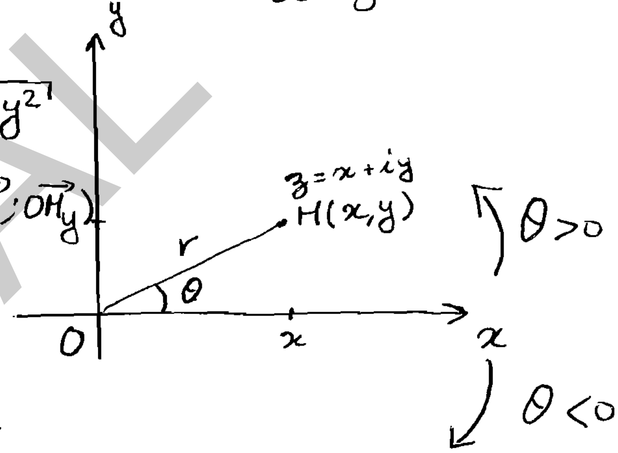
Chapter 1 First-Order Differential Equations

1.0 Complex Numbers

* A complex number is of the form $z = x + iy$
  Where x, y are real numbers
  and i is an imaginary number satisfying $i^2 = -1$



* The modulus of z is $r = |z| = \sqrt{x^2+y^2}$
* The argument of z is $\theta = \arg(z) = (\vec{Ox}; \vec{OH})$
* The conjugate of z is $\bar{z} = x - iy$
* The real part of z is $\mathrm{Re}(z) = x$
* The imaginary part of z is $\mathrm{Im}(z) = iy$.

Example: Solve the quadratic equation $5z^2 + 3z + 2 = 0$

$$\Delta = b^2 - 4ac = 9 - 40 = -31 = 31 i^2$$

$$z = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-3 \pm i\sqrt{31}}{10}$$

$$z_1 = \frac{-3}{10} + i\,\frac{\sqrt{31}}{10} \;;\; z_2 = \frac{-3}{10} - i\,\frac{\sqrt{31}}{10}$$

Definition: Let $z = x + iy$ be a complex number
The complex exponential function e^z is defined by:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

$$e^{z} = [e^{x}, y]$$

Example $\quad e^{2+3i} = ?$

In polar form : $e^{2+3i} = [e^{2}, 3]$

In cartesian form : $e^{2+3i} = e^{2}\cos 3 + i\, e^{2}\sin 3$

Complex functions: (Mathematica only)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \quad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\text{Log}\, z = \text{Ln}\, |z| + i\, \text{Arg}\, z = \text{Ln}\, r + i\theta$$

$$c^{z} = e^{z \,\text{Log}\, c}$$

M $\quad$ Log[-4]

Log[2+3i]

$(2+i)^{3-i}$

1.1 Differential Equations

Definition: A differential equation (DE) is an equation involving a variable x, a function $y = y(x)$ and at least one of its derivatives $y', y'', y''', y^{(4)}, \dots, y^{(n)}, \dots$

* The order of a DE is the order of the highest derivative appearing in the equation.
* The set of all solutions to a DE is called the General Solution

Example

$$y' + 2x y'' + x^4 y^6 = \sin(x + y'^2) \longrightarrow 2^{nd} \text{ order DE}$$

$$y' + y^2 y''' + y^{(6)} = \sin x \longrightarrow 6^{th} \text{ order DE}$$

Formula: The general solution of the 1^{st}-order DE : $\boxed{y' = ky}$ is $\boxed{y = c\, e^{kx}}$ where c, k are constant.

Proof: $y' = ky \Rightarrow \dfrac{dy}{dx} = ky \Rightarrow \dfrac{dy}{y} = k\,dx$

$$\int \frac{dy}{y} = \int k\,dx$$

1^{st} method: $\ln|y| + c_1 = kx + c_2$

$$\cancel{\ln|y| + c_1 - c_2 = kx \Rightarrow}$$

$$\ln|y| = kx + \underbrace{c_2 - c_1}_{c_3}$$

$$\ln|y| = kx + c_3 \Rightarrow |y| = e^{kx + c_3}$$

$$\Rightarrow y = \underbrace{\pm e^{c_3}}_{c} e^{kx}$$

$$\Rightarrow y = c\,e^{kx}$$

Better method: $\displaystyle\int \frac{dy}{y} = \int k\,dx$

$\ln|y| + c_3$

$\ln|y| - \ln c$

$\ln \frac{|y|}{c}$

$$\ln \frac{y}{c} = kx \Rightarrow \frac{y}{c} = e^{kx} \Rightarrow y = ce^{kx}$$

Example: The mass of a radioactive substance decreases at a rate proportional to its mass.

If $M(t)$ is the mass at the time t then $\boxed{M' = -kM}$ k is positive.

General Solution: $M(t) = ce^{-kt}$

Half-life: The Half-life of a radioactive substance is the time T needed for half the mass to disappear.

$$M(T) = \frac{1}{2} M(0)$$

Example: The mass of a radioactive substance at the time $t = 1$ year is $0.2g$. Its mass at the time $t = 2$ years is $0.06g$.

(a) Find the initial mass

(b) Find the half-life T.

a) $M(1) = 0.2$

$M(2) = 0.06$

$M' = -kM \Rightarrow M = ce^{-kt}$ (General Solution)

$M(1) = 0.2 \Rightarrow ce^{-k} = 0.2 \quad (1)$

$M(2) = 0.06 \Rightarrow ce^{-2k} = 0.06 \quad (2)$

$$\frac{(2)}{(1)} : \frac{ce^{-2k}}{ce^{-k}} = \frac{0.06}{0.2} \Rightarrow \boxed{e^{-k} = 0.3}$$

$$(1) \Rightarrow 0.3c = 0.2 \Rightarrow \boxed{c = 2/3}$$

b) $T = ?$ / $M(T) = \frac{1}{2} M(0)$

$$\frac{2}{3}(0.3)^T = \frac{1}{3} \Rightarrow (0.3)^T = \frac{1}{2} = 0.5$$

$$T \ln 0.3 = \ln 0.5 \Rightarrow \boxed{T = \frac{\ln 0.5}{\ln 0.3}} \text{ year}$$

<u>Prob 1.1.</u> HW (17, 21)

1.2 Direction Fields

Consider the 1st-order DE: $y' = f(x,y)$

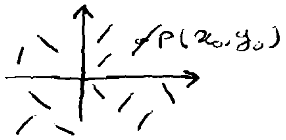
Let $y = y(x)$ be a solution passing through $P(x_0, y_0)$

The slope of this solution at P is:

$$m = y'\big|_{at\,P} = f(x,y)\big|_{at\,P} = f(x_0, y_0)$$

We can draw from P a small segment of slope m called <u>lineal element</u>.

* The set of all lineal elements is called the <u>Direction field</u> of the DE
* This direction field can be used to see the shape of the solution graphically.



<u>Definition</u>: $\left.\begin{array}{l} y' = f(x,y) \\ y' = m \end{array}\right\} \Rightarrow f(x,y) = m$

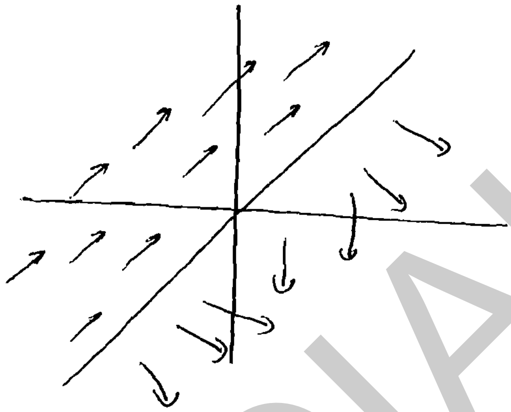
The curve $f(x,y) = m$ is called Isocline.

This curve is the set of points in the plane where the solutions have the same slope m.

Example: Draw the direction field of the DE: $y' = y - x$

step 1 Find, if possible, the region in the plane where the solutions are increasing (decreasing)

$$y' > 0 \iff f(x,y) \geq 0 \iff y - x \geq 0 \iff y \geq x \quad \left(\text{above the line } y = x\right)$$



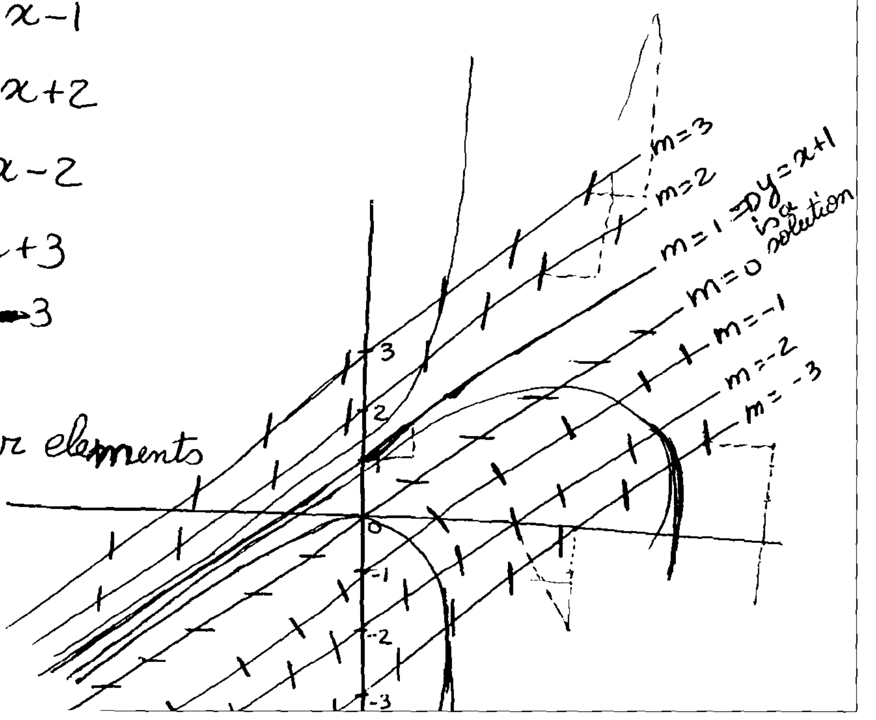
step 2 Draw some of the isoclines
(They are used to help you draw several linear elements)

$$f(x,y) = m \Rightarrow y - x = m$$

* $m = 0$; $y = x$
* $m = 1$; $y = x + 1$
* $m = -1$; $y = x - 1$
* $m = 2$; $y = x + 2$
* $m = -2$; $y = x - 2$
* $m = 3$; $y = x + 3$
* $m = -3$; $y = x - 3$

Step 3 Draw several linear elements on the isoclines

Step 4 Draw some of the solutions



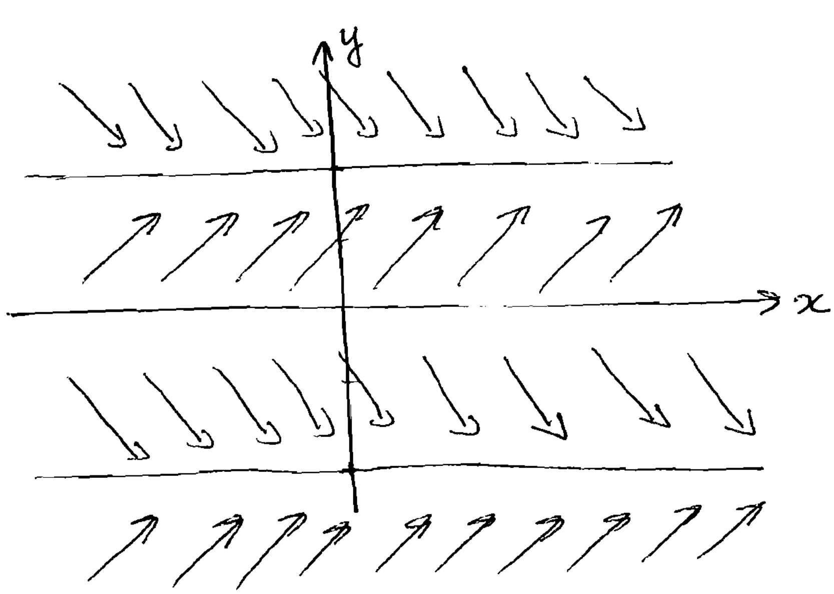
<u>Remark</u> If the equation is of the form $y' = f(y)$ then the isoclines are $f(y) = m \iff y = \underline{c}te$. The isoclines are straight lines $//$ Ox.
Instead of selecting values for m, we select values for y.

<u>Example</u> $y' = 4y - y^3$ Draw the direction field.

<u>step 1</u> $y' = 0 \iff y(4-y^2) = 0 \Rightarrow y=0, y=2, y=-2$ (3 lines)

$y' < 0 \iff y(4-y^2) < 0 \iff y(2-y)(2+y) < 0$

y	$-\infty$		-2		0		2		$+\infty$
y		$-$		$-$	0	$+$		$+$	
$2-y$		$+$		$+$		$+$	0	$-$	
$2+y$		$-$	0	$+$		$+$		$+$	
y'		$+$	0	$-$	0	$+$	0	$-$	
		↗		↘		↗		↘	

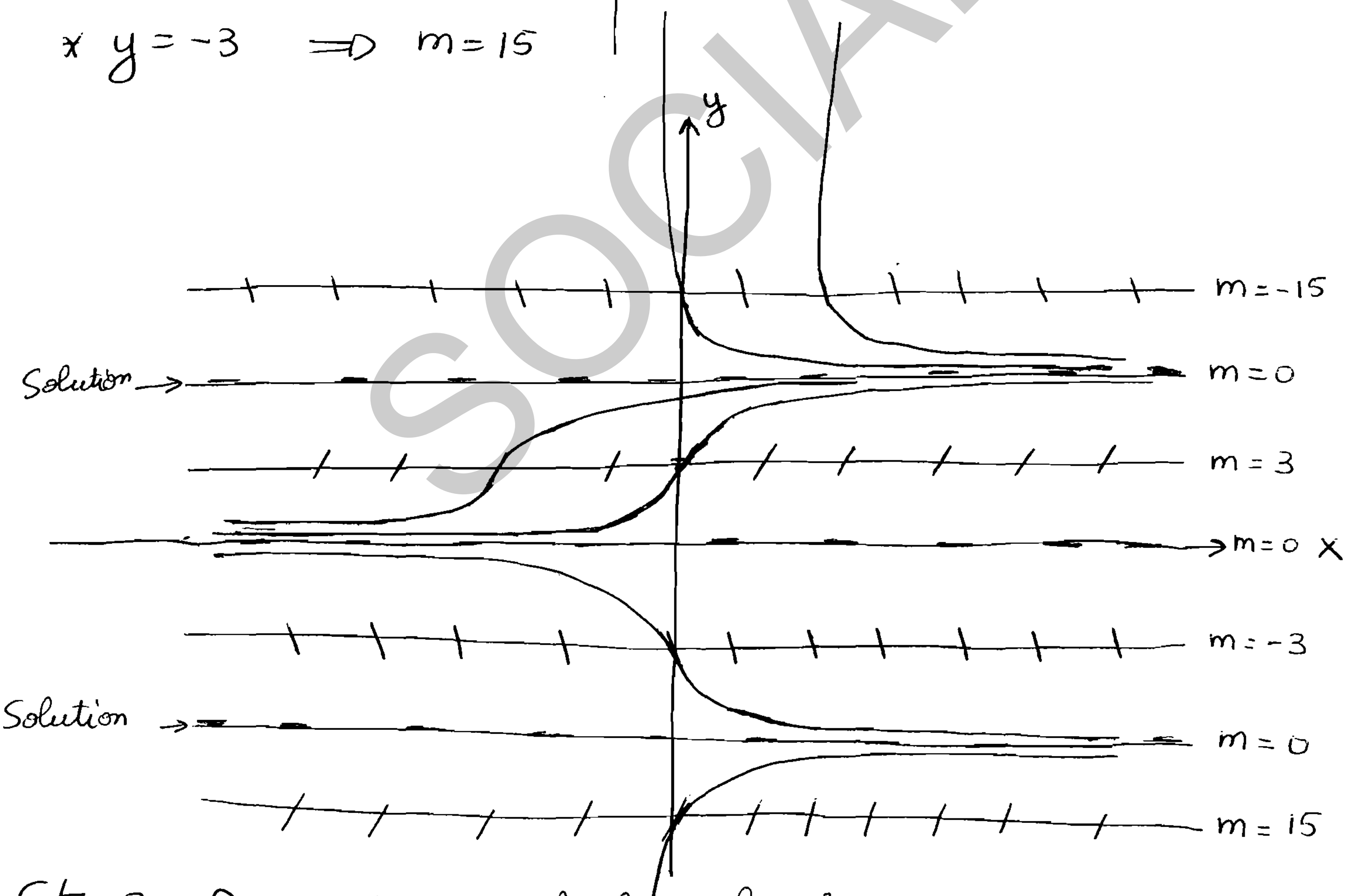


<u>Step 2</u> Draw some of the isoclines

$$f(y) = m \iff 4y - y^3 = m$$

* $y = 0 \implies m = 0$
* $y = 1 \implies m = 3$
* $y = -1 \implies m = -3$
* $y = 2 \implies m = 0$
* $y = -2 \implies m = 0$
* $y = 3 \implies m = -15$
* $y = -3 \implies m = 15$

$y = 1.5 \implies m = 6 - \frac{27}{8} = 2.6$



<u>Step 3</u> Draw several lineal elements intersecting the isoclines (Red Segments)

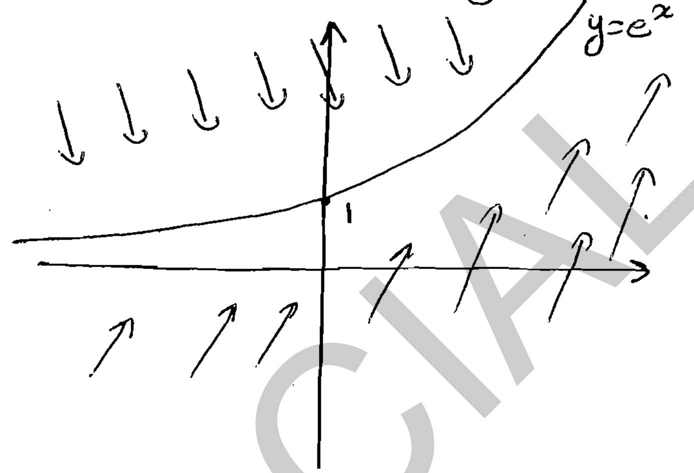
<u>Step 4</u> Sketch some of the solutions passing through:
(0,0) ;

Prob 1.2 (1-15) HW (2, 4, 13)

(1) $y' = e^{x} - y$ (Solutions through $(0,0)$; $(0,1)$)
$(-4,0)$

Step 1 $y' \geq 0 \iff e^{x} - y \geq 0 \iff y \leq e^{x}$

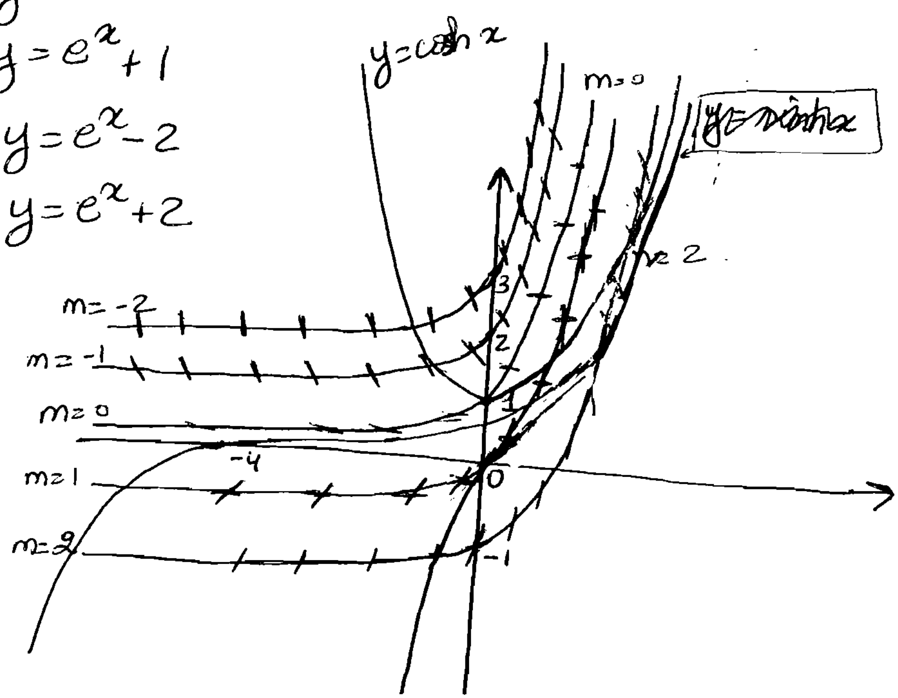
The solutions are increasing below the curve $y = e^{x}$



Step 2 Draw some of the isoclines $f(x,y) = m$

$e^{x} - y = m \Rightarrow y = e^{x} - m$

* $m = 0 \Rightarrow y = e^{x}$
* $m = 1 \Rightarrow y = e^{x} - 1$
* $m = -1 \Rightarrow y = e^{x} + 1$
* $m = 2 \Rightarrow y = e^{x} - 2$
* $m = -2 \Rightarrow y = e^{x} + 2$

