Calc 51 ... EXAM 1 KEY

Problem 1: (14%) Evaluate the following proper integrals

(a) $\int \cosh(\ln x)\, dx$ $= \int \frac{e^{\ln x} + e^{-\ln x}}{2}\, dx = \int \frac{x + 1/x}{2}\, dx$

$= x^2/4 + \frac{1}{2} \ln|x| + C$

7pts

(b) $\int \frac{1}{e^x + e^{-x}}\, dx$ $= \int \frac{e^x}{e^{2x}+1}\, dx$

Let $u = e^x$

$\int \frac{e^x}{e^{2x}+1}\, dx = \int \frac{du}{u^2+1} = \tan^{-1} u + C$

$= \tan^{-1}(e^x) + C$

7pts

Problem 2: (20%) Evaluate the following improper integrals

(a) $\displaystyle\int_3^\infty \frac{1}{x^2-6x+8}\,dx$ 8 pts

→ mixed type (pole at 4) and ∞

→ Partial fractions $x^2-6x+8=(x-4)(x-2)$

→ So split as $\displaystyle\int_3^4 + \int_4^5 + \int_5^\infty$

→ do partial fractions → evaluate limits.

→ get zero

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(9+x)}\,dx$ 10 pts

Let $u=\sqrt{x}$ $\quad du=\frac{1}{2\sqrt{x}}\,dx$

$$\int \frac{1}{\sqrt{x}(9+x)}\,dx = \int \frac{2\,du}{9+u^2} = 2\int \frac{du}{3^2+u^2}$$

$$= 2\left(\tfrac{1}{3}\tan^{-1} u/3\right) + C$$

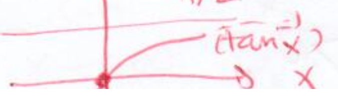
$$= \tfrac{2}{3}\tan^{-1}(\sqrt{x}/3) + C$$

Now $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(9+x)}\,dx$

$$= \lim_{c\to+\infty}\int_1^\infty \frac{1}{\sqrt{x}(9+x)}\,dx = \lim_{c\to+\infty} \left.2/3\tan^{-1}(\sqrt{x}/3)\right]$$

$$= 2/3\,\pi/2 - 2/3\tan^{-1}(1/3$$

= some number



(c) $\int_e^\infty \frac{1}{(\ln x)^3 x}\,dx$ ~~4~~ pts

Let $u = \ln x$ $\quad du = 1/x\,dx$

$$\int \frac{1}{(\ln x)^3 x}\,dx = \int \frac{du}{u^3} = \frac{u^{-3+1}}{-3+1} + C = -\frac{1}{2u^2} + C$$

back to x!!

$$= -\frac{1}{2(\ln x)^2} + C$$

Now $\int_e^\infty \frac{1}{(\ln x)^3 x}\,dx = \lim_{c\to\infty} -\frac{1}{2(\ln x)^2}\Big]_{x=e}^{x=c}$

$$\Longrightarrow 0 + \frac{1}{2(\ln e)^2} = \frac{1}{2}$$

Problem 3: (36%) Determine the convergence or divergence of the following improper integrals. Justify your answers.

1. $\int_0^\infty \frac{x}{(x^3+1)^2}\,dx$

type 1 only, but be careful when you compare to a p-integral you should NOT start from zero!

$$\int_0^\infty \frac{x}{(x^3+1)^2}\,dx$$

$$\int_0^1 \frac{x}{(x^3+1)^2}\,dx + \int_1^\infty \frac{x}{(x^3+1)^2}\,dx$$

a number

LCT with

$g(x) = \frac{1}{x^5}$, $f(x) =$ (

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x/(x^3+1)^2}{1/x^5} = \lim_{x\to\infty} \frac{x^6}{x^6+1+2x^3} =$$

4. $\int_1^\infty \frac{(\ln x)^{100}}{x^{0.2}}\,dx$ $= \underbrace{\int_1^e \frac{(\ln x)^{100}}{x^{0.2}}\,dx}_{\text{number}} + \int_e^\infty \frac{(\ln x)^{100}}{x^{0.2}}\,dx$

~~Let P be a number such that~~

$x \geq e \Rightarrow (\ln x) \geq 1 \Rightarrow (\ln x)^{100} \geq 1$

$\Rightarrow \int_1^\infty \frac{(\ln x)^{100}}{x^{0.2}}\,dx \geq \underbrace{\int_1^\infty \frac{1}{x^{0.2}}\,d}_{\text{Diverge p-integ}}$

$\Rightarrow$ Original diverges by DCT.

