $\displaystyle\int_0^{\infty} \frac{\sqrt{x}}{e^{2x}+x}\, dx$ $\boxed{\text{converges.}}$

$\displaystyle\int_1^{\infty} x e^{-2x}\, dx$

$$\begin{array}{c|c} x & e^{-2x} \\ \hline 1 & -\frac{1}{2} e^{-2x} \\ 0 & \frac{1}{4} e^{-2x} \end{array}$$

$= \displaystyle\lim_{a \to \infty} \left[-\frac{1}{2} x e^{-2x}\right]_1^a + \left[-\frac{1}{4} e^{-2x}\right]_1^a$

$= \frac{1}{2} e^{-2} + \frac{1}{4} e^{-2} = \frac{3}{4} e^{-2}$

finite [illegible]

III. **(20 Points)** Let $a$ be a real number and consider $I_a = \int \cos(ax)e^x\,dx$ and $J_a = \int \sin(ax)e^x\,dx$

a. Using an integration by parts in $I_a$, prove that $I_a = e^x\cos(ax) + aJ_a$.

b. Using an integration by parts in $J_a$, prove that $J_a = e^x\sin(ax) - aI_a$.

c. Deduce $I_a$ and $J_a$.

d. Deduce $\int \cos^2(x)e^x\,dx$.

a) $I_a = \int \cos(ax)e^x\,dx$ let $u = \cos(ax) \longrightarrow du = -a\sin(ax)dx$

$dv = e^x dx \longrightarrow v = e^x$

then $I_a = e^x\cos(ax) - a\int \sin(ax)e^x dx = e^x\cos(ax) - aJ_a$

b) $J_a = \int \sin(ax)e^x dx$ let $u = \sin(ax) \longrightarrow du = a\cos(ax)dx$

$dv = e^x dx \longrightarrow v = e^x$

then $J_a = \sin(ax)e^x - a\int \cos(ax)e^x dx = \sin(ax)e^x - aI_a$

c) $\begin{cases} I_a = e^x\cos(ax) + aJ_a \\ J_a = e^x\sin(ax) - aI_a \end{cases}$

then $I_a = e^x\cos(ax) + a(e^x\sin(ax) - aI_a)$

$\Leftrightarrow (1+a^2)I_a = e^x\cos(ax) + ae^x\sin(ax)$

$\Leftrightarrow \boxed{I_a = \dfrac{e^x\cos(ax) + ae^x\sin(ax)}{1+a^2}}$

$J_a = e^x\sin(ax) - a(e^x\cos(ax) + aJ_a) \Leftrightarrow$

$\boxed{J_a = \dfrac{e^x\sin(ax) - ae^x\cos(ax)}{1+a^2}}$

d) $\int \cos^2(x)e^x\,dx = \int \frac{1+\cos(2x)}{2}e^x\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)e^x\,dx$

$$= \frac{1}{2}x + \frac{1}{2}I_2 = \frac{1}{2}x + \frac{1}{2}\left(\frac{e^x\cos(2x) + 2e^x\sin(2x)}{5}\right)$$

$$= \frac{5x + e^x\cos(2x) + 2e^x\sin(2x)}{10}.$$

IV. **(10 Points)** Find the volume of the regular cone of circular basis of radius $r$ and height $h$ (see Figure 1).

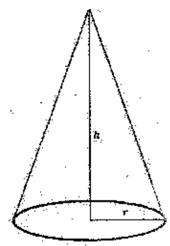
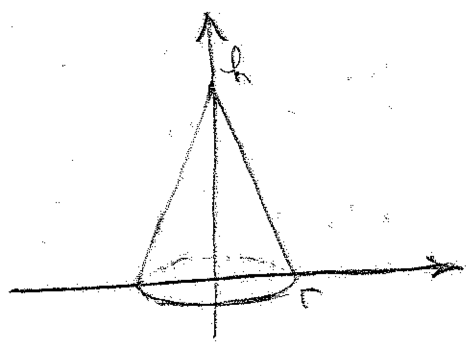



Figure 1: Cone

The cone is the result of the rotation of the ligne passing by the points $(r,0)$ & $(0,h)$ with respect to the y-axis (see figure below).