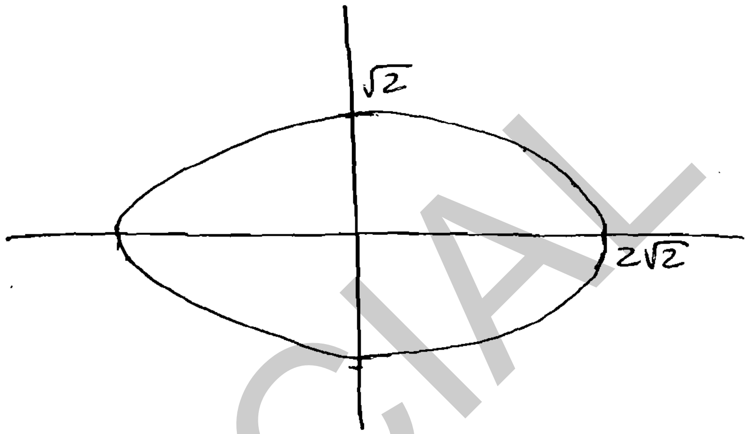


(10') $y' = \frac{x^2}{4} + y^2 - 2$, $m = 0, \pm 1, \pm 2, 7$

Step 1 $y' \leq 0 \Rightarrow \frac{x^2}{4} + y^2 - 2 \leq 0 \Rightarrow \frac{x^2}{4} + y^2 \leq 2$

$\Rightarrow \frac{x^2}{8} + \frac{y^2}{2} \leq 1$

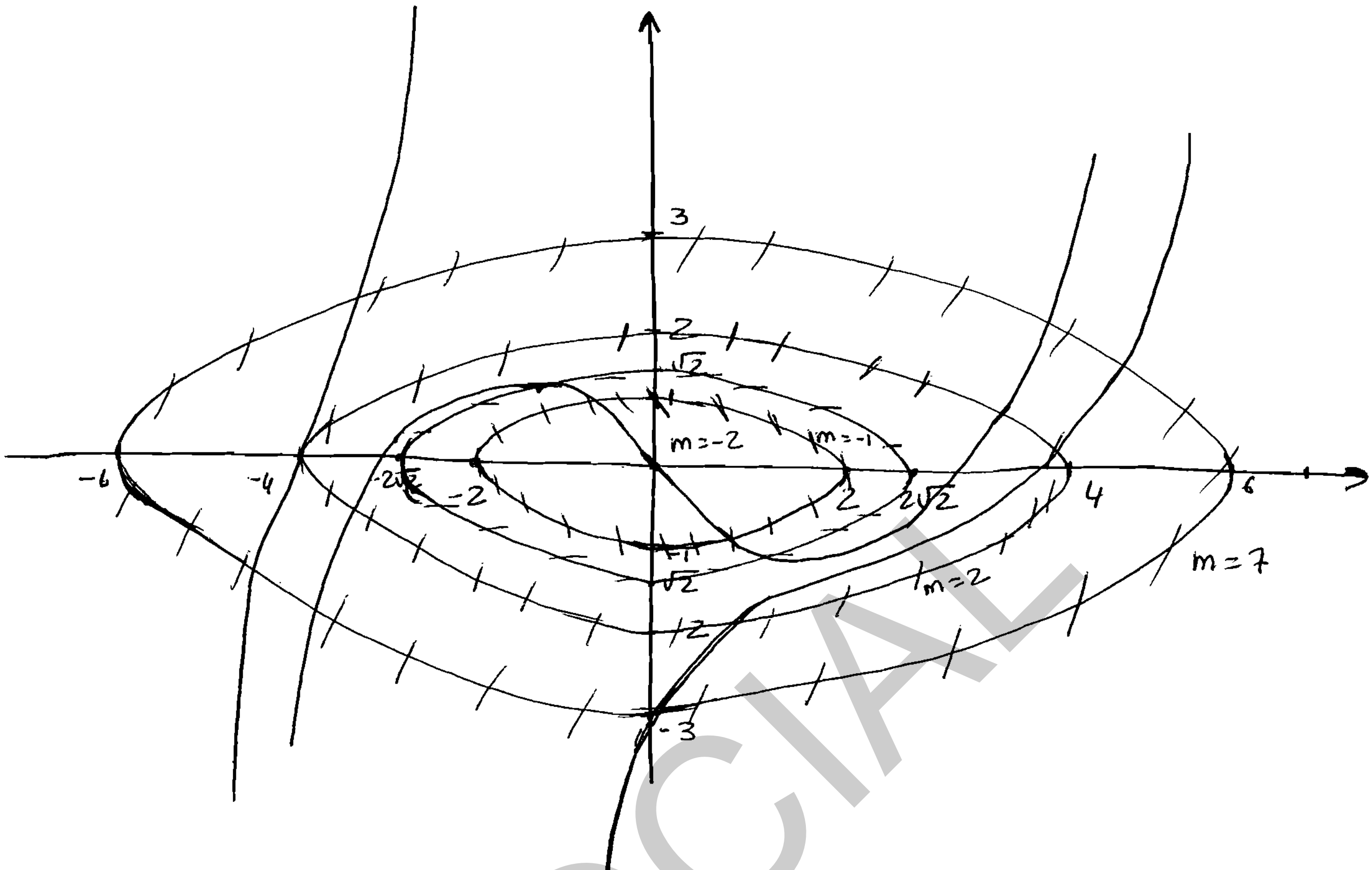
Solutions are decreasing inside the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$



Step 2 Draw some of the isoclines $f(x,y) = m$

$\frac{x^2}{4} + y^2 - 2 = m \Rightarrow \frac{x^2}{4} + y^2 = m + 2$

* $m = 0$: $\frac{x^2}{8} + \frac{y^2}{2} = 1$
* $m = 1$: $\frac{x^2}{12} + \frac{y^2}{3} = 1$ X X
* $m = -1$: $\frac{x^2}{4} + y^2 = 1$
* $m = 2$: $\frac{x^2}{16} + \frac{y^2}{4} = 1$
* $m = -2$: $\frac{x^2}{4} + y^2 = 0 \Rightarrow x = y = 0$ (1 point)
* $m = 7$: $\frac{x^2}{36} + \frac{y^2}{9} = 1$.



Solutions through: $(0,0)$, $(0,-3)$, $(-4,0)$.

1.3 Separable Differential equation

* A 1^{st}-order DE is said to be seperable if it can be Written in the form $f(y)\,dy = g(x)\,dx$.

  The general solution is $\boxed{\int f(y)\,dy = \int g(x)\,dx}$

* An initial value problem (IVP) is of the form $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

Example Solve the IVP. $\begin{cases} y'\cos x - y = 0 \\ y(0) = 2 \end{cases}$

$$\frac{dy}{dx}\cos x = y \implies \frac{1}{y}\,dy = \frac{1}{\cos x}\,dx \quad (\text{separable DE})$$

$$\int \frac{1}{y}\,dy = \int \sec x \; dx$$

$$\ln \frac{y}{c_1} = \ln \frac{\sec x + \tan x}{c_2}$$

$$\frac{y}{c_1} = \frac{\sec x + \tan x}{c_2} \Rightarrow y = \frac{c_1}{c_2}(\sec x + \tan x)$$

$$y = c(\sec x + \tan x)$$

* $y(0) = 2 \Rightarrow c(\sec 0 + \tan 0) = 2 \Rightarrow \boxed{c = 2}$

$$\boxed{y = 2(\sec x + \tan x)}$$

Newton's Law of Cooling

- The rate of change (derivative) of the temperature of an object is proportional to (is a multiple of) the difference between the temperature of the object and its surrounding medium.
  * Let $T = T(t)$ be the temperature of the object at the time t.
  * Let T_M be the temperature of the surrounding medium

$$\boxed{T' = k(T - T_M)}$$

Example: A cup of boiling water ($100°C$) is placed in a room with constant temperature $25°C$.
At the time $t = 2$ minutes the temperature of the water is $73°C$. Find $T(t)$.

$$T' = k(T - T_M) \Rightarrow \frac{dT}{dt} = k(T-25)$$

$$\frac{dT}{T-25} = k\,dt \quad \text{separate.}$$

$$\int \frac{dT}{T-25} = \int k\,dt \Rightarrow Ln\left(\frac{T-25}{c}\right) = kt$$

$$\frac{T-25}{c} = e^{kt} \Rightarrow \boxed{T = 25 + ce^{kt}}.$$

$*$ $T(0) = 100 \Rightarrow 25 + c = 100 \Rightarrow c = 75 \Rightarrow T = 25 + 75e^{kt}.$

$*$ $T(2) = 73 \Rightarrow 25 + 75e^{2K} = 73 \Rightarrow 75e^{2K} = 48 \Rightarrow$

$$e^{2K} = \frac{48}{75} = \frac{16}{25}$$

$$e^{K} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$T = 25 + 75(e^k)^t \Rightarrow \boxed{T = 25 + 75\left(\frac{4}{5}\right)^t}.$$

Reduction to a Seperable DE

If the DE can be Written in the form $y' = f\left(\frac{y}{x}\right)$ then It can be reduced to a seperable DE by applying the change of variables $z = \frac{y}{x}$

Example $\quad y' = \dfrac{xy + y^2}{x^2}$

$*$ seperable? $\quad \dfrac{dy}{dx} = \dfrac{y(x+y)}{x^2} \Rightarrow \dfrac{dy}{y} = \dfrac{x+y}{x^2}dx$

(This is not seperable)

$$* \; y' = \frac{xy}{x^2} + \frac{y^2}{x^2} \Rightarrow y' = \frac{y}{x} + \frac{y^2}{x^2} = f\left(\frac{y}{x}\right)$$

Let $z = \frac{y}{x} \Rightarrow y = xz \Rightarrow y' = u'v + uv' = z + xz'$

DE: $z + xz' = z + z^2 \Rightarrow xz' = z^2 \Rightarrow x\frac{dz}{dx} = z^2$

$$\frac{1}{z^2}\,dz = \frac{1}{x}\,dx \quad \text{seperable}$$

$$\int \frac{1}{z^2}\,dz = \int \frac{1}{x}\,dx \Rightarrow -\frac{1}{z} = \ln\frac{x}{c} \Leftrightarrow \frac{-x}{y} = \ln\frac{x}{c}$$

$$\Rightarrow \boxed{y = -\frac{x}{\ln(x/c)}}$$

Prob 1.3 : (2 – 19) + (23, 26, 29)

HW (4, 6, 9, 23, 26, 29)

Review from Calc III
* Partial Fraction method
* Heaviside Method.

(8) $xy' = \frac{1}{2}y^2 + y$

* separable ?

$$x\frac{dy}{dx} = \frac{1}{2}y^2 + y \Rightarrow \frac{dy}{\frac{1}{2}y^2+y} = \frac{dx}{x} \quad \text{separable}$$

$$\int \frac{2\,dy}{y^2+y} = \int \frac{dx}{x}$$

(I)

$$\frac{2}{y^2+2y} = \frac{2}{y(y+2)} = \frac{2/2}{y} + \frac{2/-2}{y+2}$$

$$\int\left(\frac{1}{y}-\frac{1}{y+2}\right)dy = \int \frac{1}{x}\,dx$$

$$\ln|y| - \ln|y+2| = \ln\frac{x}{c_2}$$

$$\ln\frac{y}{c_1(y+2)} = \ln\frac{x}{c_2} \Rightarrow \frac{y}{c_1(y+2)} = \frac{x}{c_2}$$

$$\Rightarrow \frac{y}{y+2} = \frac{c_1}{c_2}x.$$

$$\boxed{\frac{y}{y+2} = c\,x}$$

(12) IVP: $\begin{cases} 2xyy' = 3y^2+x^2 \\ y(1)=2 \end{cases}$

* Seperable ? $\quad 2x\,y\,\dfrac{dy}{dx} = 3y^2+x^2$

$$2y\,dy = (3y^2+x^2)\,\frac{dx}{x}$$

(It is not seperable)

* $y' = \dfrac{3y^2+x^2}{2xy} \Rightarrow y' = \dfrac{3y^2}{2xy} + \dfrac{x^2}{2xy}$

$$y' = \frac{3y}{2x} + \frac{x}{2y} = \frac{3}{2}\left(\frac{y}{x}\right) + \frac{1}{2(y/x)} = f\left(\frac{y}{x}\right)$$

Let $z = \dfrac{y}{x} \Rightarrow y = xz \Rightarrow y' = z + xz'.$

$$\therefore\; z + xz' = \frac{3}{2}z + \frac{1}{2z} \Rightarrow x\frac{dz}{dx} = \frac{1}{2}z + \frac{1}{2z}.$$

$$\ln(z^2+1) = \ln\frac{x}{c} \Rightarrow z^2+1 = \frac{x}{c} \Rightarrow \frac{y^2}{x^2}+1 = \frac{x}{c}$$

$$* \; y(1) = 2 \Rightarrow \frac{4}{1} + 1 = \frac{1}{c} \Rightarrow \frac{1}{c} = 5 \Rightarrow c = \frac{1}{5}$$

$$\boxed{\frac{y^2}{x^2} + 1 = 5x.}$$

(16) IVP : $\begin{cases} xy' = y + 4x^5\cos^2(y/x) \\ y(2) = 0. \end{cases}$

* It is not seperable.
* $y' = \frac{y}{x} + 4x^4\cos^2(y/x)$ It is not of the form $y' = f(z)$

Try the change of variables : $z = \frac{y}{x}$.

$$y = xz \Rightarrow y' = z + xz'.$$

DE : $z + xz' = z + 4x^4\cos^2 z$

$$xz' = 4x^4\cos^2 z$$

$$\frac{dz}{dx} = 4x^3\cos^2 z$$

$$\Rightarrow \frac{dz}{\cos^2 z} = 4x^3\,dx \quad \text{seperable}$$

$$\int \sec^2 z\,dz = \int 4x^3\,dx \Rightarrow \tan z = x^4 + c$$

$$\boxed{\tan\left(\frac{y}{x}\right) = x^4 + c}$$

1.4 Exact DE

Definition: The differential of $f(x,y)$ is

$$\boxed{df = f_x\,dx + f_y\,dy}$$

Example: If $f = x^2y + y^4$ then

$$df = f_x\,dx + f_y\,dy = (2xy)dx + (x^2+4y^3)\,dy$$

Remark $\boxed{f = C \iff df = 0}$

Proof:
* $f = C \Rightarrow f_x = 0,\ f_y = 0 \Rightarrow df = 0$
* $df = 0 \Rightarrow f_x = f_y = 0 \Rightarrow f$ contains neither x nor y $\Rightarrow \boxed{f = C}$

Definition The expression $w = P(x,y)\,dx + Q(x,y)\,dy$ is said to be exact if $\exists$ a function $f(x,y)$ such that $\boxed{w = df}$

Theorem: If $w = P\,dx + Q\,dy$ is exact then $\boxed{P_y = Q_x}$

Proof w exact $\Rightarrow w = df$

$\Rightarrow P\,dx + Q\,dy = f_x\,dx + f_y\,dy$

$\Rightarrow P = f_x$ and $Q = f_y$

$\Rightarrow P_y = f_{xy}$ and $Q_x = f_{yx}$

but $f_{xy} = f_{yx} \Rightarrow P_y = Q_x$

Method to solve: $P\,dx + Q\,dy = 0 \qquad (1)$

1st case: Assume that $w = P\,dx + Q\,dy$ is exact

$\Rightarrow \exists f \,/\, w = df$

DE: $w = 0 \Rightarrow df = 0 \Rightarrow f = c$

Solution: $\boxed{f = c}$

2nd case: If w is not exact We multiply the DE by a function $\rho = \rho(x,y)$ such that ρw is exact then use case 1.

The function $\rho = \rho(x,y)$ is called an Integrating factor.

Example: Solve the DE: $\underbrace{y}_{P}\,dx + \underbrace{(y-x)}_{Q}\,dy = 0$

Exact:?

$$\left.\begin{array}{l} P = y \Rightarrow P_y = 1 \\ Q = y - x \Rightarrow Q_x = -1 \end{array}\right\} \Rightarrow P_y \neq Q_x$$

$\Rightarrow w$ is not exact.

* Multiply by an integrating factor $\rho = \rho(x,y)$

$$\underbrace{\rho y}_{P}\,dx + \underbrace{\rho(y-x)}_{Q}\,dy = 0$$

w exact $\Rightarrow P_y = Q_x \Rightarrow \rho_y y + \rho = \rho_x (y-x) + \rho(-1)$

$y\rho_y + 2\rho = \rho_x (y-x)$ [This is a partial DE] PDE

* Since We need one integrating factor ρ

We try to find $\rho = \rho(x)$ or $\rho = \rho(y)$
If these 2 method don't work then a hint must be given to find ρ.

* Try $\rho = \rho(x)$: $\rho_y = 0$ and $\rho_x = \rho'$

$(3) \Rightarrow 0 + 2\rho = \rho'(y-x)$ It can't be solved because it contains 2 variables x, y.

* Try $\rho = \rho(y)$: $\rho_x = 0$ and $\rho_y = \rho'$

$(3) \Rightarrow y\rho' + 2\rho = 0$

<u>Seperable ?</u> $y \dfrac{d\rho}{dy} = -2\rho \Rightarrow \dfrac{d\rho}{\rho} = -2\dfrac{dy}{y}$

$$\int \frac{d\rho}{\rho} = \int -2\frac{dy}{y} \Rightarrow \ln\frac{\rho}{K} = -2\ln|y|$$

$$\ln\frac{\rho}{K} = \ln|y|^{-2} \Rightarrow \frac{\rho}{K} = |y|^{-2} = \frac{1}{y^2}$$

$$\Rightarrow \rho = \frac{K}{y^2}$$

Since we need 1 integrating factor we can take $K = 1$

$$\boxed{\rho = \frac{1}{y^2}}$$

* Multiply (1) by ρ: $\dfrac{1}{y^2}y\,dx + \dfrac{1}{y^2}(y-x)\,dy = 0$

$$\underbrace{\frac{1}{y}dx + \frac{y-x}{y^2}dy}_{w} = 0 \text{ (exact)}$$

(Don't simplify it)

$f = ?\ /\ f_x = P$ and $f_y = Q$

$* \quad f_x = P = \frac{1}{y} \Longrightarrow f = \int \frac{1}{y}\,dx \Longrightarrow f = \frac{x}{y} + \underbrace{g(y)}_{\text{constant with respect to } x}$

$$f = \frac{x}{y} + g(y) \Longrightarrow f_y = \frac{-x}{y^2} + g'(y)$$

but $f_y = Q = \dfrac{y-x}{y^2} = \dfrac{1}{y} - \dfrac{x}{y^2}$

$\therefore\ g'(y) = \frac{1}{y}$ (doesn't contain x)

$\Longrightarrow g(y) = \ln|y| + K$

$\Longrightarrow f = \frac{x}{y} + \ln|y| + K$

Since we need 1 f we can take $K = 0$

$$f = \frac{x}{y} + \ln|y|$$

General Solution: $f = c$.

$$\boxed{\frac{x}{y} + \ln|y| = c}$$

Prob (1–20) HW (1, 9, 11, 14)

(4) $\underbrace{(e^y - y e^x)}_{P}\,dx + \underbrace{(x e^y - e^x)}_{Q}\,dy = 0.$

$*$ Exact ?

$P_y = e^y - e^x \Big\} \Longrightarrow P_y = Q_x \Longrightarrow W$ is exact

$* \; f = ? \; / \; df = \omega$

$f = ? \; / \; f_x = P, \; f_y = Q$

$f_y = Q = xe^y - e^x \Rightarrow f = xe^y - ye^x + g(x)$

$\left.\begin{array}{l} \Rightarrow f_x = e^y - ye^x + g'(x) \\ \text{but } \; f_x = P = e^y - ye^x \end{array}\right] \Rightarrow g'(x) = 0 \Rightarrow g(x) = K$

We need one f: Take $K = 0$

$\Rightarrow f = xe^y - ye^x$

* General solution: $f = C \Rightarrow \boxed{xe^y - ye^x = C}$

(12) $\underbrace{(e^{x+y} - y)}_{P}dx + \underbrace{(xe^{x+y} + 1)}_{Q}dy = 0$

<u>Exact?</u> $\left.\begin{array}{l} P_y = e^{x+y} - 1 \\ Q_x = e^{x+y} + xe^{x+y} \end{array}\right] \Rightarrow P_y \neq Q_x \Rightarrow \omega$ is not exact

Multiply by $\rho = \rho(x,y)$

$\rho\,(e^{x+y} - y)dx + \rho\,(xe^{x+y} + 1)dy = 0$ is exact

$\Rightarrow P_y = Q_x \Rightarrow \rho_y\,(e^{x+y} - y) + \rho\,(e^{x+y} - 1)$

$= \rho_x\,(xe^{x+y} + 1) + \rho(e^{x+y} + xe^{x+y})$

$\rho_y\,(e^{x+y} - y) = \rho_x\,(xe^{x+y} + 1) + \rho(xe^{x+y} + 1)$

$$0 = \rho'(xe^{x+y}+1) + \rho(xe^{x+y}+1) \Rightarrow \rho' = -\rho.$$

$$\Rightarrow \rho = Ke^{-x} \text{ (formula in section 1)}$$

Take $K=1 \Rightarrow \boxed{\rho = e^{-x}}$

$$(1) \Rightarrow (e^y - ye^{-x})\,dx + (xe^y + e^{-x})\,dy = 0.$$

* find f / $df = \omega$ $\qquad f = xe^y + ye^{-x}$.

solution $f = c$

$$\Rightarrow xe^y + ye^{-x} = c.$$

13 $\underbrace{-3y}_{P}\,dx + \underbrace{2x}_{Q}\,dy = 0$ (1)

<u>Exact</u>: $\left.\begin{array}{l} P_y = -3 \\ Q_x = 2 \end{array}\right] \Rightarrow P_y \neq Q_x \Rightarrow \omega$ is not exact.