5. $\int_1^\infty \frac{x^3+1}{\sin x + 2 + x^3\sqrt{x}}\,dx$

$\sin x \leq 1$

$\sin x + 2 + x^{3.5} \leq 3 + x^{3.5}$

$\Rightarrow \int_1^\infty \frac{x^3+1}{\sin x + 2 + x^{3.5}}\,dx \geq \int_1^\infty \frac{x^3+1}{3+x^{3.5}}\,dx$

LCT with $\int_1^\infty \frac{1}{x^{0.5}}\,dx$ shows that this diverges

$\Rightarrow$ Original diverges by DCT

2. $\int_{-\infty}^{\infty} \frac{1}{e^{(x^2)}}\,dx$

This is an EVEN function

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{e^{(x^2)}}\,dx = 2\int_0^{\infty} \frac{1}{e^{(x^2)}}\,dx$$

Now $e^{(x^2)} \geq e^x \quad \forall x \geq 1$

$$\Rightarrow 1/e^{(x^2)} \leq 1/(e^x) \quad \forall x \geq 1$$

$$\Rightarrow \int_1^{\infty} \frac{1}{e^{(x^2)}}\,dx \leq \underbrace{\int_1^{\infty} e^{-x}\,dx}_{\text{Convergen}}$$

$\Rightarrow$ By DCT,

$\int_1^{\infty} \frac{1}{e^{(x^2)}}\,dx$ converges

$\Rightarrow \int_0^{\infty} \frac{1}{e^{(x^2)}}\,dx$ converges

Since $\int_0^1 \frac{1}{e^{(x^2)}}\,dx$ is a numbe

3. $\int_1^{\infty} \frac{|\sin x|}{x^2}\,dx$

$|\sin x| < 1$

$$\Rightarrow \frac{|\sin x|}{x^2} < 1/x^2 \quad \forall x \geq 1$$

$$\Rightarrow \int_1^{\infty} \frac{|\sin x|}{x^2}\,dx \leq$$

$$\underbrace{\int_1^{\infty} 1/x^2\,dx}_{\substack{\text{convergent}\\ \text{p-integral}\\ p=2}} \Rightarrow \text{Convergent by DCT}$$

6. $\int_1^\infty \frac{e^x}{\sqrt{1+x^2}}\,dx$ Type 1

$$\lim_{x\to\infty} \frac{e^x}{\sqrt{1+x^2}} = +\infty$$

Integral diverges by divergence test.

Problem 4: (12%)

Find the values of p for which

$$\int_0^\infty \frac{1}{(x^2+1)^p}\,dx$$

converges. Justify your answer.

The idea is to compare to a p-integral

But we need to remove the 0 first

(or we end up with $\int_0 \frac{1}{x^{2p}}\,dx$ ← pole at 0)

So $\int_0^\infty \frac{1}{(x^2+1)^p}\,dx = \underbrace{\int_0^1 \frac{1}{(x^2+1)^p}\,dx}_{\forall p\text{, this is a number since } x^2+1>0} + \int_1^\infty \frac{dx}{(x^2+1)^p}$

⇒ Enough to study

$\int_1^\infty \frac{dx}{(x^2+1)^p}$ & LCT with $\int^\infty \frac{dx}{}$

$$\lim_{x\to\infty} f/g = \lim_{x\to\infty} \frac{1}{(x^2+1)^p} \cdot x^{2p} = \lim_{x\to\infty} \left(\frac{x^2}{x^2+1}\right)^p$$

$$= 1$$

So integrals behave alike

But $\int_1^\infty \frac{dx}{x^{2p}}$. converges

whenever $2p > 1$

$$\Rightarrow \boxed{p > 1/2}$$

→ Remark DCT is not enough because it doesn't guarantee you find all $\underline{p}$. (unless you also show divergence)

Problem 5: (18%) Find the following limits if they exist. Justify your answers.

(a) $\lim_{n\to\infty} (-1)^n \frac{n}{7+n}$

$\lim_{n\to\infty} \frac{n}{7+n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(7/n+1)} = 1$

So $\lim_{n\to\infty} (-1)^n \frac{n}{7+n} = \pm 1$ whether n is even or odd

$\Rightarrow$ limit DNE

(b) $\lim_{n\to\infty} (1 + \frac{1}{\sqrt{n}})^n$

$\lim_{n\to\infty} \left\{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right\}^{\sqrt{n}} = \lim_{n\to\infty} (e)^{\sqrt{n}} = +\infty$

$e > 1$

Diverges

(c) $\lim_{n\to\infty} \frac{n!}{6^{n+2}}$

$\lim_{n\to\infty} \frac{1}{36} \frac{n!}{6^n}$

Since $\lim_{n\to\infty} \frac{x^n}{n!} =$

$\forall x \in \mathbb{R}$

$= +\infty$ Diverges