$y = ax + b$

$$\begin{cases} 0 = ar + b \\ h = b \end{cases} \Rightarrow a = -\frac{h}{r}$$

then $\boxed{y = -\frac{h}{r}x + h}$

we use the shell method:

$$V = \int_0^r 2\pi x \left(-\frac{h}{r}x + h\right) dx = \left[\frac{-2\pi h}{r}\frac{x^3}{3} + \pi h x^2\right]_0^r$$

$$= \frac{-2\pi}{3} h r^2 + \pi h r^2 = \frac{\pi}{3} h r^2$$

V. **(20 Points)** Let $C_1$ be the curve representing the function $f(x) = 3\sqrt{x-1}$, $C_2$ the curve representing $g(x) = x + 1$.

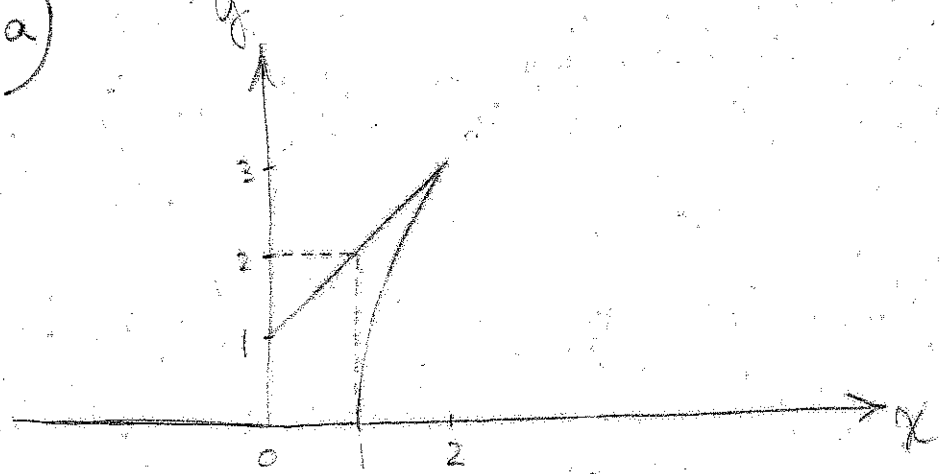
a. Sketch the graphic representation of $C_1$ and $C_2$.

b. Find the area of the surface $S$ enclosed by $C_1$, $C_2$, the $x$-axis and the $y$-axis.

c. Find the volume of the solid generated by the rotation of $S$ with respect to $y$-axis.

d. Find the volume of the solid generated by the rotation of $S$ with respect to the line $y = 1$.

a)



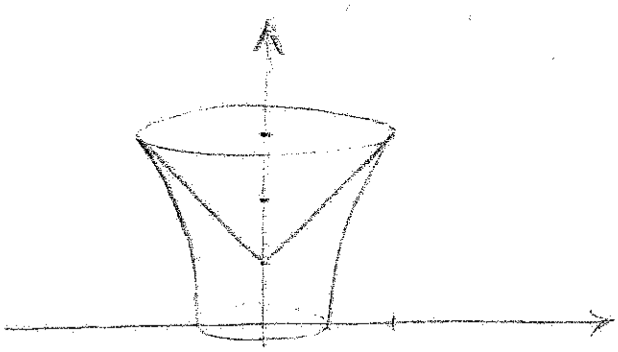

b) $S = \int_0^1 (x+1)\,dx + \int_1^2 \left((x+1) - 3\sqrt{x-1}\right) dx$

$= \left[\frac{x^2}{2} + x\right]_0^1 + \left[\frac{x^2}{2} + x - 2(x-1)^{\frac{3}{2}}\right]_1^2$

$= \frac{3}{2} + \left(\frac{4}{2} + 2 - 2 - \frac{1}{2} - 1 + 0\right)$

$= \frac{3}{2} + \frac{1}{2} = 2$

c)