<u>Multiply by $\rho = \rho(x,y)$</u>

$$-3y\rho\,dx + 2x\rho\,dy = 0 \quad (2)$$

ω exact $\Rightarrow P_y = Q_x \Rightarrow -3\rho - 3y\rho_y = 2\rho + 2x\rho_x$

$$-3y\rho_y = 5\rho + 2x\rho_x.$$

Try: $\rho = \rho(x) \Rightarrow \rho_x = \rho'$ and $\rho_y = 0$.

$$0 = 5\rho + 2x\rho' \Rightarrow 2x\frac{d\rho}{dx} = -5\rho \Rightarrow \frac{d\rho}{\rho} = \frac{-5\,dx}{2x}$$

$$\ln\frac{\rho}{c_1} = \ln\left(\frac{x}{c_2}\right)^{-5/2} \Rightarrow \rho = c_1\left(\frac{x}{c_2}\right)^{-5/2} = c_1\,\frac{|x|^{-5/2}}{c_2^{-5/2}} = c\,|x|^{-5/2}$$

Take $c=1 \Rightarrow \rho = |x|^{-5/2}$

* Assume $x>0$ then $\rho = x^{-5/2}$

$$(2) \Rightarrow \underbrace{-3y\,x^{-5/2}}_{P}\,dx + \underbrace{2\,x^{-3/2}}_{Q}\,dy = 0.$$

$f = ? \;/\; df = W$

$f = ? \;/\; f_x = P$ and $f_y = Q$.

* $f_x = P = -3y\,x^{-5/2} \Rightarrow f = -3y\,\dfrac{x^{-3/2}}{-3/2} + g(y) = 2y\,x^{-3/2} + g(y).$

$$\left.\begin{array}{l} \Rightarrow f_y = 2x^{-3/2} + g'(y) \\ \text{but } f_y = Q = 2x^{-3/2} \end{array}\right\} \Rightarrow g'(y) = 0 \Rightarrow g(y) = K \rightarrow 0$$

(need 1 f).

$$f = 2y\,x^{-3/2}$$

Solution: $2y\,x^{-3/2} = c \Rightarrow \boxed{y = \dfrac{c}{2x^{-3/2}} = \dfrac{c}{2}\,x^{3/2}}$

1.5 Linear DE

Definition: A 1^{st}-order DE is said to be Linear if it is of the form:

$$\boxed{y' + a(x)y = r(x)}$$

Theorem The general solution to the LDE is:

$$\boxed{y = \frac{1}{\rho}\int \rho\, r\, dx}$$

Where $\boxed{\rho = e^{\int a(x)dx}}$ called an integrating factor.

Proof: Multiply (1) by $\rho = \rho(x)$:

$$\underset{v}{\rho}\, \underset{u'}{y'} + a\, \underset{u}{y\rho} = \rho r \quad (2)$$

Let ρ be a function such that $\rho y' + a y \rho = (\rho y)'$ (3)

$$\rho y' + a y \rho = \rho' y + \rho y' \Rightarrow a y \rho = \rho' y$$

$$\Rightarrow a\rho = \rho'$$

$$a\rho = \frac{d\rho}{dx} \Rightarrow \frac{d\rho}{\rho} = a(x)dx \Rightarrow \int \frac{d\rho}{\rho} = \int a(x)dx.$$

$$\mathrm{Ln}\,\frac{\rho}{K} = \int a(x)dx \Rightarrow \frac{\rho}{K} = e^{\int a(x)dx}.$$

We need 1 ρ : Take $K = 1 \Rightarrow \rho = e^{\int a(x)dx}$

(2), (3) $\Rightarrow (\rho y)' = \rho r \Rightarrow \rho y = \int \rho r\, dx$

$$\boxed{y = \frac{1}{\rho}\int \rho\, r\, dx}$$

Bernoulli equation: $y' + a(x)y = r(x)y^m$ where $m \neq 0$, $m \neq 1$

This Bernoulli equation can be transformed into a LDE as follows:

Step 1: Divide by y^m: $y^{-m}y' + a(x)y^{1-m} = r(x)$

Step 2: Let $z = y^{1-m} \Rightarrow z' = (1-m)y^{-m}y' \Rightarrow y^{-m}y' = \frac{z'}{1-m}$

$$\therefore \frac{z'}{1-m} + a(x)z = r(x)$$

$$\Rightarrow z' + a(x)(1-m)z = (1-m)r(x) \quad \text{LDE}$$

Example Solve $y' = ay - by^2$ (called Logistic equation) (a, b) constant.

$$y' - ay = -by^2 \quad \text{(Bernoulli)}$$

Step 1 Divide by y^2

$$y^{-2}y' - ay^{-1} = -b$$

Step 2 Let $z = y^{-1} \Rightarrow z' = -y^{-2}y' \Rightarrow y^{-2}y' = -z'$

$$\therefore -z' - az = -b \Rightarrow z' + az = b \quad \text{(LDE)}$$

Integrating factor $\rho = e^{\int a(x)dx} = e^{\int a\,dx} = e^{ax + \overset{0}{K}} \Rightarrow \boxed{\rho = e^{ax}}$

General Solution $z = \frac{1}{\rho}\int \rho r\,dx = \frac{1}{e^{ax}}\int e^{ax} b\,dx$

Chap 7.

$$* \quad \rho = e^{\int a(x)dx} = e^{\int 3x^2 dx} = e^{x^3 + k} \Rightarrow \boxed{\rho = e^{x^3}}$$

$k \to 0$

$$* \text{ Solution: } \quad z = \frac{1}{\rho}\int \rho\, r\, dx = \frac{1}{e^{x^3}} \int e^{x^3} e^{-x^3} \sinh x \, dx$$

$$z = \frac{1}{e^{x^3}} \int \sinh x \, dx \Rightarrow$$

$$\boxed{y^3 = \frac{1}{e^{x^3}}(\cosh x + c)}$$

Chapter 2 Second Order LDE

2.3 Differential Operator

Definition: A 2nd order DE is said to be linear if it is of the form:

$$\boxed{y'' + a(x)y' + b(x)y = r(x)} \quad (1)$$

If $r(x) = 0$ then $y'' + a(x)y' + b(x)y = 0$ is called a Homogeneous LDE. (HLDE).

Notation: If we denote y' by Dy and y'' by D^2y

then $D^2y + a(x)Dy + b(x)y = r(x)$

$$(D^2 + aD + b)y = r(x)$$

Let $L = D^2 + aD + b$

then $\boxed{L(y) = r(x)}$ $(1')$

Theorem: The operator L is linear. That is:

1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

Proof: $L(y_1+y_2) = (D^2+aD+b)(y_1+y_2)$

$$= (y_1''+y_2'') + a(y_1'+y_2') + b(y_1+y_2)$$
$$= (y_1''+ay_1'+by_1) + (y_2''+ay_2'+by_2)$$
$$= L(y_1) + L(y_2)$$

$$L(\alpha y) = (D^2+aD+b)(\alpha y)$$
$$= \alpha y'' + a\alpha y' + b\alpha y = \alpha(y''+ay'+by) = \alpha L(y)$$

Delete Prob 2.3

2.1 2^{nd}-order HLDE

Theorem: Consider the 2^{nd}-order HLDE: $y''+a(x)y'+b(x)y=0$ (1)
If y_1 and y_2 are 2 solutions to (1) then any linear combination of y_1 and y_2 is also a solution to (1).
If c_1 and c_2 are constant then $c_1y_1+c_2y_2$ is called a linear combination of y_1, y_2.

Proof: y_1 is a solution of (1) $\Rightarrow y_1''+a(x)y_1'+b(x)y_1=0 \Rightarrow L(y_1)=0$

y_2 is a solution of (1) $\Rightarrow L(y_2)=0$.

$$L(c_1y_1+c_2y_2) = L(c_1y_1)+L(c_2y_2)$$
$$= c_1L(y_1) + c_2L(y_2)$$
$$= c_1\cdot 0 + c_2\cdot 0 = 0$$

$\Rightarrow y = c_1y_1+c_2y_2$ is a solution to (1).

Definition Two functions y_1 and y_2 are said to be linearly dependent if $\exists$ 2 constants c_1 and c_2 (not both 0) such that

$$\boxed{c_1 y_1 + c_2 y_2 = 0}$$

Theorem: y_1 and y_2 are linearly dependent iff

$$\frac{y_1}{y_2} \left(\text{or } \frac{y_2}{y_1}\right) \text{ is constant.}$$

Theorem: If y_1, y_2 are two linearly independent solutions to the HLDE: $y'' + a(x)y' + b(x)y = 0$ (1) then the general solution is $\boxed{y = c_1 y_1 + c_2 y_2}$

Knowing y_1 how to find y_2?

Let y_1 be a solution to (1) then $y_1'' + a y_1' + b y_1 = 0$

Try to find a 2nd solution of the form $y_2 = y_1 v(x)$

y_2 solution $\Rightarrow y_2'' + a y_2' + b y_2 = 0$ (3)

$y_2 = y_1 v \Rightarrow y_2' = y_1' v + y_1 v'$

$\Rightarrow y_2'' = y_1'' v + y_1' v' + y_1' v' + y_1 v''$

Replace in (3): $(y_1'' v + 2y_1' v' + y_1 v'') + a(y_1' v + y_1 v') + b(y_1 v) = 0$

$$\Rightarrow v\,(\underbrace{y_1'' + a y_1' + b y_1}_{0}) + v'(2y_1' + a y_1) + v'' y_1 = 0$$

$$v'' y_1 = -v'(2y_1' + a y_1) \Rightarrow \frac{v''}{v'} = -\frac{2y_1' + a y_1}{y_1}$$

$$\int \frac{v''}{v'} dx = -\int \left(\frac{2y_1'}{y_1} + a\right) dx \Rightarrow \ln \frac{v'}{C} = -2\ln|y_1| - \int a(x)\,dx$$

$$\ln\frac{v'}{c} + \ln y_1^2 = -\int a(x)\,dx \Rightarrow \ln\left(\frac{v' y_1^2}{c}\right) = -\int a(x)\,dx$$

$$\frac{v' y_1^2}{c} = e^{-\int a(x)dx} \Rightarrow v' = \frac{c}{y_1^2} e^{-\int a(x)dx}$$

* If $c=0$ then $v'=0 \Rightarrow v = \underline{cte} \Rightarrow \frac{y_2}{y_1} = v$ is constant

$\Rightarrow y_1$ and y_2 are linearly dependent.

* If $c \neq 0 \Rightarrow v' \neq 0 \Rightarrow \frac{y_2}{y_1} = v$ is not constant

$\Rightarrow y_1, y_2$ are linearly independent.

Take $c=1$ then $v' = \frac{1}{y_1^2} e^{-\int a(x)dx} \Rightarrow v(x) = \int \frac{1}{y_1^2} e^{-\int a(x)dx} dx$

$$y_2 = y_1 v \Rightarrow y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int a(x)dx} dx$$

__Theorem__ If y_1 is a particular solution to the HLDE:
$y'' + a(x)y' + b(x)y = 0$ then the general solution is

$\boxed{y = c_1 y_1 + c_2 y_2}$ Where $\boxed{y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int a(x)dx} dx}$

__Example__: Solve the HLDE $4x^2 y'' + 8xy' + y = 0 \quad (x>0)$

Hint: Find a particular solution of the form $y_1 = x^m$

$y = x^m \Rightarrow y' = m x^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

$(1) \Rightarrow 4x^2(m)(m-1)x^{m-2} + 8x\, m x^{m-1} + x^m = 0.$

$\Rightarrow x^m [4m(m-1) + 8m + 1] = 0$

$-b' \pm \sqrt{\Delta'}$

* $y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int a(x)dx} dx$ Where $a(x) = \frac{8x}{4x^2} = \frac{2}{x}$

$$y_2 = x^{-\frac{1}{2}} \int \frac{1}{x^{-1}} e^{-\int \frac{2}{x}dx} dx = \frac{1}{\sqrt{x}} \int x\, e^{-2\ln|x| + \cancel{K}^{\,0}} dx$$

$$y_2 = \frac{1}{\sqrt{x}} \int x\, e^{\ln x^{-2}} dx = \frac{1}{\sqrt{x}} \int \frac{1}{x} dx = \frac{1}{\sqrt{x}} (\ln x + \underset{\downarrow\\0}{K})$$

$$y_2 = \frac{\ln x}{\sqrt{x}}$$

General Solution: $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = \frac{c_1}{\sqrt{x}} + c_2 \frac{\ln x}{\sqrt{x}}}$$

Remark: 1) $F(x, y', y'') = 0$ can be transformed into a 1^{st} order DE by letting $\boxed{z = y'}$

$z = y' \Rightarrow y'' = z' \Rightarrow \boxed{F(x, z, z') = 0} \rightarrow$ Chap 1

2) $F(y, y', y'') = 0$ can be transformed into a 1^{st} order DE by letting $\boxed{z = y'}$

$y' = z \Rightarrow y'' = z' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = z'_y\, y' = z z'$

$$\boxed{F(y, z, zz') = 0}$$

Prob 2.1 (1 – 22) HW (9, 11, 19, 20)

10) $y_1 = \cos^2 x$, $y_2 = \sin^2 x$ (dep. or indep.)?

$\frac{y_1}{y_2} = \cot^2 x$ is not constant $\Rightarrow y_1, y_2$ are lin. indep.

(21) $y'' + y'^3 \sin y = 0$ This is not a Linear DE

It doesnt contain x: Let $z = y' \Rightarrow y'' = z'_x = z z'_y$

$$\therefore z z' + z^3 \sin y = 0 \Rightarrow z(z' + z^2 \sin y) = 0 \Rightarrow z = 0 \text{ or } z' = -z^2 \sin y$$

* If $z = 0 \Rightarrow y' = 0 \Rightarrow \boxed{y = c}$

* $\frac{dz}{dy} = -z^2 \sin y \Rightarrow \frac{dz}{z^2} = -\sin y \, dy$ seperable

$$\int \frac{dz}{z^2} = \int -\sin y \, dy \Rightarrow -\frac{1}{z} = \cos y + c_1$$

$$\Rightarrow -\frac{1}{y'} = \cos y + c_1 \Rightarrow -\frac{dx}{dy} = \cos y + c_1$$

$-dx = (\cos y + c_1) dy$ seperable

$$\int -dx = \int (\cos y + c_1) dy$$

$$\Rightarrow \boxed{-x = \sin y + c_1 y + c_2}$$

(22) $(1 - x^2) y'' - 2x y' + 2y = 0$

$y'' - \frac{2x}{1-x^2} y' + \frac{2}{1-x^2} y = 0$ LDE

Hint: Verify that $y_1 = x$ is a solution

$$y = x \Rightarrow y' = 1 \Rightarrow y'' = 0 .$$

$(1) \Rightarrow 0 - 2x(1) + 2x = 0$ ✓

$y_2 = y_1 \int \frac{1}{\ldots} e^{-\int a(x) dx} \; \ldots \int 1 \; e^{-\int \frac{-2x}{1-x^2} dx}$

$$y_2 = x\int \frac{1}{x^2}\, e^{-\ln\frac{1-x^2}{k}}\,dx = x\int \frac{1}{x^2}\,\frac{k}{1-x^2}\,dx$$

$$k = 1 \Longrightarrow y_2 = x\int \frac{1}{x^2(1-x^2)}\,dx$$

Partial Fractions Method;

Heaviside Method

1) $$\frac{1}{x^2(1-x^2)} = \frac{1}{x^2(1-x)(1+x)} = \frac{a}{x} + \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

Replace x by any value $\neq 0, 1, -1$

$x = 2 \Rightarrow$ find a $(a = 0)$

2) If the powers of x are even we let $X = x^2$ in the fraction

$$\frac{1}{x^2(1-x^2)} = \frac{1}{X(1-X)} = \frac{1}{X} + \frac{1}{1-X}$$

$$= \frac{1}{x^2} + \frac{1}{1-x^2} = \frac{1}{x^2} + \frac{1}{(1-x)(1+x)}$$

$$= \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

$$I = \int\left(\frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x}\right)dx = -\frac{1}{x} - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+K| + K$$

$$I = -\frac{1}{x} + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + K$$

$$y_2 = xI = x\left(-\frac{1}{x} + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + \underset{0}{\underset{\downarrow}{K}}\right)$$

(We need 1 y_2)

$$y_2 = -1 + \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right|$$

General Solution : $y = c_1 y_1 + c_2 y_2$

2.2 2^{nd} order HLDE with Constant Coefficients

Consider the 2^{nd} order HLDEC: $\boxed{y'' + ay' + by = 0}$ (1)

Where a and b are constant

Find a particular solution of the form $y_1 = e^{\lambda x}$

$$y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}$$

Replace in (1): $\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \Rightarrow \boxed{\lambda^2 + a\lambda + b = 0} \quad (2)$$

(2) is called the characteristic equation.

If λ_1 is a root to (2) then $y_1 = e^{\lambda_1 x}$ is a solution to (1)

1st case: If $\Delta = a^2 - 4b > 0$ then (2) has 2 distinct real roots $\boxed{\lambda_1 \neq \lambda_2} \Rightarrow y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are 2 solutions.

$$\frac{y_1}{y_2} = \frac{e^{\lambda_1 x}}{e^{\lambda_2 x}} = e^{(\lambda_1 - \lambda_2)x}$$ is not constant because $\lambda_1 \neq \lambda_2$

$\Rightarrow y_1, y_2$ are linearly independent.

General Solution: $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

2nd case: If $\Delta = a^2 - 4b = 0$ then (2) has a double root $\lambda_1 = \lambda_2 = \frac{-a}{2}$

$\Rightarrow y_1 = e^{\lambda_1 x}$ is a solution to (1).

* 2^{nd} solution: $y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int a(x)dx} dx = e^{\lambda_1 x} \int \frac{1}{\ldots} e^{-\int a\, dx}$

General Solution $y = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$

$$\boxed{y = (c_1 + c_2 x) e^{\lambda_1 x}}$$

3rd case: If $\Delta < 0$ then (2) has 2 complex roots: $\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \alpha \pm i\beta$

$\Rightarrow z_1 = e^{\lambda_1 x}$ and $z_2 = e^{\lambda_2 x}$ are 2 solutions $\boxed{\beta \neq 0}$

but they are complex solutions (They involve the complex exponential function)

The real solutions can be obtained as follows:

$$z_1 = e^{\lambda_1 x} = e^{(\alpha + i\beta)x} = e^{\alpha x + i\beta x} = e^{\alpha x}(\cos\beta x + i \sin\beta x)$$

$$z_2 = e^{\lambda_2 x} = e^{\alpha x - i\beta x} = e^{\alpha x}(\cos\beta x - i \sin\beta x)$$

(2.1) any linear combination of 2 solutions is also a solution to (1)

* $z_1 + z_2 = 2e^{\alpha x}\cos\beta x \Rightarrow \frac{1}{2} z_1 + \frac{1}{2} z_2 = e^{\alpha x}\cos\beta x$ is a linear combination of z_1 and z_2 $\Rightarrow y_1 = e^{\alpha x}\cos\beta x$ is a real solution.
* $z_1 - z_2 = 2i e^{\alpha x}\sin\beta x \Rightarrow \frac{z_1}{2i} - \frac{z_2}{2i} = e^{\alpha x}\sin\beta x$ is another lin. comb. $\Rightarrow y_2 = e^{\alpha x}\sin\beta x$ is another real solution.
* $\frac{y_2}{y_1} = \tan\beta x$ is not constant because $\beta \neq 0 \Rightarrow y_1, y_2$ are lin. indep.

General solution $y = c_1 y_1 + c_2 y_2$

$$\boxed{y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)}.$$

Prob 2.2 (1–32) HW(3, 7, 9, 10, 23)

(2) $10y'' - 7y' + 1.2y = 0$ HLDEC

* Char. equ: $10\lambda^2 - 7\lambda + 1.2 = 0$

$$\Delta = b^2 - 4ac = 49 - 48 = 1$$

Roots: $\lambda = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{7 \pm 1}{20} \Rightarrow \lambda_1 = \dfrac{3}{10} = 0.3$

$$\lambda_2 = \frac{4}{10} = 0.4$$

Sol: $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$$\boxed{y = c_1 e^{0.3x} + c_2 e^{0.4x}}$$

(6) $y'' + 2y' + 5y = 0$ HLDEC

Char. equ $\lambda^2 + 2\lambda + 5 = 0$

$$\Delta' = b'^2 - ac = 1 - 5 = -4 = 4i^2$$

$$\lambda = \frac{-b' \pm \sqrt{\Delta'}}{a} = \frac{-1 \pm 2i}{1} = \alpha \pm i\beta$$

Gen. sol. $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

(8) $y'' + 2.6y' + 1.69y = 0$ HLDEC.

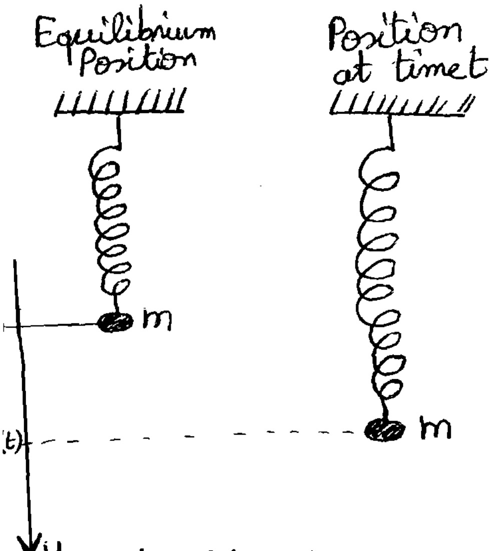
Char. equ $\lambda^2 + 2.6\lambda + 1.69 = 0$.

$$\Delta' = (1.3)^2 - 1.69 = 0.$$

$$\lambda_1 = \lambda_2 = \frac{-b' \pm 0}{a} = -1.3 \text{ (double}$$

General Solution $y = (c_1 + c_2 x)e^{\lambda_1 x}$

2.4 Mass - Spring System



A mass m is attached to the lower end of a spring. If we push the mass m y_0 units and give it an initial velocity of v_0 then m will oscillate about the equilibrium position.
Let $y(t)$ to the position of m at the time t.

* <u>Hookes Law</u>: The force applied from the spring on the mass m is proportional to the displacement ΔL.

$$\boxed{F = -k\,y(t)}$$

Where k is a positive constant called the spring constant.

* <u>By Newton's Law</u>: $\boxed{F = m\,a(t)}$ Where $a(t) = y''$ is the acceleration.

$$F = F \Rightarrow m\,a(t) = -k\,y(t) \Rightarrow m y'' + k y = 0$$

$$y'' + \frac{k}{m} y = 0 \quad \text{HLDEC}$$

* char equ: $\lambda^2 + \frac{k}{m} = 0$.

Let $\boxed{\omega^2 = \frac{k}{m}}$ then $\lambda^2 + \omega^2 = 0 \Leftrightarrow \lambda^2 = -\omega^2 = i^2 \omega^2$

$$\Rightarrow \begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \pm i\omega = \alpha \pm i\beta$$

Solution: $y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$

$= \boxed{y = c_1 \cos(\omega t) + c_2 \sin(\omega t)}$

Initial Position: $y(0) = y_0 \Rightarrow c_1 = y_0$

$$\Rightarrow y = y_0 \cos \omega t + c_2 \sin \omega t$$

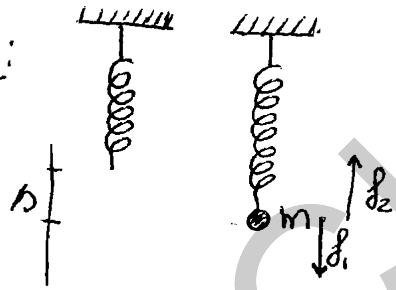
Initial Velocity: $y'(0) = v_0$

$$y(t) = -y_0 \omega \sin \omega t + c_2 \omega \cos \omega t$$

$$\Rightarrow y'(0) = c_2 \omega = v_0 \Rightarrow c_2 = \frac{v_0}{\omega}$$

Position of m at time t: $y(t) = y_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t.$

<u>Spring Constant:</u>



A mass of m kg stretches a spring s meters.

* At the equilibrium position, the sum of all forces is 0. The force due to gravity is $f_1 = mg$.

The force from the spring on m is: $f_2 = -k\Delta L$ (Hookes Law)

$$\Delta L = s \Rightarrow f_2 = -ks$$

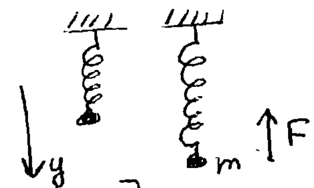
$$f_1 + f_2 = 0 \Rightarrow mg - ks = 0 \Rightarrow \boxed{k = \frac{mg}{s}}$$

<u>Example</u> A ball of weight $W = mg = 89$ Newtons stretches a spring 0.1 meter.

a) Find the spring constant k.

b) We pull down the ball an additional 0.15 m and release it with an initial velocity $v_0 = 0.35\sqrt{5}$ m/sec. Find the motion of the ball.

c) Find the period T and the frequency f.



b) $y(0) = 0.15 \, m$

$y'(0) = 0.35\sqrt{6} \; m/sec.$

By Hookes Law: $F = -k\, y(t)$ [If $y(t) > 0$ then $F < 0$; If $y(t) < 0$ then $F > 0$]

$\boxed{F = -890\, y}$.

By Newton's Law: $F = m\,a(t) = m y'' \Rightarrow \boxed{F = \frac{89}{g} y''}$

$$F = F \Rightarrow \frac{89}{g} y'' = -890 y \Rightarrow y'' = -10 g y.$$

$$y'' + 98 y = 0 \quad \text{HLDEC}.$$

* char. equ.: $\lambda^2 + 98 = 0 \Rightarrow \lambda^2 = -98 = 98 i^2 \Rightarrow \lambda_1, \lambda_2 = \pm i 7\sqrt{2}$

$$y = C_1 \cos 7\sqrt{2} t + C_2 \sin 7\sqrt{2} t \Rightarrow y' = -7\sqrt{2} C_1 \sin 7\sqrt{2} t + 7\sqrt{2} C_2 \sin 7\sqrt{2} t$$

* $y(0) = 0.15 \Rightarrow C_1 = 0.15$
* $y'(0) = 0.35\sqrt{6} \Rightarrow 7\sqrt{2}\, C_2 = 0.35\sqrt{6} \Rightarrow C_2 = 0.05\sqrt{3}$

$$\boxed{y = 0.15 \cos 7\sqrt{2} t + 0.05\sqrt{3} \sin 7\sqrt{2} t}.$$

c) The period T is the time needed to finish 1 oscillation

$$\boxed{T = \frac{2\pi}{w}} \Rightarrow \boxed{T = \frac{2\pi}{7\sqrt{2}}}$$

The frequency is the number of oscillations per second

1 oscillation $\longrightarrow$ T seconds

$f \longrightarrow 1$ second

$$\boxed{f = \frac{1}{T} \text{ hertz}} \Rightarrow \boxed{f = \frac{7\sqrt{2}}{2\pi} \text{ hertz}}$$

Prob 2.4 HW (1', 2(i, ii), 5)

2.5 Euler - Cauchy Equation

The 2^{nd} order Euler-cauchy equation is of the form

$\boxed{x^2 y'' + a x y' + by = 0}$. Where a and b are constant.

This is a LDE but the coefficients $a(x) = \frac{a}{x}$, $b(x) = \frac{b}{x^2}$ are not constant

1^{st} method: Find a particular solution of the form $y = x^m$

↳ for 2^{nd} order and higher order Cauchy equ.

$y = x^m \Rightarrow y' = m x^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

$(1) \Rightarrow m(m-1)x^m + a m x^m + b x^m = 0$.

$\Rightarrow m^2 - m + am + b = 0$

$\Rightarrow \boxed{m^2 + (a-1)m + b = 0}$ (2)

If m is a root to (2) then x^m is a solution to (1)

* If $\Delta > 0$ then (2) has 2 distinct real roots m_1, m_2
$\Rightarrow y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ are 2 lin. indep. solutions.
Gen. Sol: $y = c_1 x^{m_1} + c_2 x^{m_2}$

* If $\Delta = 0$ then (2) has a double root $m_1 = m_2 \Rightarrow y_1 = x^{m_1}$ is a solution to (1).
2^{nd} solution: $y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int a(x)dx} dx$
Gen. Sol. $y = c_1 x^{m_1} + c_2 y_2$

* If $\Delta < 0$ then (2) has 2 complex roots $\begin{matrix} m_1 \\ m_2 \end{matrix} = \alpha \pm i\beta$.
$\Rightarrow y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ are 2 solutions to (1). but

2nd method — for 2nd order cauchy equ. only

The cauchy-Euler equation can be transformed into a LDE by letting $x = e^t$ if $x > 0$ or $x = -e^t$ if $x < 0$

$$\Rightarrow |x| = e^t \Rightarrow t = \ln|x|$$

$$x^2 y'' + a x y' + b y = 0$$

Assume $x > 0$: $x = e^t$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{e^t}$$

$$y'' = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{\left(\frac{\dot{y}}{e^t}\right)'}{e^t} = \frac{\frac{\ddot{y}e^t - \dot{y}e^t}{e^{2t}}}{e^t}$$

$$y'' = \frac{\ddot{y} - \dot{y}}{e^{2t}}$$

$$x^2 y'' + a x y' + b y = 0 \Rightarrow e^{2t}\left(\frac{\ddot{y} - \dot{y}}{e^{2t}}\right) + a e^t\left(\frac{\dot{y}}{e^t}\right) + b y = 0$$

$$\ddot{y} - \dot{y} + a\dot{y} + b y = 0.$$

$$\Rightarrow \boxed{\ddot{y} + (a-1)\dot{y} + b y = 0} \text{ HLDEC}$$

If $x < 0$ we obtain the same formula

Prob 2.5 (1–15) HW(3,5,7,8) (2 methods)

② $4x^2 y'' + 4 x y' - y = 0$

Write it in standard form: $x^2 y'' + x y' - \frac{1}{4} y = 0$ (Euler)

Let $|x| = e^t \Rightarrow t = \ln|x|$

$$\ddot{y} + (1-1)\dot{y} - \frac{1}{4} y = 0 \Rightarrow \ddot{y} - \frac{1}{4} y = 0 \text{ HLDEC.}$$

* characteristic equation: $\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda_1, \lambda_2 = \pm\frac{1}{2}$
* Solution: $y = c_1 e^{1/2 t} + c_2 e^{-1/2 t}$

) $x^2y'' + 3xy' + y = 0$ Cauchy

$a = 3$ $\qquad$ $b = 1$

Let $|x| = e^t \Rightarrow \ddot{y} + 2\dot{y} + y = 0$ HLDEC

* Char. equ $\quad \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$

$$y = (c_1 + c_2 t)e^{-t} \Rightarrow y = (c_1 + c_2 \ln|x|)\,|x|^{-1}$$

$$\boxed{y = \frac{c_1}{|x|} + \frac{c_2}{|x|}\ln|x|}$$

2.6 Wronskian

Definition: The Wronskian of 2 functions y_1 and y_2 is the determinant $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$

Theorem y_1 and y_2 are linearly dependent iff $W = 0$

Proof y_1, y_2 linearly dependent $\Leftrightarrow \frac{y_2}{y_1} = c \Leftrightarrow y_2 = cy_1$

$$W = y_1y_2' - y_2y_1' = y_1(cy_1') - (cy_1)y_1' = 0$$

Prob 2.6 (1 – 17) HW(7, 9, 13, 17)

⑥ $y_1 = e^{3.4x}$, $y_2 = e^{-2.5x}$

a) Lin. dep.?

$$W = \begin{vmatrix} e^{3.4x} & e^{-2.5x} \\ 3.4e^{3.4x} & -2.5e^{-2.5x} \end{vmatrix} = -2.5e^{0.9x} - 3.4e^{0.9x} \neq 0$$

b) Find a LDE whose solutions are y_1, y_2

y_1, y_2 are solutions to the HLDEC whose char. equation has 2 distinct real roots $\lambda_1 = 3.4$, $\lambda_2 = -2.5$

Char. equ: $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \Rightarrow (\lambda - 3.4)(\lambda + 2.5) = 0$

$\Rightarrow \lambda^2 - 0.9\lambda - 8.5 = 0$.

LDE: $\boxed{y'' - 0.9y' - 8.5y = 0}$

(10) $y_1 = x^{-3}$, $y_2 = x^{-3} \ln x$

a) $\frac{y_2}{y_1} = \ln x$ is not constant $\Rightarrow y_1$ and y_2 are lin. indep.

b) y_1, y_2 are solutions to a Cauchy equation.

$x = e^t \Rightarrow y_1 = e^{-3t}$ and $y_2 = te^{-3t}$

y_1, y_2 are solutions to HLDEC whose characteristic equ. has a double root $\lambda_1 = \lambda_2 = -3$.

Char. equ: $(\lambda + 3)^2 = 0 \Rightarrow \lambda^2 + 6\lambda + 9 = 0$.

$\Rightarrow$ LDE: $\ddot{y} + \underset{\downarrow}{6}\dot{y} + 9y = 0$.
$\qquad\qquad a-1$

Cauchy equ: $x^2 y'' + 7x y' + 9y = 0$.

(14) $y_1 = x^{-1} \cos \ln x$, $y_2 = x^{-1} \sin \ln x$.

a) $\frac{y_2}{y_1} = \tan(\ln x) \neq$ cte $\Rightarrow$ lin. indep.

b) $x = e^t \Rightarrow y_1 = e^{-t} \cos t$, $y_2 = e^{-t} \sin t$.

They are solutions to the HLDEC whose char. equ. has 2 co...

Char. equ: $(\lambda-\lambda_1)(\lambda-\lambda_2)=0$

$(\lambda+1+i)(\lambda+1-i)=0 \Rightarrow (\lambda+1)^2-i^2=0$

$\Rightarrow \lambda^2+2\lambda+1+1=0 \Rightarrow \lambda^2+2\lambda+2=0$

LDE: $\ddot{y}+2\dot{y}+2y=0$.

$a-1 \longrightarrow a=3$

Cauchy equ: $x^2y''+3xy'+2y=0$

2.7 Nonhomogeneous LDE

Theorem 1: Consider the LDE: $y''+a(x)y'+b(x)y=r(x)$ (1)

Let y_h be the solution to the HLDE: $y''+a(x)y'+b(x)y=0$ (2)

Let y_p be be a particular solution to (1), then the general solution to (1) is $\boxed{y=y_h+y_p}$

Proof: 1) $y=y_h+y_p$ is a solution to (1)?

$L(y)=r(x)$?

y_h is a sol. to (2) $\Rightarrow L(y_h)=0$

y_p is a sol. to (1) $\Rightarrow L(y_p)=r(x)$

$L(y)=L(y_h+y_p)=L(y_h)+L(y_p)=0+r(x)$

$\Rightarrow y$ is a sol. to (1).

2) Any solution to (1) is of the form $y=y_h+y_p$?

Let y be a solution to (1) $\Rightarrow L(y)=r(x)$

y_p is a sol. to (1) $\Rightarrow L(y_p)=r(x)$

$\Rightarrow L(y)-L(y_p)=r-r \Rightarrow L(y-y_p)=0 \Rightarrow y-y_p$ is a sol. to (2)

Example 1 $y''+4y=x \quad (1)$

Hom. equ : $y''+4y=0 \ (2)$ HLDEC

$\lambda^2+4=0 \Rightarrow \lambda^2=-4=4i^2 \Rightarrow \begin{matrix}\lambda_1\\ \lambda_2\end{matrix}=\pm 2i=\alpha\pm i\beta$

$y_h=e^{0x}(c_1\cos 2x+c_2\sin 2x)$

$$\boxed{y_h=c_1\cos 2x+c_2\sin 2x}$$

Part. sol. (Guess y_p)

$4y=x \Rightarrow \boxed{y=\frac{x}{4}} \Rightarrow y'=\frac{1}{4} \Rightarrow \boxed{y''=0}$

Replace in (1) : $0+4\left(\frac{x}{4}\right)=x$ ✓

$$\boxed{y_p=\frac{x}{4}}$$

Gen. Sol $y=y_h+y_p \Rightarrow \boxed{y=c_1\cos 2x+c_2\sin 2x+\frac{x}{4}}$.

Example 2 $x^2y''-2xy'+2y=x^{-4} \quad (1)$

$a=-2$

Hom. Sol.: $x^2y''-2xy'+2y=0 \ (2)$ Cauchy

Let $|x|=e^t \Rightarrow t=\ln|x|$

$\Rightarrow \ddot{y}-3\dot{y}+2y=0$ HLDEC

$a-1=-3$

* Char. equ : $\lambda^2-3\lambda+2=0 \Rightarrow \lambda_1=1,\ \lambda_2=2$
* Sol: $y=K_1e^{\lambda_1 t}+K_2e^{\lambda_2 t} \Rightarrow y=K_1e^t+K_2e^{2t}$

$y=K_1|x|+K_2|x|^2=\underbrace{\pm K_1}_{C_1}x+\underbrace{K_2}_{C_2}x^2$

$$\boxed{y_h=C_1x+C_2x^2}$$

$RHS = x^{-4} \Rightarrow y = ax^{-4} \Rightarrow y' = -4ax^{-5} \Rightarrow y'' = 20ax^{-6}$

Replace in (1) : $20a\,x^{-4} + 8a\,x^{-4} + 2a\,x^{-4} = x^{-4}$

$\Rightarrow 30a = 1 \Rightarrow a = \frac{1}{30} \Rightarrow \boxed{y_p = \frac{1}{30}x^{-4}}$

Gen. Sol. $\boxed{y = c_1 x + c_2 x^2 + \frac{1}{30}x^{-4}}$

3rd method to obtain y_p :

Undetermined Coefficients Method:

Consider the LDE : $y'' + ay' + by = r(x)$ (1)

Where a and b are constants.

Let $y_h = c_1 y_1 + c_2 y_2$ be the solution to the HLDE:

$y'' + ay' + by = 0$ (2)

* Assume that $r(x)$ has one of the following forms:

Type 1 : $r(x) = P_n(x) = a_0 + a_1 x + \dots + a_n x^n$ is a polynomial of degree n.

Type 2 : $r(x) = e^{\alpha x} P_n(x)$

Type 3 : $r(x) = e^{\alpha x}\left[P_n(x)\cos\beta x + Q_n(x)\sin\beta x\right]$

Where $n = \text{Max}\left[\text{degree}(P_n), \deg(Q_n)\right]$

Then (1) has a particular solution of the form $y_p = x^m g(x)$ Where $g(x)$ has the same form as $r(x)$ and m is the smallest nonnegative integer such that $y_p = x^m g(x)$ is a linear combination of functions that are not solutions to (2).

Example 1 $y'' - 2y' = x^2$ (1)

Hom. Sol.: $y'' - 2y' = 0$ (2) HLDE

$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \Rightarrow \boxed{y_h = c_1 + c_2 e^{2x}}$

$y_1 = 1 \,,\, y_2 = e^{2x}$

Part. Sol.:

1st method Guess y_p : $-2y' = x^2 \Rightarrow y' = \frac{-x^2}{2} \Rightarrow y'' = -x$

$(1) \Rightarrow (-x) - 2\left(-\frac{x^2}{2}\right) = x^2$ XX

2nd method LHS is not a Cauchy equ.

3rd method Undetermined coef.: $a = -2$, $b = 0$ are constant.

$r(x) = x^2 = P_2(x)$ (Type 1)

$\therefore$ (1) has a particular solution of the form $y = x^m P_2(x)$

$$y = x^m(ax^2 + bx + c)$$

Where $m \neq 0$ because $y = ax^2 + bx + c$ is a linear combination of x^2, x and 1, and 1 is a solution to (2)

* $\boxed{m = 1}$ because x^3, x^2 and x are not solutions to (2)

$y = ax^3 + bx^2 + cx \Rightarrow y' = 3ax^2 + 2bx + c \Rightarrow$

$y'' = 6ax + 2b.$

Replace in (1): $(6ax + 2b) - 2(3ax^2 + 2bx + c) = x^2$

$$\begin{cases} \text{coef of } x^2: -6a = 1 \Rightarrow a = -\frac{1}{6} \\ \text{coef of } x: 6a - 4b = 0 \Rightarrow b = \frac{3a}{2} = -\frac{1}{4} \\ \text{constant}: 2b - 2c = 0 \Rightarrow c = b = -\frac{1}{4} \end{cases}$$

$$y_p = -\frac{1}{6}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x$$

Example 2 $\quad y'' - 2y' = e^{2x} \quad (1)$

Hom. Sol $\quad y'' - 2y' = 0 \quad (2)$

$$y_h = c_1 + c_2 e^{2x}$$

Part. Sol. undet. coef. $\quad a = -2,\ b = 0$ (constants)

$r(x) = e^{2x} = e^{2x} P_0(x) \quad$ (Type 2)

(1) has a part. sol. of the form $y = x^m a e^{2x}$

Where $m \neq 0$ because e^{2x} is a sol. to (2)

$m = 1$ because xe^{2x} is not a sol. to (2)

$$y = axe^{2x} \Rightarrow y' = ae^{2x} + 2axe^{2x}$$

$$\Rightarrow y'' = 2ae^{2x} + 2ae^{2x} + 4axe^{2x} = 4ae^{2x} + 4axe^{2x}$$

$$(1) \Rightarrow (4ae^{2x} + \underbrace{4axe^{2x}}) - 2(ae^{2x} + \underbrace{2axe^{2x}}) = e^{2x}$$

$$\Rightarrow 2ae^{2x} = e^{2x} \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

$$\boxed{y_p = \frac{1}{2} x e^{2x}}$$

Gen. Sol: $\quad \boxed{y = c_1 + c_2 e^{2x} + \frac{1}{2} x e^{2x}}$

Sum Rule (Principle of Superposition)

Let y_{p_1} be a particular solution to $y'' + a(x)y' + b(x)y = r_1(x)$

y_{p_2} be a part. sol. to $y'' + a(x)y' + b(x)y = r_2(x)$

Then $y_p = y_{p_1} + y_{p_2}$ is a part. sol. to

$$y'' + a(x)y' + b(x)y = r_1(x) + r_2(x)$$

Example 3 $\quad y'' - 2y' = x^2 + e^{2x} \quad (1)$

* Hom. Sol: $y_h = c_1 + c_2 e^{2x}$

* Part. Sol: $y_{p_1} = -\frac{1}{6}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x$ is a sol. to $y'' - 2y' = x^2$

$y_{p_2} = \frac{1}{2} x e^{2x}$ is a sol. to $y'' - 2y' = e^{2x}$

by the Sum rule $y_p = y_{p_1} + y_{p_2}$

* Gen. Sol: $y = y_h + y_p$

$$\boxed{y = c_1 + c_2 e^{2x} - \frac{1}{6}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{2} x e^{2x}}$$

Prob 2.7 $(1', 2') + (1-20)$ HW $(1', 2', 3, 13, 16)$

(1') $x^2 y'' - 2xy' + 2y = 4 + \frac{1}{\sqrt{x}}$

(2') $x^2 y'' - xy' + y = \ln(x)$

(3') $x^2 y'' - 2y = x + \sqrt{x} \quad (1)$

Hom. Sol. $\quad x^2 y'' - 2y = 0 \; (2)$ Cauchy $a = 0, b = -2$

Let $|x| = e^t \Rightarrow t = \ln|x|$

$\ddot{y} - \dot{y} - 2y = 0$ HLDEC

$a - 1 = -1$

$\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1, \; \lambda_2 = 2$

$y_h = k_1 e^{-t} + K_2 e^{2t} = k_1 |x|^{-1} = \pm \frac{K_1}{x} + K_2 x^2$

$y_h = \frac{c_1}{x} + c_2 x^2$

Part. Sol.: $\quad x^2 y'' - 2y = x \; (3)$

$x^2y'' - 2y = \sqrt{x}$ (4)

$RHS = \sqrt{x} = x^{1/2}$ Try to find a sol. of the form $y = a\,x^{1/2}$

$y' = \frac{1}{2}a\,x^{-1/2} \Rightarrow y'' = -\frac{1}{4}a\,x^{-3/2}$

(4) $\Rightarrow -\frac{1}{4}a\,x^{1/2} - 2a\,x^{1/2} = x^{1/2} \Rightarrow -\frac{9}{4}a = 1 \Rightarrow a = \frac{-4}{9}$

$\boxed{y_{p_2} = -\frac{4}{9}x^{1/2}}$ is a part. sol. to (4)

By the sum Rule: $y_p = y_{p_1} + y_{p_2} = -\frac{x}{2} - \frac{4}{9}\sqrt{x}$

Gen. Sol. $y = y_h + y_p$

$$y = \frac{c_1}{x} + c_2x^2 - \frac{x}{2} - \frac{4}{9}\sqrt{x}$$

(14) $y'' + 2y' + y = 2x\sin x$ (1)

Hom. Sol. $y'' + 2y' + y = 0$ (2) HLDEC

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$

$\boxed{y_h = (c_1 + c_2x)e^{-x}}$ $y_1 = e^{-x}$, $y_2 = xe^{-x}$

Part. Sol. (Undet. Coef.)