$V = \int_0^1 2\pi x(x+1)\,dx + \int_1^2 2\pi x\left(x+1-3\sqrt{x-1}\right)$

$= \int_0^2 2\pi x(x+1)\,dx - \int_1^2 6\pi x\sqrt{x-1}\,dx$

$= \left[\frac{2\pi}{3}x^3 + \pi x^2\right]_0^2 - 6\pi \int_1^2 x\sqrt{x-1}\,dx$

$$= \frac{16\pi}{3} + 4\pi - 6\pi \int_1^2 x\sqrt{x-1}\,dx \qquad \begin{array}{l} u = x \to du = dx \\ dv = (x-1)^{\frac{1}{2}}dx \to v = \frac{2}{3}(x-1)^{\frac{3}{2}} \end{array}$$

$$= \frac{28\pi}{3} - 6\pi\left( \left[ \frac{2x}{3}(x-1)^{\frac{3}{2}} \right]_1^2 - \frac{2}{3}\int_1^2 (x-1)^{\frac{3}{2}}\,dx \right)$$

$$= \frac{28\pi}{3} - 6\pi\left( \frac{4}{3} - \frac{2}{3}\left[ \frac{2}{5}(x-1)^{\frac{5}{2}} \right]_1^2 \right)$$

$$= \frac{28\pi}{3} - \frac{24\pi}{3} + 4\pi\frac{2}{5}$$

$$= \frac{4\pi}{3} + \frac{8\pi}{5} = \frac{44\pi}{15}$$

2nd method: Using the disk-washer method $\qquad y = 3\sqrt{x-1} \Rightarrow x = \frac{y^2}{9} + 1$

$\qquad y = x+1 \Rightarrow x = y-1$

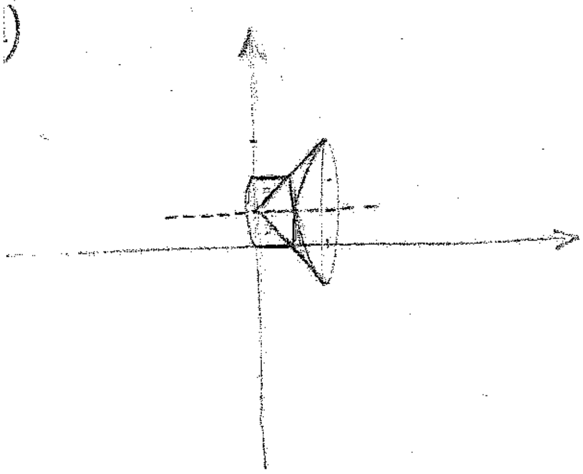
$$V = \int_0^1 \pi\left(\frac{y^2}{9}+1\right)^2 dy + \int_1^3 \pi\left(\left(\frac{y^2}{9}+1\right)^2 - (y-1)^2\right) dy$$

$$= \int_0^3 \pi\left(\frac{y^2}{9}+1\right)^2 dy - \int_1^3 \pi (y-1)^2\,dy = \pi\int_0^3 \left(\frac{y^4}{81} + \frac{2y^2}{9} + 1\right) dy - \pi\left[\frac{1}{3}(y-1)^3\right]_1^3$$

$$= \pi\left[\frac{y^5}{5\times 81} + \frac{2y^3}{27} + y\right]_0^3 - \frac{8\pi}{3} = \pi\left[\frac{3\times 81}{5\times 81} + \frac{2\times 27}{27} + 3\right] - \frac{8\pi}{3}$$

$$= \frac{28\pi}{5} - \frac{8\pi}{3} = \frac{44\pi}{15} \qquad \blacksquare$$

)



$$V = \int_1^2 2\pi(y-1)\left(\frac{y^2}{9}+1\right) dy + \int_2^3 2\pi(y-1)\left(\frac{y^2}{9}+1-(y-1)\right)$$

VI. **(15 Points)** Consider the improper integral $I(\alpha,\beta) = \int_0^{\infty} \frac{x^{\alpha}}{1+x^{\beta}}\,dx$.

a. Study in function of $\alpha$ and $\beta$ the convergence of $I(\alpha,\beta)$.