$r(x) = 2x\sin x = e^{0x}(P_1\cos x + Q_1\sin x)$ Type 3

(1) has a part. sol. of the form

$$y = x^m g(x) = x^m\left[(ax+b)\cos x + (cx+d)\sin x\right]$$

where $\boxed{m=0}$ because ...

$$y = (ax+b)\cos x + (cx+d)\sin x$$

$$y' = a\cos x - (ax+b)\sin x + c\sin x + (cx+d)\cos x.$$

$$y'' = -2a\sin x - (ax+b)\cos x + 2c\cos x - (cx+d)\sin x.$$

$$(1) \Rightarrow \; -2a\sin x - (ax\cancel{+b})\cos x + 2c\cos x - (cx\cancel{+d})\sin x$$
$$+2a\cos x - 2(ax+b)\sin x + 2c\sin x + 2(cx+d)\cos x$$
$$+(ax\cancel{+b})\cos x + (cx\cancel{+d})\sin x = 2x\sin x.$$

$$\left[\begin{array}{l} \text{Coef. of } x\cos x:\; 2c = 0 \Rightarrow c = 0 \\ \text{coef. of } x\sin x:\; -2a = 2 \Rightarrow a = -1 \\ \text{coef. of } \cos x:\; 2c + 2a + 2d = 0 \Rightarrow d = 1 \\ \text{Coef. of } \sin x:\; -2a - 2b + 2c = 0 \Rightarrow b = 1 \end{array}\right.$$

$$\boxed{y_p = (-x+1)\cos x + \sin x}$$

<u>Gen. Sol.</u>: $\boxed{y = (c_1 + c_2 x)e^{-x} + (1-x)\cos x + \sin x}$

(18) IVP: $\left\{\begin{array}{l} y'' - 2y' = 12e^{2x} - 8e^{-2x} \\ y(0) = -2, \; y'(0) = 12 \end{array}\right.$

<u>Hom. Sol.</u> $y'' - 2y' = 0$ HLDEC.

$$\lambda^2 - 2\lambda = 0 \Rightarrow \lambda_1 = 0, \; \lambda_2 = 2$$

$$\boxed{y_h = c_1 + c_2 e^{2x}} \quad y_1 = 1, \; y_2 = e^{2x}$$

Part. Sol $\quad y'' - 2y' = 12e^{2x} \quad (3)$

$r(x) = 12e^{2x} = P_0 e^{2x}$ (Type 2)

(3) has a part sol. of the form $y = x^m g(x) = x^m a e^{2x}$.

Where $m \neq 0$ because e^{2x} is a sol. to (2).

$m = 1$ because xe^{2x} is not a sol. to (2).

$y = axe^{2x} \Rightarrow y' = ae^{2x} + 2axe^{2x}$.

$y'' = 2ae^{2x} + 2ae^{2x} + 4axe^{2x}$.

$(3) \Rightarrow (4ae^{2x} + \underline{4axe^{2x}}) - 2(ae^{2x} + \underline{2axe^{2x}}) = 12e^{2x}$.

$2ae^{2x} = 12e^{2x} \Rightarrow \boxed{a = 6} \Rightarrow \boxed{y_{P_1} = 6xe^{2x}}$

* $y'' - 2y' = -8e^{-2x} \quad (4)$

$r(x) = P_0 e^{-2x} \Rightarrow$ (4) has a part. sol. of the form.

$y = x^m a e^{-2x}$ Where $m = 0$ bec. e^{-2x} is not a sol. to (2)

$y = ae^{-2x} \Rightarrow y' = -2ae^{-2x} \Rightarrow y' = 4ae^{-2x}$

$(4) \Rightarrow 4ae^{-2x} + 4ae^{-2x} = -8e^{-2x} \Rightarrow 8a = -8 \Rightarrow a = -1$

$$\boxed{y_{P_2} = -e^{-2x}}$$

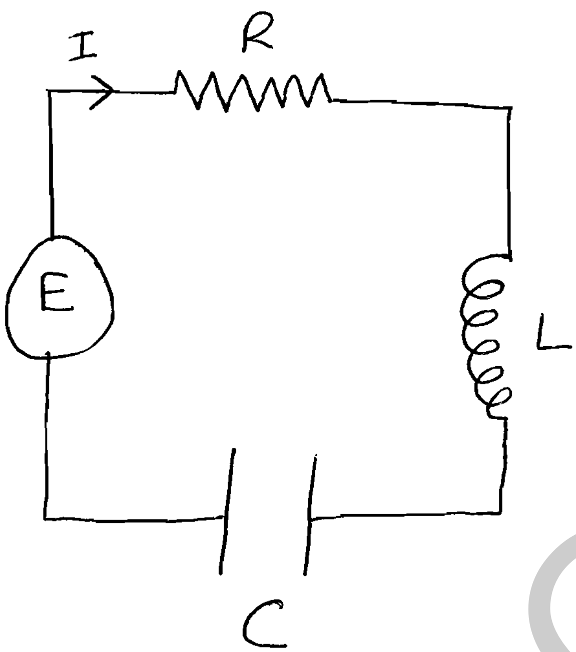
$y_P = y_{P_1} + y_{P_2} = 6xe^{2x} - e^{-2x}$.

Gen. Sol. $\quad \boxed{y = c_1 + c_2 e^{2x} + 6xe^{2x} - e^{-2x}}$

$y(0) = -2 \Rightarrow c_1 + c_2 - 10$

2.9 Electric Circuits

RLC - circuits: A resistance of R ohms, an inductance of L Henrys, and a capacitance of C farads are connected in series to an electromotive force (emf) of E volts.



The voltage drop across:

1) the resistance R is $\boxed{E_R = RI}$
2) the inductance L is $\boxed{E_L = LI'}$ where $I' = \frac{dI}{dt}$.
3) the capacitance C is $\boxed{E_C = \frac{Q}{C}}$ where Q is the Charge $\boxed{I = Q'}$

Kirchhoff's Law: If we assume that E is a voltage drop of $-E$, then the Sum of all voltage drops around any closed loop is 0

Example (RLC - circuit) $R = 4\ \Omega$, $L = 1\ H$, $C = 0.25$ farad. $E = 25 \sin t$ volts. $Q(0) = 0$, $I(0) = 0$

a) Find the current I.

$$E_R + E_L + E_C + (-E) = 0.$$

$$RI + LI' + \frac{Q}{C} = E \Rightarrow 4I + I' + 4Q = 25 \sin t.$$

1st method: Differentiate: $4I' + I'' + 4Q' = 25 \cos t$

$$4I' + I'' + 4I = 25 \cos t.$$

Hom. Sol. $Q'' + 4Q' + 4Q = 0$ (2) HLDEC

$\lambda^2 + 4\lambda + 4 = 0 \Rightarrow (\lambda+2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -2$

$Q_h = (c_1 + c_2 t)\, e^{-2t}$

Part. Sol. Undet. coef.

$r(t) = 25 \sin t = e^{\alpha t}\left[P_0 \cos t + Q_0 \sin t\right]$ type 3

(1) has a part. sol. of the form

$= t^m g(t) = t^m (a \cos t + b \sin t)$

Where $m = 0$ bec. $\cos t$ an $\sin t$ are not solutions to (2).

$Q = a \cos t + b \sin t \Rightarrow Q' = -a \sin t + b \cos t$

$\Rightarrow Q'' = -a \cos t - b \sin t.$

Replace in (1):

$(-a \cos t - b \sin t) + 4(-a \sin t + b \cos t) + 4(a \cos t + b \sin t) = 25 \sin t.$

$$\begin{cases} \text{coef. of } \cos t: -a + 4b + 4a = 0. \Rightarrow 3a + 4b = 0 \Rightarrow b = \frac{-3a}{4} \\ \text{coef. of } \sin t: -b - 4a + 4b = 25 \Rightarrow -4a + 3b = 25 \end{cases}$$

$-4a - \frac{9a}{4} = 25 \Rightarrow \frac{-25a}{4} = 25 \Rightarrow \boxed{a = -4}$

$\boxed{b = 3}$

$Q_p = -4 \cos t + 3 \sin t$

Gen. sol. $Q = (c_1 + c_2 t)\, e^{-2t} - 4 \cos t + 3 \sin t.$

$Q(0) = 0 \Rightarrow c_1 - 4 = 0 \Rightarrow c_1 = 4$

$$\boxed{Q = (4+5t)e^{-2t} - 4\cos t + 3\sin t}$$

$* \; I = Q' = 5e^{-2t} - 2(4+5t)e^{-2t} + 4\sin t + 3\cos t$

$$\boxed{I = (-3-10t)e^{-2t} + 4\sin t + 3\cos t}$$

b) Show that I is the sum of 2 functions I_T and I_S where

$\lim\limits_{t \to +\infty} I_T = 0$

I_T is called the Transient current.

I_S is called the steady-state current.

$* \; \lim\limits_{t \to +\infty} (-3-10t)e^{-2t} = \lim \dfrac{-3-10t}{e^{2t}} = \dfrac{\infty}{\infty} \overset{HR}{=} \lim \dfrac{-10}{2e^{2t}} = 0.$

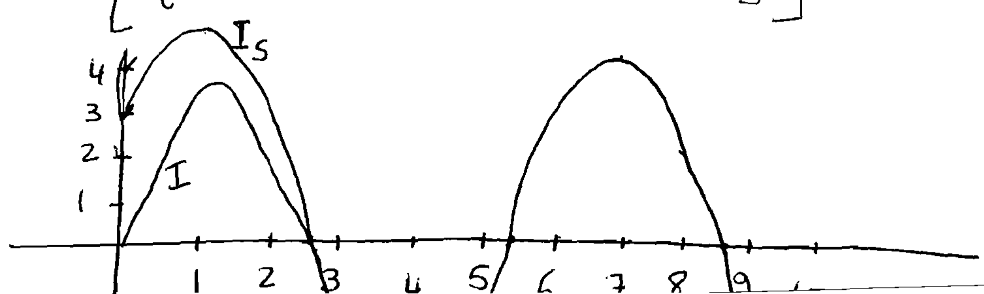
$* \; \lim\limits_{t \to +\infty} 4\sin t$ doesn't exist $\Rightarrow \lim 4\sin t \neq 0$

$* \; \lim 3\cos t \neq 0.$

$\therefore \boxed{I_T = (-3-10t)e^{-2t}}$ and $\boxed{I_S = 4\sin t + 3\cos t}$

c) Using Mathematica plot the graphs of I and I_S

```
i[t_] = (-3-10t)*E^(-2t) + 4Sin[t] + 3Cos[t];
IS[t_] = 4 Sin[t] + 3 Cos[t];
Plot[{i[t], IS[t]}, {t, 0, 10}];
```



Prob 2.9 (1-18) HW (1, 5, 11, 17)

⑥ LC-circuit $L = 0.5H$, $C = 8\times10^{-4}$ farad

$$E = t^2 \text{ volts}, \quad I(0) = Q(0) = 0.$$

$$E_L + E_C - E = 0 \Rightarrow LI' + \frac{Q}{C} = E$$

$$0.5\,Q'' + \frac{10^4}{8}Q = t^2$$

$$Q'' + \frac{10{,}000}{4}Q = 2t^2 \Rightarrow Q'' + 2500Q = 2t^2 \quad (1)$$

* Hom. sol. $Q'' + 2500Q = 0 \Rightarrow \lambda^2 + 2500 = 0 \Rightarrow \lambda_1, \lambda_2 = \pm 50i$

$$Q_h = c_1\cos 50t + c_2 \sin 50t$$

Part. $r(t) = 2t^2 = P_2$ type I

(1) has a part. sol. of the form

$$Q = t^m P_2 = t^m(at^2 + bt + c)$$

Where $m = 0$ bec t^2, t and 1 are not sol. to (2)

$$Q = at^2 + bt + c \Rightarrow Q' = 2at + b \Rightarrow Q'' = 2a$$

(1): $2a + 2500(at^2 + bt + c) = 2t^2$.

$$\left[\begin{array}{l} \text{coef of } t^2: \; 2500a = 2 \Rightarrow a = \frac{1}{1250} \\ \text{coef of } t: \; 2500b = 0 \Rightarrow b = 0 \\ \text{const}: \; 2a + 2500c = 0 \Rightarrow c = -\frac{a}{1250} = -\frac{1}{(1250)^2} \end{array}\right.$$

$$Q = c_1 \cos 50t + c_2 \sin 50t + \frac{1}{1250}\left(t^2 - \frac{1}{1250}\right)$$

$* \; Q(0) = 0 \Rightarrow c_1 - \frac{1}{1250^2} = 0 \Rightarrow c_1 = \frac{1}{1250^2}$

$* \; I(0) = Q'(0) = 0 \Rightarrow c_2 = ?$

----

⑩ RLC-circuit: $R = 8\Omega$, $L = 0.5H$, $C = 0.1F$, $E = 100 \sin 2t$ volts

$$E_R + E_L + E_c - E = 0$$

$$RI + LI' + \frac{Q}{C} = E \Rightarrow 0.5I' + 8I + 10Q = 100 \sin 2t$$

$$Q'' + 16Q' + 20Q = 200 \sin 2t \quad (1)$$

$*$ Hom. Sol: $\quad Q'' + 16Q' + 20Q = 0 \quad (2)$

$$\left.\begin{array}{l} \lambda^2 + 16\lambda + 20 = 0 \\ \Delta' = 64 - 20 = 44 \end{array}\right] \Rightarrow \lambda = \frac{-8 \pm \sqrt{44}}{1} \Rightarrow \begin{array}{l} \lambda_1 = -8 + 2\sqrt{11} \\ \lambda_2 = -8 - 2\sqrt{11} \end{array}$$

$$Q_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$*$ Part. Sol.: undet. coef.

$$r(t) = 200 \sin 2t = e^{0t}(P_0 \cos 2t + Q_0 \sin 2t)$$

(1) has a part. sol. of the form

$$Q = t^m (a \cos 2t + b \sin 2t)$$

Where $m = 0$ bec. $\cos 2t$ and $\sin 2t$ are not sol. to (2)

$$Q = a \cos 2t + b \sin 2t \Rightarrow Q' = -2a \sin 2t + 2b \cos 2t.$$

$$Q'' = -4a \cos 2t - 4b \sin 2t.$$

$$\Rightarrow \begin{cases} \text{coef. of } \cos 2t: -4a + 32b + 20a = 0 \Rightarrow a = -2b \\ \text{coef. of } \sin 2t: -4b - 32a + 20b = 200 \Rightarrow 16b + 64b = 200 \end{cases}$$

$$\Rightarrow b = \frac{200}{80} = \frac{5}{2}$$

$$a = -5$$

$$Q_p = -5\cos 2t + \frac{5}{2}\sin 2t$$

Gen. Sol. $Q = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} - 5\cos 2t + \frac{5}{2}\sin 2t$.

Assume $Q(0) = 0 \Rightarrow c_1 + c_2 - 5 = 0$ (3)

$$I = Q' = \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t} + 10\sin 2t + 5\cos 2t.$$

$I(0) = 0 \Rightarrow \lambda_1 c_1 + \lambda_2 c_2 + 5 = 0$ (4)

(3), (4) $\Rightarrow c_1 = ?,\ c_2 = ?$

$$I = (-8 + 2\sqrt{11})\, c_1 \underbrace{e^{(-8+2\sqrt{11})t}}_{\lim_{t\to+\infty} = 0} + (-8 - 2\sqrt{11})\, c_2 \underbrace{e^{(-8-2\sqrt{11})t}}_{\lim_{t\to+\infty} = 0} + 10\sin 2t + 5\cos 2t.$$

Transient current: $I_T = \lambda_1 c_1 e^{\lambda_1 t} + \lambda_2 c_2 e^{\lambda_2 t}$

Steady-state current: $I_S = 10\sin 2t + 5\cos 2t$.

2.10 Variation of Parameters: Method to find y_p

Consider the LDE: $y'' + a(x)y' + b(x)y = r(x)$ (1)

Let $\boxed{y_h = K_1 y_1 + K_2 y_2}$ be the solution to the HLDE

$$y'' + a(x)y' + b(x)y = 0 \quad (2)$$

Then (1) has a part. sol. of the form $\boxed{y = c_1(x)y_1 + c_2(x)y_2}$

satisfying $\boxed{c_1' y_1 + c_2' y_2 = 0}$ (3)

$$y = c_1 y_1 + c_2 y_2 \Rightarrow y' = \underbrace{c_1' y_1} + c_1 y_1' + \underbrace{c_2' y_2} + c_2 y_2'$$

$$\Rightarrow y' = c_1 y_1' + c_2 y_2'$$

$$y'' = c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2''$$

Replace in (1):

$$(c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2'') + a(x)(c_1 y_1' + c_2 y_2') + b(x)(c_1 y_1 + c_2 y_2) = r(x)$$

$$\Rightarrow c_1 \underbrace{(y_1'' + a y_1' + b y_1)}_{L(y_1)=0} + c_2 \underbrace{(y_2'' + a y_2' + b y_2)}_{L(y_2)=0} + c_1' y_1' + c_2' y_2' = r(x)$$

$$\Rightarrow \boxed{c_1' y_1' + c_2' y_2' = r(x)} \quad (4)$$

$$(3),(4): \begin{cases} y_1 c_1' + y_2 c_2' = 0 \\ y_1' c_1' + y_2' c_2' = r(x) \end{cases}$$

$\Rightarrow$ the system has a unique solution:

$$c_1' = \frac{\Delta_1}{\Delta}, \quad c_2' = \frac{\Delta_2}{\Delta}$$

where $\Delta_1 = \begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix} = -y_2 r$

$\Delta_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix} = y_1 r$

$$c_1' = \frac{\Delta_1}{\Delta} = \frac{-y_2 r}{W} \Rightarrow c_1 = \int \frac{-y_2 r}{W}\,dx + K_1^{\nearrow 0}$$ need 1 sol.

$$c_2' = \frac{\Delta_2}{\Delta} = \frac{y_1 r}{W} \Rightarrow c_2 = \int \frac{y_1 r}{W}\,dx + K_2^{\nearrow 0}$$

$$y_p = c_1 y_1 + c_2 y_2$$

Gen. sol. $y = y_h + y_p = (K_1 y_1 + K_2 y_2) + (c_1 y_1 + c_2 y_2)$

$$y = (c_1 + K_1) y_1 + (c_2 + K_2) y_2$$

Theorem: Let $y_h = K_1 y_1 + K_2 y_2$ be the solution to the HLDE(2).
The solution to the nonhomogeneous LDE: $y'' + a(x) y' + b(x) y = r(x)$ (1)
is $\boxed{y = c_1 y_1 + c_2 y_2}$ where $\boxed{c_1 = \int \frac{-y_2 r}{W}\,dx + K_1}$

and $\boxed{c_2 = \int \frac{y_1 r}{W}\,dx + K_2}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Prob 2.10 (1–17) HW (1, 3, 7, 14)

5) $y'' + y = \tan x$ (1)

$$\boxed{y_h = K_1\cos x + K_2 \sin x}$$

tan x is not of type 1, 2 or 3

Gen. Sol

$$* \; W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$* \; c_1 = \int \frac{-y_2 r}{W} dx = \int \frac{-\sin x \cdot \tan x}{1} dx = \int \frac{-\sin^2 x}{\cos x} dx$$

$$c_1 = \int \frac{-1+\cos^2 x}{\cos x} dx = \int -\sec x \, dx + \int \cos x \, dx$$

$$c_1 = -\ln|\sec x + \tan x| + \sin x + K_1$$

$$* \; c_2 = \int \frac{y_1 r}{W} dx = \int \frac{\cos x \tan x}{1} dx = \int \sin x \, dx = -\cos x + K_2$$

$$y = c_1 y_1 + c_2 y_2 \Rightarrow y = \cos x\left[K_1 + \sin x - \ln|\sec x + \tan x|\right]$$
$$+ \sin x \left[K_2 - \cos x\right]$$

$$y = \cos x\left(K_1 - \ln|\sec x + \tan x|\right) + K_2 \sin x.$$

(8) $y'' - 4y' + 4y = \dfrac{12e^{2x}}{x^4}$

* $r(x) = \dfrac{12}{x^4}e^{2x}$ is not of type 1, 2 or 3

Hom. Sol. $y'' - 4y' + 4y = 0 \quad (2)$

$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2$$

$$y_h = (K_1 + K_2 x)e^{2x} = K_1 e^{2x} + K_2 x e^{2x}$$

Gen. Sol. variation of parameters.

$$* \; W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & \ldots \end{vmatrix} = e^{4x}$$

* $c_1 = \int \frac{-y_2 r}{W} dx = \int \frac{-x e^{2x} \frac{12 e^{2x}}{x^4}}{e^{4x}} dx = \int \frac{-12}{x^3} dx$

$c_1 = \int -12 x^{-3} dx = -12 \frac{x^{-2}}{-2} = \frac{6}{x^2} + K_1$

* $c_2 = \int \frac{y_1 r}{W} dx = \int \frac{e^{2x} \frac{12 e^{2x}}{x^4}}{e^{4x}} dx = \int 12 x^{-4} dx = 12 \frac{x^{-3}}{-3} + K_2$

$c_2 = \frac{-4}{x^3} + K_2$

* $y = c_1 y_1 + c_2 y_2 = \left(\frac{6}{x^2} + K_1\right) e^{2x} + \left(\frac{-4}{x^2} + K_2\right) x e^{2x}$

(12) $(x^2 D^2 + x D - \frac{1}{4} I) y = 3x^{-1} + 3x$

$x^2 y'' + x y' - \frac{1}{4} y = 3x^{-1} + 3x$ (1)

* Hom.Sol. $x^2 y'' + x y' - \frac{1}{4} y = 0$ (2) Cauchy

Let $|x| = e^t \Rightarrow t = \ln|x|$ ⟶ $y = x^m \Rightarrow y' = m x^{m-1}$
$y'' = m(m-1) x^{m-2}$

$\ddot{y} + 0 . \dot{y} - \frac{1}{4} y = 0$ HLDEC

(2) $\Rightarrow (m^2 - \frac{1}{4}) x^m = 0$
$\Downarrow$
$m = \pm 1/2$

$\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \pm \frac{1}{2}$

$y_1 = x^{1/2}, y_2 = x^{-1/2}$
$y = K_1 x^{1/2} + K_2 x^{-1/2}$

$y_h = K_1 e^{\frac{1}{2} t} + K_2 e^{-\frac{1}{2} t} = K_1 |x|^{1/2} + K_2 |x|^{-1/2}$

Assume $x > 0$ $\boxed{y_h = K_1 x^{1/2} + K_2 x^{-1/2}}$

* Part. Sol. $x^2 y'' + x y' - \frac{1}{4} y = 3x^{-1}$ (3)

$y = a x^{-1} \Rightarrow y' = -a x^{-2} \Rightarrow y'' = 2 a x^{-3}$

(3) $\Rightarrow 2 a x^{-1} - a x^{-1} - \frac{1}{4} a x^{-1} = 3x^{-1}$

$\frac{3}{4} a = 3 \Rightarrow a = 4 \Rightarrow \boxed{y_{P_1} = 4 x^{-1}}$

$* \quad x^2 y'' + x y' - \frac{1}{4} y = 3x \quad (4)$

$y = ax \Rightarrow y' = a \Rightarrow y'' = 0$

$(4) \quad 0 + ax - \frac{1}{2}ax = 3x \Rightarrow \frac{3}{4}ax = 3x \Rightarrow \boxed{a = 4}$

$y_{P_2} = 4x.$

$\Rightarrow y_p = y_{P_1} + y_{P_2} = 4x^{-1} + 4x$ is a part. sol to (1)

Gen. Sol. $\boxed{y = K_1 x^{1/2} + K_2 x^{-1/2} + \frac{4}{x} + 4x}$

Other Method Variation of parameter.

$* \quad W = \begin{vmatrix} x^{1/2} & x^{-1/2} \\ \frac{1}{2}x^{-1/2} & -\frac{1}{2}x^{-3/2} \end{vmatrix} = -\frac{1}{2}x^{-1} - \frac{1}{2}x^{-1} = -x^{-1}$

$(1) \Rightarrow y'' + \frac{1}{x} y' - \frac{1}{4x^2} y = \frac{3x^{-1} + 3x}{x^2}$

$\Rightarrow r(x) = 3x^{-3} + 3x^{-1}$

$* \quad C_1 = \int \frac{-x^{-1/2}(3x^{-3} + 3x^{-1})}{-x^{-1}} dx = \int (3x^{-5/2} + 3x^{-1/2}) dx = \cdots + K_1$

$* \quad C_2 = \int \frac{x^{1/2}(3x^{-3} + 3x^{-1})}{-x^{-1}} dx = \int -(3x^{-3/2} + 3x^{1/2}) dx = \cdots + K_2$

$\vdots$

(15) $y'' + y = \sec x - 10 \sin 5x \quad (1)$

Hom. sol. $\quad y'' + y = 0 \; (2)$

$\lambda^2 + 1 = 0 \Rightarrow \lambda_1, \lambda_2 = \pm i \Rightarrow \boxed{y_h = K_1 \cos x + K_2 \sin x}$

Part. Sol.: $\quad y'' + y = \sec x \; (3)$

Variation of parameters:

$$* \; W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$* \; c_1 = \int \frac{-y_2 r}{W}\, dx = \int \frac{-\sin x \sec x}{1}\, dx = \int -\tan x \, dx .$$

$$c_1 = -\ln|\sec x| + \cancelto{0}{K_1}$$

$$* \; c_2 = \int \frac{y_1 r}{W}\, dx = \int \frac{\cos x \sec x}{1}\, dx = \int dx = x + \cancelto{0}{K_2} .$$

$$\boxed{c_2 = x}$$

$$\boxed{y_{P_1} = -\ln|\sec x| \cos x + x \sin x .}$$

$y'' + y = -10 \sin 5x \; (4)$

undet. coef.: $r(x) = -10 \sin 5x = e^{0x}(P_0 \cos 5x + Q_0 \sin 5x)$

(4) has a part. sol. of the form $y = x^m (a \cos 5x + b \sin 5x)$

where $m = 0$ bec. $\cos 5x$ and $\sin 5x$ are not sol. to (2).

$y = a \cos 5x + b \sin 5x \Rightarrow y' = -5a \sin 5x + 5b \cos 5x .$

$\Rightarrow y'' = -25 a \cos 5x - 25 b \sin 5x .$

(4) $\Rightarrow a = ?, \; b = ?$

$* \; y_p = y_{P_1} + y_{P_2}$

<u>Gen. Sol.</u> $y = y_h + y_p$

$$y = K_1 \cos x + K_2 \sin x - \ln|\sec x| \cos x + x \sin x + a \cos 5x + b \sin 5x .$$

Chapter 6 Laplace Transforms

6.0 Gamma Function (page 192) section 5.5.

Definition: Let $x > 0$ be a positive real number.
The Gamma function is defined by:

$$\boxed{\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt}$$

Properties. 1) $\Gamma(1) = 1$
2) $\Gamma(x+1) = x\Gamma(x)$
3) If $n \in \mathbb{N}$ then $\Gamma(n+1) = n!$

Proof 1) $\Gamma(1) = \int_0^{+\infty} e^{-t} t^0 dt = -e^{-t}\Big|_0^{+\infty} = -e^{-\infty} + e^0 = 1$

2) $\Gamma(x+1) = \int_0^{+\infty} \underbrace{e^{-t} \underbrace{t^x}_{u}}_{dv} dt$

$$\left| \begin{array}{l} u = t^x \Rightarrow du = x t^{x-1} dt \\ dv = e^{-t} dt \Rightarrow v = -e^{-t} \end{array} \right.$$

$$\Gamma(x+1) = \underbrace{-t^x e^{-t}\Big|_0^{+\infty}}_{0} - \int_0^{+\infty} -x t^{x-1} e^{-t} dt$$

$$\Gamma(x+1) = x \int_0^{+\infty} e^{-t} t^{x-1} dt = x\Gamma(x)$$

3) $\Gamma(n+1) = n\,\Gamma(n)$
$= n(n-1)\,\Gamma(n-1)$
$= n(n-1)(n-2)\,\Gamma(n-2)$
$\vdots$
$= n(n-1)(n-2) \cdots \Gamma(1) = n!$

Notation: If n is an integer then $n! = \Gamma(n+1)$
If x is a real number then $x!$ is defined by $\boxed{x! = \Gamma(x+1)}$

6.1 Laplace Transforms

Definition: Let $f(t)$ be a function defined $\forall t \geqslant 0$.
The Laplace transform of $f(t)$ is defined by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt.$$

* $F(s)$ is defined for all values of s for which the improper integral is convergent. $t^2 \underset{\text{integral}}{\overset{\text{derivative}}{\rightleftarrows}} 2t$ $\quad f(t) \underset{\text{inverse laplace}}{\overset{\text{laplace Transform}}{\rightleftarrows}} F(t)$
* If $F(s) = \mathcal{L}\{f(t)\}$ then $f(t)$ is called the Inverse Laplace of $F(s)$.

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Formula: If c is constant then $\mathcal{L}\{c\} = \frac{c}{s}$
It is defined if $s > 0$

Proof: $\mathcal{L}\{c\} = \int_0^{+\infty} e^{-st} c\, dt = c \frac{e^{-st}}{-s}\Big|_0^{+\infty} = \underset{\substack{\downarrow \\ s>0}}{0} - \frac{c}{-s} e^0 = \frac{c}{s}$

Formula: If $a > -1$ is a real number then

$$\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}} = \frac{a!}{s^{a+1}} \quad (s > 0)$$

Proof: $\mathcal{L}\{t^a\} = \int_0^{+\infty} e^{-st} t^a\, dt$

Let $T = st \Rightarrow dT = s\,dt$
$t = 0 \Rightarrow T = 0$
$t = +\infty \Rightarrow T = +\infty$

$$\mathcal{L}\{t^a\} = \int_0^{+\infty} e^{-T} \left(\frac{T}{s}\right)^a \frac{dT}{s} = \frac{1}{s^{a+1}} \int_0^{+\infty} e^{-T} T^a\, dT$$

$$= \frac{1}{s^{a+1}} \Gamma(a+1) = \frac{a!}{s^{a+1}}$$

Theorem: (The 1st Shifting Property) (or s-shifting)

Formula 3: $\boxed{\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}}$ defined if $s-a>0$

Proof: $\mathcal{L}\{1\} = \dfrac{1}{s} \Rightarrow \mathcal{L}\{e^{at}\cdot 1\} = F(s-a) = \dfrac{1}{s-a}$.

Linearity: The Laplace Transform is Linear:

1) $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$.

2) $\mathcal{L}\{c\cdot f(t)\} = c\ \mathcal{L}[f(t)]$.

Formula 4: $\boxed{\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}}$

$\boxed{\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}}$

Proof: * $\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\dfrac{e^{at}+e^{-at}}{2}\right\} = \dfrac{1}{2}\left(\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}\right)$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{1}{2}\,\frac{s+a+s-a}{(s-a)(s+a)} = \frac{s}{s^2-a^2}$$

$s-a>0 \quad s+a>0$

* $\mathcal{L}\{\sinh at\} \overset{HW}{=} \dots$

Formula 5: $\boxed{\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}}$ $\boxed{\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}}$

Proof $e^{iat} = \cos at + i \sin at$

$\Rightarrow \mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at\} + i\,\mathcal{L}\{\sin at\}$.

$\Rightarrow \dfrac{1}{s-ia} = \mathcal{L}\{\cos at\} + i\,\mathcal{L}\{\sin at\} \quad (1)$

But $\dfrac{1}{s-ia} = \dfrac{s+ia}{(s-ia)(s+ia)} = \dfrac{s+ia}{s^2-i^2a^2} = \dfrac{s+ia}{s^2+a^2} = \dfrac{s}{s^2+a^2} + i\dfrac{a}{s^2+a^2} \quad (2)$

Formulas

1) $\mathcal{L}\{c\} = \dfrac{c}{s}$

2) $\mathcal{L}\{t^a\} = \dfrac{\Gamma(a+1)}{s^{a+1}}$

3) $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

4) $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$

5) $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

6) $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$

1') $\mathcal{L}\{e^{bt}\} = \dfrac{1}{s-b}$

2') $\mathcal{L}\{t^a e^{bt}\} = \dfrac{\Gamma(a+1)}{(s-b)^{a+1}}$

3') $\mathcal{L}\{e^{bt}\cos at\} = \dfrac{s-b}{(s-b)^2+a^2}$

4') $\mathcal{L}\{e^{bt}\sin at\} = \dfrac{a}{(s-b)^2+a^2}$

5') $\mathcal{L}\{e^{bt}\cosh at\} = \dfrac{s-b}{(s-b)^2-a^2}$

6') $\mathcal{L}\{e^{bt}\sinh at\} = \dfrac{a}{(s-b)^2-a^2}$

Prob 6.1 (1', 2') (1–20) (29–54)

HW (1, 1', 7, 9, 19, 31, 37, 45, 47, 51, 53)

(2) $f(t) = (t^2-3)^2 = t^4+9-6t^2$

$$\mathcal{L}\{f\} = \mathcal{L}[t^4-6t^2+9]$$

$$= \mathcal{L}[t^4] - 6\mathcal{L}[t^2] + \mathcal{L}[9]$$

$$= \frac{\Gamma(5)}{s^5} - 6\,\frac{\Gamma(3)}{s^3} + \frac{9}{s}$$

$$= \frac{4!}{s^5} - \frac{6(2!)}{s^3} + \frac{9}{s} = \frac{24}{s^5} - \frac{12}{s^3} + \frac{9}{s}$$

(1') $f(t) = \dfrac{t^3+e^{4t}}{\quad}$ find $F(s)$. Given $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(21) $f(t) = \sqrt{t}\,(t + e^{-2t} + t^2 e^t) = t^{3/2} + e^{-2t}\sqrt{t} + t^{5/2} e^t$

$$\mathcal{L}\{f\} = \mathcal{L}[f^{3/2}] + \mathcal{L}[e^{-2t} t^{1/2}] + \mathcal{L}[e^t t^{5/2}]$$

$$* \ \mathcal{L}[t^{3/2}] = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{\frac{3}{2}\Gamma(3/2)}{s^{5/2}} = \frac{\frac{3}{2}\times\frac{1}{2}\times\Gamma(1/2)}{s^{5/2}}$$

$$= \frac{3\sqrt{\pi}}{4 s^{5/2}}$$

$$* \ \mathcal{L}[t^{1/2}] = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2 s^{3/2}}$$

By the 1^{st}-SP: $\mathcal{L}[e^{-2t} t^{1/2}] = \dfrac{\sqrt{\pi}}{2(s+2)^{3/2}}$

$$* \ \mathcal{L}[t^{5/2}] = \frac{\Gamma(7/2)}{s^{7/2}} = \frac{\frac{5}{2}\Gamma(5/2)}{s^{7/2}} = \frac{\frac{5}{2}\,\frac{3\sqrt{\pi}}{4}}{s^{7/2}} = \frac{15\sqrt{\pi}}{8 s^{7/2}}$$

$$\mathcal{L}[e^t t^{5/2}] = \frac{15\sqrt{\pi}}{8(s-1)^{7/2}}$$

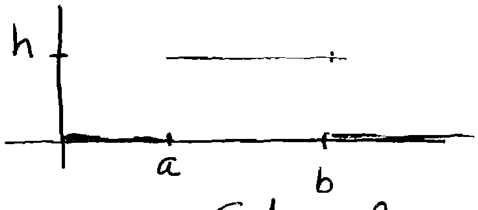
$$\Longrightarrow F(s) = \mathcal{L}[f(t)] = \frac{3\sqrt{\pi}}{4 s^{5/2}} + \frac{\sqrt{\pi}}{2(s+2)^{3/2}} + \frac{15\sqrt{\pi}}{8(s-1)^{7/2}}$$

(8) $f(t) = \sin(3t - \frac{1}{2}) = \sin 3t \cos\frac{1}{2} - \cos 3t \sin\frac{1}{2}$

$$\mathcal{L}\{f\} = \cos\tfrac{1}{2}\,\mathcal{L}[\sin 3t] - \sin\tfrac{1}{2}\,\mathcal{L}[\cos 3t]$$

$$= \cos\tfrac{1}{2}\,\frac{3}{s^2+9} - \sin\tfrac{1}{2}\,\frac{s}{s^2+9}$$

(14)



see section 6.3

$$f(t) = \begin{cases} h & \text{if } a < t < b \\ 0 & \text{elsewhere} \end{cases}$$

$$F(s) = \mathcal{L}\{f\} = \int^{+\infty} e^{-st} f(t)\,dt = \int^{a} e^{-st}\cdot 0\,dt + \int^{b} e^{-st} h\,dt + \int^{+\infty} e^{-st}\cdot 0\,dt.$$

$$F(s) = h\frac{e^{-st}}{-s}\Big|_a^b = -\frac{h}{s}e^{-bs} + \frac{h}{s}e^{-as}$$

(34) $F(s) = \dfrac{20}{(s-1)(s+4)}$

Partial Fractions : $F(s) = \dfrac{20}{(s-1)(s+4)} \overset{\text{Heaviside}}{=} \dfrac{\frac{20}{5}}{s-1} + \dfrac{\frac{20}{-5}}{s+4}$

$$F(s) = \frac{4}{s-1} - \frac{4}{s+4} = \mathcal{L}[4e^{-t}] - \mathcal{L}[4e^{-4t}]$$

$$= \mathcal{L}[4e^{t} - 4e^{-4t}]$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}[F(s)] = 4e^{t} - 4e^{-4t}$$

(42) $f(t) = -3t^4 e^{-0.5t}$

$$\mathcal{L}[-3t^4] = -3\mathcal{L}[t^4] = -3\frac{\Gamma(5)}{s^5} = -3\frac{4!}{s^5} = \frac{-72}{s^5}$$

$$\Rightarrow \mathcal{L}[f] = \frac{-72}{(s+0.5)^5}$$

(48) $F(s) = \dfrac{\pi}{(s+\pi)^2}$

$$\frac{\pi}{s^2} = \frac{\pi}{\Gamma(2)}\frac{\Gamma(2)}{s^2} = \frac{\pi}{1!}\mathcal{L}[t] = \mathcal{L}[\pi t]$$

$$\Rightarrow F(s) = \frac{\pi}{(s+\pi)^2} = \mathcal{L}[\pi t e^{-\pi t}] \Rightarrow f(t) = \pi t e^{-\pi t}$$

(52) $F(s) = \dfrac{4s-2}{s^2-6s+18}$

* Try to factorize the denominator : $\Delta' = b'^2 - ac = 9 - 18 = -9 < 0$

The roots are not real.

* Complete the square :

$$F(s) = \frac{4s-2}{\ldots} = \frac{4s-2}{\ldots} = \frac{4(s-3)+10}{\ldots}$$

$$F(s) = 4\,\frac{s-3}{(s-3)^2+9} + \frac{10}{3}\,\frac{3}{(s-3)^2+9}$$

$$F(s) = 4\,\mathcal{L}\{\cos 3t \cdot e^{3t}] + \frac{10}{3}\,\mathcal{L}[\sin 3t \cdot e^{3t}]$$

$$= \mathcal{L}\left[4e^{3t}\cos 3t + \frac{10}{3}e^{3t}\sin 3t\right]$$

$$\Rightarrow \mathcal{L}^{-1}(F) = f(t) = e^{3t}\left(4\cos 3t + \frac{10}{3}\sin 3t\right).$$

(54) $F(s) = \dfrac{2s-56}{s^2-4s-12}$

<u>1st method</u>: Partial Fractions

$$F(s) = \frac{2s-56}{(s+2)(s-6)} = \frac{-60/-8}{s+2} + \frac{-44/8}{s-6} = \frac{15}{2}\mathcal{L}\{e^{-2t}\} - 11\,\mathcal{L}\{e^{6t}\}$$

$$= \mathcal{L}\left\{\frac{15}{2}e^{-2t} - \frac{11}{2}e^{6t}\right\} \Rightarrow \mathcal{L}^{-1}(F) = f(t) = \frac{15}{2}e^{-2t} - \frac{11}{2}e^{6t}$$

<u>2nd method</u>: Complete the square

$$F(s) = \frac{2s-56}{s^2-4s+4-16} = \frac{2(s-2)-52}{(s-2)^2-16} = 2\,\frac{s-2}{(s-2)^2-16} - 13\,\frac{4}{(s-2)^2-16}$$

$$F(s) = 2\,\mathcal{L}\{\cosh 4t\, e^{2t}\} - 13\,\mathcal{L}\{\sinh 4t\, e^{2t}$$

$$\mathcal{L}^{-1}(F) = 2e^{2t}\cosh 4t - 13e^{2t}\sinh 4t.$$

6.2 Laplace Transforms of Derivatives and Integrals

Theorem Let $f(t)$ be a continuous function whose derivative $f'(t)$ is piecewise continuous. If $|f(t)| \leq ce^{kt}$.

then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ if $s > k$

Proof: $\mathcal{L}[f'] = \int_0^{+\infty} \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv}$

$$\left| \begin{array}{l} u = e^{-st} \Rightarrow du = -se^{-st}dt \\ dv = f'dt \Rightarrow v = f. \end{array} \right.$$

$$\mathcal{L}\{f'\} = uv - \int v\,du = e^{-st} f(t)\Big|_0^{+\infty} - \int_0^{+\infty} -se^{-st} f(t)\,dt.$$

$$= \lim_{t \to +\infty} e^{-st} f(t) - e^0 f(0) + s\int_0^{+\infty} e^{-st} f(t)\,dt.$$

$$= s\mathcal{L}\{f\} - f(0) + L$$

* $\left| e^{-st} f(t) \right| \leq e^{-st} c\, e^{kt} = ce^{-(s-k)t}$

$s > k \Rightarrow s - k > 0 \Rightarrow \lim\limits_{t \to +\infty} e^{-(s-k)t} = 0$

By Sandwich Th. $\lim e^{-st} f(t) = 0 \Rightarrow L = 0$.