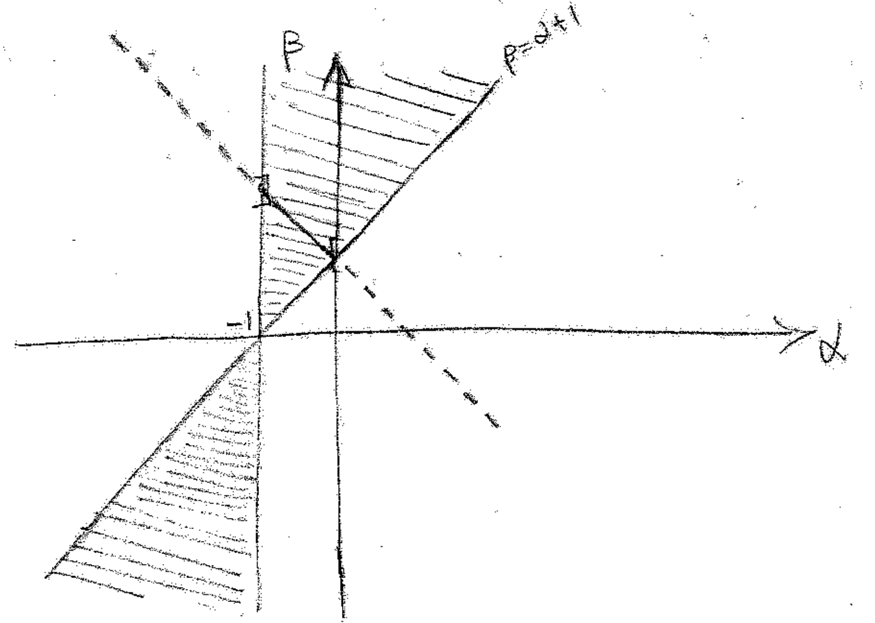
b. Sketch the graphic representation of the couples $(\alpha,\beta)$ for which $I(\alpha,\beta)$ converges.

c. Deduce the behavior of the improper integrals $\int_0^{\infty} \frac{dx}{\sqrt{x}+x^3}$ and $\int_0^{\infty} \frac{dx}{x^{-\alpha}+x^{1-2\alpha}}$.

a) $I(\alpha,\beta) = \int_0^{\infty} \frac{x^{\alpha}}{1+x^{\beta}}\,dx = \int_0^{1} \frac{x^{\alpha}}{1+x^{\beta}}\,dx + \int_1^{\infty} \frac{x^{\alpha}}{1+x^{\beta}}\,dx$

| | $\widetilde{\text{at } 0}$ | $\widetilde{\text{at } \infty}$ | $\int_0^1$ | $\int_1^{\infty}$ |
|---|---|---|---|---|
| $\beta>0$ | $x^{\alpha}$ | $x^{\alpha-\beta}$ | $\alpha>-1$ | $\alpha-\beta<-1$ |
| $\beta=0$ | $\frac{x^{\alpha}}{2}$ | $\frac{x^{\alpha}}{2}$ | $\alpha>-1$ | $\alpha<-1$ |
| $\beta<0$ | $x^{\alpha-\beta}$ | $x^{\alpha}$ | $\alpha-\beta>-1$ | $\alpha<-1$ |

b)




c) $\int_0^{\infty} \frac{dx}{\sqrt{x}+x^3} = \int_0^{\infty} \frac{dx}{\sqrt{x}\,(1+x^{\frac{5}{2}})} = \int_0^{\infty} \frac{x^{-\frac{1}{2}}}{1+x^{\frac{5}{2}}}\,dx$ particular case where $\alpha=-\frac{1}{2}$ & $\beta=\frac{5}{2}$ hence it $\boxed{\text{Converges}}$

$\int_0^{\infty} \frac{dx}{x^{-\alpha}+x^{1-2\alpha}} = \int_0^{\infty} \frac{x^{\alpha}}{1+x^{1-\alpha}}\,dx$ particular case where $\beta=1-\alpha$ hence it converges when $\alpha\in\,]-1,0[$