* Formulas:

1) $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$
2) $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$
3) $\mathcal{L}\{y'''\} = s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0)$
4) $\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y\} - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)$

Example Let $f(t) = t \sin t$

Find a relation between f'' and f, and use it to find $\mathcal{L}\{f\}$

$$f = t\sin t \Rightarrow f' = \sin t + t\cos t$$
$$\Rightarrow f'' = \cos t + \cos t - t\sin t$$
$$\Rightarrow f'' = 2\cos t - f.$$

$$\Rightarrow \mathcal{L}\{f''\} = \mathcal{L}\{2\cos t\} - \mathcal{L}\{f\}$$

$$s^2\mathcal{L}\{f\} - sf(0) - f'(0) = 2\frac{s}{s^2+1} - \mathcal{L}\{f\}.$$

$$\Rightarrow \mathcal{L}\{f\}(s^2+1) = s\underbrace{f(0)}_{0} + \underbrace{f'(0)}_{0} + \frac{2s}{s^2+1}$$

$$(s^2+1)\,F(s) = \frac{2s}{s^2+1} \Rightarrow F(s) = \frac{2s}{(s^2+1)^2}$$

Laplace Transform Method

Consider the IVP: $\begin{cases} y'' + ay' + by = r(t) \\ y(0) = \alpha \;;\; y'(0) = \beta \end{cases}$

Step1: Find the Laplace of the DE

$$\mathcal{L}\{y''\} + a\mathcal{L}\{y'\} + b\mathcal{L}\{y\} = \mathcal{L}\{r(t)\}$$

$$(s^2Y - s\alpha - \beta) + a(sY - \alpha) + bY = R.$$

Step2: Solve the equation obtained for Y

$$Y(s^2 + as + b) = R + \alpha s + \beta + a\alpha$$

$$Y = \frac{R + \alpha s + \beta + a\alpha}{s^2 + as + b}$$

Step3: Solution : $y = \mathcal{L}^{-1}(Y)$

Example: Solve the IVP: $\begin{cases} y''-4y'+4y=e^{2t} \\ y(0)=1,\ y'(0)=2 \end{cases}$ → HW Solve it using chap 2

Step 1 $\mathcal{L}\{y''\}-4\mathcal{L}\{y'\}+4\mathcal{L}\{y\}=\mathcal{L}\{e^{2t}\}$

$$[s^2Y-sy(0)-y'(0)]-4[sY-y(0)]+4Y=\frac{1}{s-2}$$

Step 2 $Y(s^2-4s+4)=\frac{1}{s-2}+s+2-4$

$$Y(s-2)^2=\frac{1}{s-2}+s-2 \Rightarrow Y=\frac{1}{(s-2)^3}+\frac{1}{s-2}$$

Step 3 $Y=\frac{1}{2!}\frac{\Gamma(3)}{(s-2)^3}+\frac{1}{s-2}=\frac{1}{2}\mathcal{L}\{t^2e^{2t}\}+\mathcal{L}\{e^{2t}\}$

$$=\mathcal{L}\left\{\frac{1}{2}t^2e^{2t}+e^{2t}\right\}$$

Solution: $y=\frac{1}{2}t^2e^{2t}+e^{2t}$

Laplace of an integral:

Theorem: $\boxed{\mathcal{L}\left\{\int_0^t f(u)\,du\right\}=\frac{1}{s}\mathcal{L}\{f(t)\}}$

Proof Let $g(t)=\int_0^t f(u)\,du$ then $g'(t)=f(t)$

$$\Rightarrow \mathcal{L}\{g'\}=\mathcal{L}[f] \Rightarrow s\mathcal{L}\{g\}-\underbrace{g(0)}_{0}=\mathcal{L}\{f\}$$

$$\Rightarrow \mathcal{L}\{g\}=\frac{1}{s}\mathcal{L}\{f\}$$

Example: Find the inverse Laplace of $F(s)=\frac{1}{s(s^2+4)}$

1st method: Partial fraction

$$F(s)=\frac{1}{s(s^2+4)}=\frac{1/4}{s}+\frac{bs+c}{s^2+4}$$

Multiply by $s(s^2+4)$: $1=\frac{1}{4}(s^2+4)+s(bs+c)$

$$F(s) = \frac{1/4}{s} + \frac{-\frac{1}{4}s}{s^2+4} = \mathcal{L}\left\{\frac{1}{4}\right\} - \frac{1}{4}\mathcal{L}\{\cos 2t\} = \mathcal{L}\left\{\frac{1}{4} - \frac{1}{4}\cos 2t\right\}.$$

$$f = \mathcal{L}^{-1}(F) = \frac{1}{4} - \frac{1}{4}\cos 2t.$$

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

2nd method:

$$F(s) = \frac{1}{s}\,\frac{1}{s^2+4} = \frac{1}{s}\cdot\frac{1}{2}\cdot\frac{2}{s^2+4} = \frac{1}{s}\cdot\frac{1}{2}\mathcal{L}\{\sin 2t\}.$$

$$F(s) = \frac{1}{2}\,\frac{1}{s}\,\mathcal{L}\{\underbrace{\sin 2t}_{g(t)}\} = \frac{1}{2}\mathcal{L}\left\{\int_0^t g(u)\,du\right\}$$

$$F(s) = \frac{1}{2}\mathcal{L}\left\{\int_0^t \sin 2u\,du\right\} = \frac{1}{2}\mathcal{L}\left\{\left[\frac{-\cos 2u}{2}\right]_0^t\right\}$$

$$F(s) = \mathcal{L}\left\{-\frac{1}{4}\cos 2t + \frac{1}{4}\right\} \Rightarrow f(t) = \frac{1}{4} - \frac{1}{4}\cos 2t.$$

Prob 6.2 (10-20) + (27-35)   HW (13, 17, 20, 27, 31, 33, 35)

(12) IVP: $\begin{cases} y'' - y' - 6y = 0 \\ y(0) = 6,\ y'(0) = 13. \end{cases}$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$$

$$(s^2Y - 6s - 13) - (sY - 6) - 6Y = 0.$$

$$Y(s^2 - s - 6) = 6s + 7 \Rightarrow Y = \frac{6s+7}{s^2 - s - 6}$$

$$Y = \frac{6s+7}{(s+2)(s-3)} = \frac{-5/-5}{s+2} + \frac{25/5}{s-3} = \frac{1}{s+2} + \frac{5}{s-3}$$

$$= \mathcal{L}\{e^{-2t}\} + \mathcal{L}\{5e^{3t}\}$$

$$= \mathcal{L}\{e^{-2t} + 5e^{3t}\}.$$

$$y = e^{-2t} + 5e^{3t}$$

(15) IVP: $\begin{cases} y'' + 2y' + 2y = 0 \\ y'(0) = -3,\ y(0) = 1 \end{cases}$ $*(s^2Y - s + 3) + 2(sY - 1) + 2Y = 0$

$(s^2 + 2s + 2)Y = s - 1 \Rightarrow Y = \frac{s-1}{\ldots}$

$$Y=\frac{s-1}{s^2+2s+1+1}=\frac{s-1}{(s+1)^2+1}=\frac{s+1-2}{(s+1)^2+1}=\frac{s+1}{(s+1)^2+1}-2\,\frac{1}{(s+1)^2+1}$$

$$Y=\mathcal{L}\{\cos t\, e^{-t}\}-2\,\mathcal{L}\{\sin t\, e^{-t}\}$$

$$y=e^{-t}\cos t-2e^{-t}\sin t$$

(18) IVP: $\begin{cases} y''+9y=10e^{-t} \\ y(0)=0,\ y'(0)=0. \end{cases}$

$$(s^2Y-s.0-0)+9\,Y=10\,\frac{1}{s+1}$$

$$(s^2+9)\,Y=\frac{10}{s+1}\Rightarrow Y=\frac{10}{(s^2+9)(s+1)}$$

$$Y=\frac{1}{s+1}+\frac{bs+c}{s^2+9}\Rightarrow 10=(s^2+9)+(s+1)(bs+c)$$

coef of s^2: $1+b=0\Rightarrow b=-1$

coef of s : $b+c=0$

const : $10=9+c\Rightarrow c=1$

$$Y=\frac{1}{s+1}+\frac{-s+1}{s^2+9}=\mathcal{L}\{e^{-t}\}-\frac{s}{s^2+9}+\frac{1}{3}\,\frac{3}{s^2+9}$$

$$Y=\mathcal{L}\{e^{-t}\}-\mathcal{L}\{\cos 3t\}+\frac{1}{3}\mathcal{L}\{\sin 3t\}.$$

$$y=e^{-t}-\cos 3t+\frac{1}{3}\sin 3t.$$

(28) $F(s)=\dfrac{10}{s^3-\pi s^2}$ Find $\mathcal{L}^{-1}(F)$

1st method: $F(s)=\dfrac{10}{s^2(s-\pi)}=\dfrac{a}{s}+\dfrac{\frac{10}{-\pi}}{s^2}+\dfrac{10/\pi^2}{s-\pi}$

$$F(s) = \mathcal{L}\{a\} - \frac{10}{\pi}\mathcal{L}\{t\} + \frac{10}{\pi^2}\mathcal{L}\{e^{\pi t}\}$$

$$f = a - \frac{10}{\pi}t + \frac{10}{\pi^2}e^{\pi t}$$

2nd method: $\quad \dfrac{10}{s-\pi} = \mathcal{L}\{10e^{\pi t}\}$

$$\frac{1}{s}\frac{10}{s-\pi} = \mathcal{L}\left\{\int_0^t 10e^{\pi u}\,du\right\} = \mathcal{L}\left\{\frac{10}{\pi}e^{\pi u}\Big|_0^t\right\}$$

$$= \mathcal{L}\left\{\frac{10}{\pi}(e^{\pi t}-1)\right\}.$$

$$F(s) = \frac{1}{s}\left(\frac{1}{s}\frac{10}{s-\pi}\right) = \mathcal{L}\left\{\int_0^t \frac{10}{\pi}(e^{\pi u}-1)\,du\right\}.$$

$$f(t) = \int_0^t \frac{10}{\pi}(e^{\pi u}-1)\,du = \frac{10}{\pi}\left(\frac{e^{\pi u}}{\pi} - u\right)\Big|_0^t$$

$$f(t) = \frac{10}{\pi}\left(\frac{e^{\pi t}}{\pi} - t\right) - \frac{10}{\pi}\left(\frac{1}{\pi}\right).$$

(32) $\quad F(s) = \dfrac{2}{s^3+9s}$

1st method: $\quad F(s) = \dfrac{2}{s(s^2+9)} = \dfrac{2/9}{s} + \dfrac{bs+c}{s^2+9}$

$$2 = \frac{2}{9}(s^2+9) + s(bs+c) \Rightarrow \left(b+\frac{2}{9}\right)s^2 + cs = 0$$

$$\Rightarrow b = \frac{-2}{9},\ c = 0$$

$$F(s) = \frac{2/9}{s} - \frac{2}{9}\frac{s}{s^2+9} = \mathcal{L}\left\{\frac{2}{9}\right\} - \frac{2}{9}\mathcal{L}\{\cos 3t\}$$

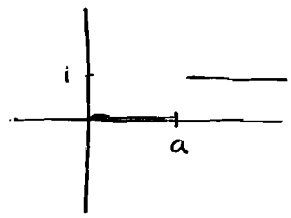
$$f = \frac{2}{9} - \frac{2}{9}\cos 3t.$$

2nd method: $\quad F(s) = \dfrac{1}{s}\dfrac{2}{s^2+9} = \dfrac{1}{s}\,\dfrac{2}{3}\,\dfrac{3}{s^2+9}$

$F(s) = \dfrac{2}{3}\dfrac{1}{s}\mathcal{L}\{\sin 3t\}$ $\qquad f(t) = \dfrac{2}{3}\displaystyle\int_0^t \sin 3u\,du$

6.3 Unit Step Function

Definition: The unit step function $U(t-a)$ is defined by:



$$U(t-a)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } t>0\end{cases} \quad \text{where } \boxed{a>0}$$

2nd Shifting Property (or t-shifting)

1) $\mathcal{L}\{U(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$ used to find $\mathcal{L}$

2) $e^{-as}\mathcal{L}\{f(t)\} = \mathcal{L}\{U(t-a)f(t-a)\}$ used to find $\mathcal{L}^{-1}$

Proof 1) $\mathcal{L}\{U(t-a)f(t)\} = \int_0^{+\infty} e^{-st}U(t-a)f(t)\,dt$

$$= \int_0^a e^{-st}(0)f(t)\,dt + \int_a^{+\infty} e^{-st}(1)f(t)\,dt.$$

$$= \int_a^{+\infty} e^{-st}f(t)\,dt \quad \text{Let } T=t-a \Rightarrow dT=dt$$

$$= \int_0^{+\infty} e^{-s(T+a)}f(T+a)\,dT = e^{-as}\int_0^{+\infty} e^{-sT}f(T+a)\,dT$$

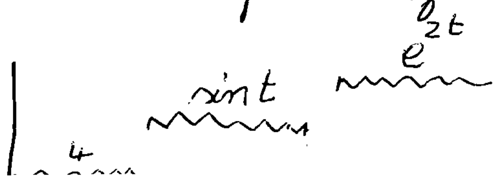
$$= e^{-as}\mathcal{L}\{f(t+a)\}$$

Formula $$\boxed{\mathcal{L}\{U(t-a)\} = \frac{e^{-as}}{s}}$$

Proof: $f(t)=1 \Rightarrow f(t+a)=1 \Rightarrow \mathcal{L}\{f(t+a)\}=\frac{1}{s}$

$$(1) \Rightarrow \mathcal{L}\{U(t-a)\cdot 1\} = e^{-as}\frac{1}{s}$$

Example: Find the Laplace of $f(t)=\begin{cases}4 & \text{if } t<\frac{\pi}{2}\\ \sin t & \text{if } \frac{\pi}{2}<t<4\pi\\ e^{2t} & \text{if } t>4\pi\end{cases}$



$$f(t) = 4 + (\sin t - 4)\,U\!\left(t - \tfrac{\pi}{2}\right) + (e^{2t} - \sin t)\,U(t - 4\pi)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4\} + \mathcal{L}\{U(t-\tfrac{\pi}{2})\underbrace{[\sin t - 4]}_{g(t)}\} + \mathcal{L}\{U(t-4\pi)\underbrace{[e^{2t} - \sin t]}_{h(t)}\}$$

$$= \frac{4}{s} + e^{-\frac{\pi}{2}s}\mathcal{L}\{g(t+\tfrac{\pi}{2})\} + e^{-4\pi s}\mathcal{L}\{h(t+4\pi)\} \quad (2^{nd}\,SP)$$

$$* g(t) = \sin t - 4 \Rightarrow g(t+\tfrac{\pi}{2}) = \sin(t+\tfrac{\pi}{2}) - 4 = \cos t - 4.$$

$$\mathcal{L}\{g(t+\tfrac{\pi}{2})\} = \frac{s}{s^2+1} - \frac{4}{s}$$

$$* h(t) = e^{2t} - \sin t \Rightarrow h(t+4\pi) = e^{2(t+4\pi)} - \sin(t+4\pi)$$

$$= e^{8\pi}e^{2t} - \sin t$$

$$\mathcal{L}\{h(t+4\pi)\} = e^{8\pi}\frac{1}{s-2} - \frac{1}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{4}{s} + e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1} - \frac{4}{s}\right) + e^{-4\pi s}\left(\frac{e^{8\pi}}{s-2} - \frac{1}{s^2+1}\right)$$

<u>Example</u>: Find the inverse laplace of $F(s) = \dfrac{s + e^{-\frac{\pi s}{2}}}{s^2+4}$

$$F(s) = \frac{s}{s^2+4} + \frac{e^{-\frac{\pi s}{2}}}{2}\,\frac{2}{s^2+4}$$

$$F(s) = \mathcal{L}\{\cos 2t\} + \frac{1}{2}e^{-\frac{\pi}{2}s}\mathcal{L}\{\underbrace{\sin 2t}_{g(t)}\}.$$

$$= \mathcal{L}\{\cos 2t\} + \frac{1}{2}\mathcal{L}\{U(t-\tfrac{\pi}{2})\,g(t-\tfrac{\pi}{2})\}.$$

$$f(t) = \cos 2t + \frac{1}{2}U(t-\tfrac{\pi}{2})\sin 2(t-\tfrac{\pi}{2})$$

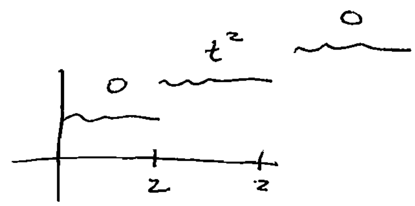
$$\boxed{f(t) = \cos 2t - \frac{1}{2}U(t-\tfrac{\pi}{2})\sin 2t}$$

$$\Rightarrow f(t) = \begin{cases} \cos 2t & \text{if } t < \frac{\pi}{2} \\ \cos 2t - \frac{1}{2}\sin 2t & \text{if } t > \frac{\pi}{2} \end{cases}$$

Prob 6.3 (1–47) HW (7, 15, 37, 39, 45)

(5) $f(t)=\begin{cases} t^2 & \text{if } 1<t<2 \\ 0 & \text{elsewhere.}\end{cases}$

$$f(t)=\begin{cases} 0 & \text{if } t<1 \\ t^2 & \text{if } 1<t<2 \\ 0 & \text{if } t>2.\end{cases}$$



$$f(t)=0+(t^2-0)u(t-1)+(0-t^2)u(t-2)$$

$$\mathcal{L}\{f(t)\}=\mathcal{L}\{\underset{\substack{\uparrow\\ g(t)}}{t^2}u(t-1)\}-\mathcal{L}\{\underset{\substack{\uparrow\\ g(t)}}{t^2}u(t-2)\}$$

$$=e^{-s}\mathcal{L}\{g(t+1)\}-e^{-2s}\mathcal{L}\{g(t+2)\}.$$

$$=e^{-s}\mathcal{L}\{(t+1)^2\}-e^{-2s}\mathcal{L}\{(t+2)^2\}$$

$$=e^{-s}\mathcal{L}\{t^2+2t+1\}-e^{-2s}\mathcal{L}\{t^2+4t+4\}$$

$$=e^{-s}\left(\frac{2!}{s^3}+2\frac{1}{s^2}+\frac{1}{s}\right)-e^{-2s}\left(\frac{2}{s^3}+4\frac{1}{s^2}+\frac{4}{s}\right)$$

(18) $$F(s)=\frac{e^{-\pi s}}{s^2+2s+2}=e^{-\pi s}\frac{1}{(s+1)^2+1}=e^{-\pi s}\mathcal{L}\{\underbrace{\sin t\, e^{-t}}_{g(t)}\}$$

$$=\mathcal{L}\{u(t-\pi)\,g(t-\pi)\}$$

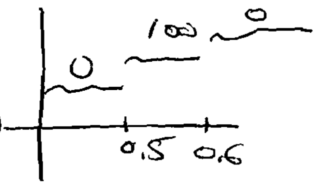
$$f(t)=u(t-\pi)\sin(t-\pi)e^{-(t-\pi)}=\boxed{-u(t-\pi)\sin t\, e^{\pi-t}}$$

$$f(t)=\begin{cases} 0 & \text{if } t<\pi \\ -\sin t\, e^{\pi-t} & \text{if } t>\pi.\end{cases}$$

(36) RC-circuit : $R=10\,\Omega$, $C=10^{-2}F$, $i(0)=0$ current $i(t)$

$$v(t)=\begin{cases} 100 \text{ volts} & \text{if } 0.5<t<0.6 \\ 0 & \text{elsewhere.}\end{cases}$$

By Kirchhoff's Law: $E_R + E_C - E = 0$.



$$Ri + \frac{q}{c} = E \Rightarrow \boxed{10i + 100q = v(t)} \quad (1)$$

$$\Rightarrow q' + 10q = \frac{1}{10} v(t)$$

Where $v(t) = 0 + (100 - 0)U(t-0.5) + (0 - 100)U(t-0.6)$

$$\mathcal{L}\{q'\} + 10\mathcal{L}\{q\} = \frac{1}{10}\mathcal{L}\{v\}$$

$$s\mathcal{L}\{q\} - q(0) + 10\mathcal{L}\{q\} = \frac{1}{10}\mathcal{L}\{v\}$$

$$(s+10)Q = q(0) + \frac{1}{10}\mathcal{L}\{v\}$$

$$*(1) \Rightarrow \underset{\downarrow \atop 0}{i(0)} + 10q(0) = \frac{1}{10} \underset{\downarrow \atop 0}{v(0)} \Rightarrow q(0) = 0$$

$$\therefore Q = \frac{1}{10(s+10)}\mathcal{L}\{v\}$$

Where $\mathcal{L}\{v\} = 100\mathcal{L}\{U(t-0.5)\} - 100\mathcal{L}\{U(t-0.6)\}$

$$= 100\frac{e^{-0.5s}}{s} - 100\frac{e^{-0.6s}}{s}$$

$$Q = \frac{10}{s(s+10)}\left(e^{-0.5s} - e^{-0.6s}\right)$$

$$* \frac{10}{s(s+10)} = \frac{1}{s} + \frac{-1}{s+10} = \mathcal{L}\{1\} - \mathcal{L}\{e^{-10t}\}$$

$$\frac{10}{s(s+10)} = \mathcal{L}\{\underbrace{1 - e^{-10t}}_{f(t)}\}$$

$$* e^{-0.5s}\frac{10}{s(s+10)} = e^{-0.5s}\mathcal{L}\{f(t)\} \overset{2^{nd}\ SP}{=} \mathcal{L}\{U(t-0.5)f(t-0.5)\}$$

$$= \mathcal{L}\left\{U(t-0.5)\left[1 - e^{-10(t-0.5)}\right]\right\}$$

$$e^{-0.6s}\ 10 \quad \ldots \quad e^{-10(t-0.6)}]$$

$$q = U(t-0.5)\left(1-e^{5-10t}\right) - U(t-0.6)\left(1-e^{6-10t}\right)$$

$$q = \begin{cases} 0 & \text{if } t<0.5 \\ 1-e^{5-10t} & \text{if } 0.5<t<0.6 \\ -e^{5-10t}+e^{6-10t} & \text{if } t>0.6 \end{cases}$$

$$i = q' = \begin{cases} 0 & \text{if } t<0.5 \\ 10e^{5-10t} & \text{if } 0.5<t<0.6 \\ 10e^{5-10t}-10e^{6-10t} & \text{if } t>0.6 \end{cases}$$

(46) RLC-circuit ; $R = 4\,\Omega$, $L = 1\,H$, $C = 0.05\,F$, $i(0) = q(0) = 0$.

$$v = \begin{cases} 34e^{-t} \text{ volts} & \text{if } 0<t<4 \\ 0 & \text{if } t>4 \end{cases}$$

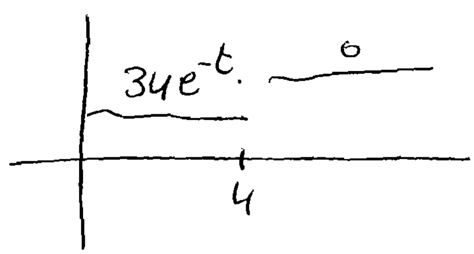
$$E_R + E_L + E_C - E = 0 \Rightarrow Ri + Li' + \frac{q}{C} = E$$

$$4i + i' + \frac{q}{0.05} = v \Rightarrow q'' + 4q' + 20q = v$$

$$s^2\mathcal{L}\{q\} - s\underbrace{q(0)}_{0} - \underbrace{q'(0)}_{i(0)=0} + 4\left(s\mathcal{L}\{q\} - q(0)\right) + 20\mathcal{L}\{q\} = \mathcal{L}\{v\}$$

$$(s^2+4s+20)Q = \mathcal{L}\{v\} \Rightarrow Q = \frac{\mathcal{L}\{v\}}{s^2+4s+20}$$

$* \; v = 34e^{-t} + (0-34e^{-t})\,U(t-4)$



$$\mathcal{L}\{v\} = \frac{34}{s+1} - 34\mathcal{L}\{e^{-t}U(t-4)\}$$

$$\mathcal{L}\{v\} = \frac{34}{s+1} - 34e^{-4s}\mathcal{L}\{f(t+4)\}$$

$$\mathcal{L}\{v\} = \frac{34}{s+1} - 34e^{-4s}\mathcal{L}\{e^{-(t+4)}\}$$

$$\mathcal{L}\{v\} = \frac{34}{s+1}\left(1 - e^{-4}e^{-4s}\right)$$

$$Q = \frac{34}{(s+1)(s^2+4s+20)}\left(1 - e^{-4}e^{-4s}\right)$$

$$* \quad G(s) = \frac{34}{(s+1)(s^2+4s+20)} = \frac{2}{s+1} + \frac{bs+c}{s^2+4s+20}$$

$$34 = 2(s^2+4s+20) + (bs+c)(s+1)$$

$$\begin{cases} s^2: \; 2+b=0 \Rightarrow b=-2 \\ s: \; 8+b+c=0 \; \checkmark \\ \text{const}: \; 34 = 40+c \Rightarrow c=-6. \end{cases}$$

$$G(s) = \frac{2}{s+1} + \frac{-2s-6}{(s+2)^2+16} = \frac{2}{s+1} + \frac{-2(s+2)}{(s+2)^2+16}$$

$$G(s) = \frac{2}{s+1} - 2\frac{s+2}{(s+2)^2+16} - \frac{1}{2}\frac{4}{(s+2)^2+16}$$

$$G(s) = \mathcal{L}\{2e^{-t}\} - 2\mathcal{L}\{\cos 4t\, e^{-2t}\} - \frac{1}{2}\mathcal{L}\{\sin 4t\, e^{-2t}\}.$$

$$G(s) = \mathcal{L}\{2e^{-t} - 2e^{-2t}\cos 4t - \tfrac{1}{2}e^{-2t}\sin 4t\}.$$

$$Q = G(s) - e^{-4}e^{-4s}G(s)$$

$$e^{-4s}G(s) = e^{-4s}\mathcal{L}\{g(t)\} = \mathcal{L}\{U(t-4)g(t-4)\}$$

$$Q = \mathcal{L}\{g(t)\} - e^{-4}\mathcal{L}\{U(t-4)\,g(t-4)\}.$$

$$q(t) = g(t) - e^{-4}U(t-4)g(t-4)$$

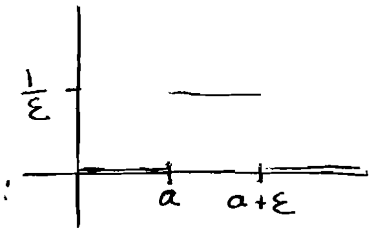
$$q(t) = 2e^{-t} - e^{-2t}\left(2\cos 4t + \tfrac{1}{2}\sin 4t\right) - e^{-4}U(t-4)\Big[2e^{-(t-4)} - e^{-2(t-4)}\left(2\cos 4(t-4) + \tfrac{1}{2}\sin 4(t-4)\right)\Big]$$

$$q(t) = \begin{cases} \text{——} & \text{if } t < 4 \end{cases} \qquad i'(t) = q''(t) = \{ \text{—}$$

6.4 Dirac Delta function

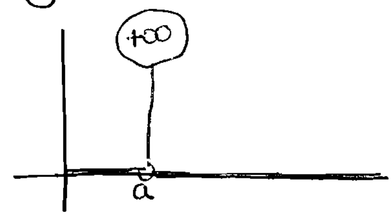
Definition: Let a, ε be 2 positive numbers.

Let $\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & \text{if } a < t < a+\varepsilon \\ 0 & \text{elsewhere} \end{cases}$



* The Dirac Delta function is defined by:

$$\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)$$



Formula: $\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}}$

Proof * $\mathcal{L}\{\delta_\varepsilon(t-a)\} = \int_0^{+\infty} e^{-st}\delta_\varepsilon(t-a)\,dt$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}\,dt + \int_{a+\varepsilon}^{+\infty} e^{-st}\cdot 0\,dt$$

$$= \frac{1}{\varepsilon}\,\frac{e^{-st}}{-s}\Big|_a^{a+\varepsilon} = -\frac{1}{\varepsilon s}\left(e^{-s(a+\varepsilon)} - e^{-as}\right)$$

$$= -\frac{e^{-as}}{\varepsilon s}\left(e^{-\varepsilon s} - 1\right)$$

* $\mathcal{L}\{\delta(t-a)\} = \lim_{\varepsilon \to 0} \mathcal{L}\{\delta_\varepsilon(t-a)\}$

$$= \lim_{\varepsilon \to 0} e^{-as}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = \frac{0}{0}$$

$$\overset{HR}{=} \lim_{\varepsilon \to 0} e^{-as}\left(\frac{s e^{-\varepsilon s}}{s}\right) = e^{-as}$$

Prob 6.4 (1–12) HW (1, 6, 11)

④ IVP: $\begin{cases} y'' + 3y' + 2y = 10\sin t + 10\delta(t-1) \\ y(0) = 1, \quad y'(0) = -1 \end{cases}$

$$(s^2 Y - s.1 + 1) + 3(sY - 1) + 2Y = 10\frac{1}{s^2+1} + 10e^{-s}$$

$$(s^2 + 3s + 2)Y = s + 2 + \frac{10}{s^2+1} + 10e^{-s}$$

$$Y = \frac{s+2}{(s+1)(s+2)} + \frac{10}{\quad} + \frac{10e^{-s}}{\quad}$$

⑩ IVP: $\begin{cases} y'' + 5y = 25t - 100\,\delta(t-\pi) \\ y(0) = -2 \quad , \; y'(0) = 5 \end{cases}$

$$s^2 Y - s(-2) - 5 + 5Y = 25\,\frac{\Gamma(2)}{s^2} - 100\,e^{-\pi s}$$

$$Y(s^2+5) = -2s + 5 + \frac{25}{s^2} - 100e^{-\pi s}$$

$$= -2s + \frac{5s^2+25}{s^2} - 100e^{-\pi s}$$

$$\Rightarrow Y = \frac{-2s}{s^2+5} + \frac{5}{s^2} - \frac{100e^{-\pi s}}{\sqrt{5}}\,\frac{\sqrt{5}}{s^2+5}$$

$$Y = -2\mathcal{L}\{\cos\sqrt{5}t\} + 5\mathcal{L}\{t\} - 20\sqrt{5}\,e^{-\pi s}\mathcal{L}\{\underbrace{\sin\sqrt{5}t}_{f(t)}\}$$

$$* \; e^{-\pi s}\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-\pi)\,f(t-\pi)\}$$
$$= \mathcal{L}\{u(t-\pi)\sin\sqrt{5}(t-\pi)\}.$$

$$Y = \mathcal{L}\{-2\cos\sqrt{5}t + 5t - 20\sqrt{5}\,u(t-\pi)\sin\sqrt{5}(t-\pi)\}.$$

$$\boxed{y = 5t - 2\cos\sqrt{5}t - 20\sqrt{5}\,u(t-\pi)\sin\sqrt{5}(t-\pi)}$$

6.5 Convolution Product

Definition: The convolution product of 2 functions $f(t)$ and $g(t)$, denoted by $f*g$, is defined by.

$$\boxed{f * g = \int_0^t f(t-u)\,g(u)\,du}$$

$\mathcal{L}\{f\} + \mathcal{L}\{g\} = \mathcal{L}\{f+g\}$

$\mathcal{L}\{f\}\cdot\mathcal{L}\{g\} \neq \mathcal{L}\{f\cdot g\}$

Formulas: 1) $\boxed{g * f = f * g}$ $\Rightarrow$ used to put the easier function first.

Example: Find the inverse Laplace of $F(s) = \dfrac{8}{(s^2+4)^2}$

$$F(s) = 2\,\frac{2}{s^2+4}\,\frac{2}{s^2+4} = 2\,\mathcal{L}\{\sin 2t\}\,\mathcal{L}\{\sin 2t\}$$

$$F(s) = 2\,\mathcal{L}\{\sin 2t * \sin 2t\}$$

$$f(t) = \mathcal{L}^{-1}(F) = 2\sin 2t * \sin 2t$$

$$f(t) = 2\int_a^t \sin 2(t-u)\,\sin 2u\,du$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b \quad (1)$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b \quad (2)$$

$(2)-(1)$: $2\sin a\sin b = \cos(a-b) - \cos(a+b)$

$$f(t) = \int_0^t [\cos(2t-2u-2u) - \cos(2t-2u+2u)]\,du$$

$$f(t) = \int_0^t (\cos(2t-4u) - \cos 2t)\,du$$

$$= \frac{\sin(2t-4u)}{-4} - u\cos 2t\,\Big|_{u=0}^{t}$$

$$f(t) = \left(\frac{\sin(-2t)}{-4} - t\cos 2t\right) - \left(\frac{\sin 2t}{-4} - 0\right)$$

$$\boxed{f(t) = \frac{\sin 2t}{2} - t\cos 2t}$$

Example: Solve for $y(t)$ the integral equation:

$$y(t) = \sqrt{t} + \int_0^t \sin(t-u)\,y(u)\,du$$

$$y(t) = \sqrt{t} + (\sin t * y(t))$$

$$\Rightarrow \mathcal{L}\{y\} = \mathcal{L}\{t^{1/2}\} + \mathcal{L}\{\sin t * y(t)\}$$

$$Y = \frac{\Gamma(3/2)}{s^{3/2}} + \mathcal{L}\{\sin t\}\,\mathcal{L}\{y\}$$

$Y(\ldots)\ \ldots\ \Gamma(3/2)$

$$Y = \frac{s^2+1}{s^2 \cdot s^{3/2}}\,\Gamma(3/2) = \Gamma\left(\tfrac{3}{2}\right)\left(\frac{1}{s^{3/2}} + \frac{1}{s^{7/2}}\right)$$

$$Y = \frac{\Gamma(3/2)}{s^{3/2}} + \frac{\Gamma(3/2)}{\Gamma\left(\frac{7}{2}\right)}\;\frac{\Gamma(7/2)}{s^{7/2}} = \mathcal{L}\{t^{1/2}\} + \frac{\Gamma(3/2)}{\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}\mathcal{L}\{t^{5/2}\}$$

$$Y = \mathcal{L}\left\{t^{1/2} + \frac{\Gamma(3/2)}{\frac{5}{2}\,\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)}\,t^{5/2}\right\}$$

$$\boxed{y(t) = t^{1/2} + \frac{4}{15}t^{5/2}}$$

Prob 6.5 (1 – 34) HW (14, 15, 19, 21, 28, 33)

(1) $1 * 1 = \int_0^t f(t-u)g(u)\,du = \int_0^t 1.1\,du = u\Big|_0^t = t$

(13) $F(s) = \dfrac{1}{s^2(s^2+1)}$

1st method: Partial Fractions.

$$F(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{S(S+1)} = \frac{1}{S} + \frac{-1}{S+1} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$F(s) = \mathcal{L}\{t\} - \mathcal{L}\{\sin t\} \Rightarrow f(t) = t - \sin t$$

2nd method: $F(s) = \dfrac{1}{s^2}\,\dfrac{1}{s^2+1} = \mathcal{L}\{t\}\,\mathcal{L}\{\sin t\} = \mathcal{L}\{t * \sin t\}$

$$f(t) = t * \sin t = \int_0^t (t-u)\sin u\,du$$

$$f(t) = -(t-u)\cos u - \sin u\Big|_0^t$$

$$= (0 - \sin t) - (-t - 0) = t - \sin t.$$

$t-u$	$\sin u$
-1 (+)	$-\cos u$
0 (−)	$-\sin u$

(18) IVP: $\begin{cases} y'' + y = \sin t \\ y(0) = 0,\; y'(0) = 0. \end{cases}$

$s^2Y - s\cdot 0 - 0 + Y = \frac{1}{\ldots} \Rightarrow Y(s^2+1) = \frac{1}{\ldots}$

$$Y = c\,\mathcal{L}\{1 - \tfrac{1}{6}t^3 + \tfrac{3}{2}t^2 - 3t\} \Rightarrow y = c(1 - 3t + \tfrac{3}{2}t^2 - \tfrac{1}{6}t^3)$$

$* \; y(0) = 1 \Rightarrow c = 1 \Rightarrow \boxed{y = 1 - 3t + \tfrac{3}{2}t^2 - \tfrac{1}{6}t^3}$

$* \; y' = -3 + 3t - \tfrac{1}{2}t^2 \Rightarrow y'(0) = -3$

Prob 6.6 (1-20) HW (5, 7, 11, 14, 16, 19)

(4) $f(t) = t\cos(t+k) = t\,g(t)$

$\mathcal{L}\{t\,g(t)\} = -G'(s)$?

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\cos(t+k)\}$$
$$= \mathcal{L}\{\cos t \cos k - \sin t \sin k\}$$
$$= \cos k \frac{s}{s^2+1} - \sin k \frac{1}{s^2+1} = \frac{s\cos k - \sin k}{s^2+1}$$

$$F(s) = \mathcal{L}\{t\,g(t)\} = -G'(s) = -$$

$$F(s) = -\frac{\cos k\,(s^2+1) - (s\cos k - \sin k)(2s)}{(s^2+1)^2}$$

$$F(s) = -\left[\frac{(1-s^2)\cos k + 2s\sin k}{(s^2+1)^2}\right]$$

(12) $f(t) = t e^{-kt}\sin t$

$* \; \mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

$* \; \mathcal{L}\{t\sin t\} = -\left(\dfrac{1}{s^2+1}\right)' = -\left(\dfrac{0-2s}{(s^2+1)^2}\right) = \dfrac{2s}{(s^2+1)^2}$

$* \; \mathcal{L}\{e^{-kt}t\sin t\} = \dfrac{2(s+k)}{[(s+k)^2+1]^2}$

5) $F(s) = \dfrac{2(s+2)}{\left[(s+2)^2+1\right]^2}$

$* \dfrac{2s}{(s^2+1)^2} = \mathcal{L}\{?\}$

<u>1st method</u>: $G(s) = \dfrac{2s}{(s^2+1)^2} = \dfrac{2}{s^2+1} \cdot \dfrac{s}{s^2+1}$

$= \mathcal{L}\{2\sin t\} \cdot \mathcal{L}\{\cos t\}$

$G(s) = \mathcal{L}\{2\sin t * \cos t\}$

$g(t) = 2\sin t * \cos t = \int^t 2\sin(t-u)\cos u \, du = \dots$

<u>2nd method</u>: $G(s) = \dfrac{2s}{(s^2+1)^2} \Rightarrow \int_s^{+\infty} G(s)\, ds = \int_s^{+\infty} \dfrac{2s}{(s^2+1)^2} ds$

$\Rightarrow \mathcal{L}\left\{\dfrac{g(t)}{t}\right\} = \dfrac{-1}{s^2+1}\Big|_s^{+\infty} \Rightarrow \mathcal{L}\left\{\dfrac{g(t)}{t}\right\} = 0 - \dfrac{-1}{s^2+1}$

$= \dfrac{1}{s^2+1} = \mathcal{L}\{\sin t\}$

$\Rightarrow \dfrac{g(t)}{t} = \sin t. \Rightarrow g(t) = t\sin t$

$\Rightarrow f(t) = e^{-2t} g(t) = e^{-2t} t \sin t$

(18) $F(s) = \operatorname{Ln}\left(\dfrac{s+a}{s+b}\right) = \operatorname{Ln}(s+a) - \operatorname{Ln}(s+b)$

$F'(s) = \dfrac{1}{s+a} - \dfrac{1}{s+b}$

$-F'(s) = \dfrac{1}{s+b} - \dfrac{1}{s+a}$

$\mathcal{L}\{t f(t)\} = \mathcal{L}\{e^{-bt}\} - \mathcal{L}\{e^{-at}\} = \mathcal{L}\{e^{-bt} - e^{-at}\}.$

$t f(t) = e^{-bt} - e^{-at} \Rightarrow \boxed{f(t) = \dfrac{e^{-bt} - e^{-at}}{t}}$

20) $F(s) = \cot^{-1}\frac{s}{w}$

$$F'(s) = \frac{-u'}{1+u^2} = \frac{-1/w}{1+\frac{s^2}{w^2}} \Rightarrow -F'(s) = \frac{w}{s^2+w^2}$$

$$\Rightarrow \mathcal{L}\{t f(t)\} = \mathcal{L}\{\sin wt\}$$

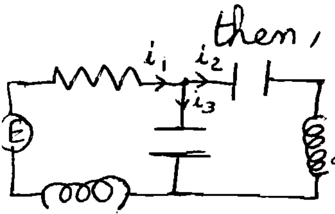
$$\Rightarrow t f(t) = \sin wt$$

$$\Rightarrow f(t) = \frac{\sin wt}{t}.$$

6.7 Systems of LDE

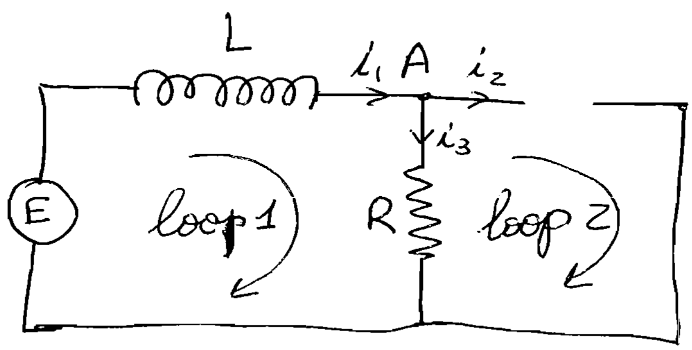
A) Electric Circuits

Kirchhoff's Law: If a circuit consist of more than 1 loop then,



1) The sum of all voltage drops around any closed loop is zero
2) The "algebraic Sum" of all currents at any junction is 0.

Example: Consider the following network where $R = 1\Omega$, $L = 2$ henrys, $C = 0.5$ farad, $E = 12$ volts. Assume all currents are initially 0.
a) Find the current i_2
b) Find i_1 and i_3



a) Loop 1: $E_L + E_R - E = 0 \Rightarrow L i_1' + R i_3 = E$

$$\boxed{2i_1' + i_3 = 12} \quad (1)$$

junction A : $\boxed{i_1 - i_2 - i_3 = 0}$ (3) $\Rightarrow$ $i_3 = i_1 - i_2$

$$\begin{cases} (1) \to 2i_1' + i_1 - i_2 = 12 & (4) \\ (2) \to 2i_2 - i_1' + i_2' = 0 & (5) \end{cases}$$

<u>Method of elimination</u> (Don't use it in all exams)

(4) $\Rightarrow i_2 = 2i_1' + i_1 - 12 \Rightarrow i_2' = 2i_1'' + i_1'$

Replace in (5) :

$$2(2i_1' + i_1 - 12) - i_1' + (2i_1'' + i_1') = 0$$

2^{nd}-order LDE

<u>2^{nd} Method</u>: (Use Laplace Transforms)

(4) $\to 2\mathcal{L}\{i_1'\} + \mathcal{L}\{i_1\} - \mathcal{L}\{i_2\} = \mathcal{L}\{12\}$.

(5) $\to 2\mathcal{L}\{i_2\} - \mathcal{L}\{i_1'\} + \mathcal{L}\{i_2'\} = 0$

$$\begin{cases} 2s I_1 + I_1 - I_2 = \frac{12}{s} \\ 2I_2 - sI_1 + sI_2 = 0 \end{cases} \Rightarrow \begin{cases} (2s+1)I_1 - I_2 = \frac{12}{s} \\ -sI_1 + (s+2)I_2 = 0 \end{cases}$$

Use Cramer's Rule to solve this linear system:

* $\Delta = \begin{vmatrix} 2s+1 & -1 \\ -s & s+2 \end{vmatrix} = (2s+1)(s+2) - s = 2s^2 + 4s + 2$

$$\Delta = 2(s+1)^2$$

* $\Delta_2 = \begin{vmatrix} 2s+1 & \frac{12}{s} \\ -s & 0 \end{vmatrix} = 0 + 12$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{12}{2(s+1)^2} = 6\frac{1}{(s+1)^2} = 6\mathcal{L}\{te^{-t}\}$$

$$\boxed{i_2 = 6te^{-t}}$$

b) $\Delta_1 = \begin{vmatrix} \frac{12}{s} & -1 \\ 0 & s+2 \end{vmatrix} = \frac{12(s+2)}{s}$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{6(s+2)}{s(s+1)^2} = \frac{12}{s} + \frac{b}{s+1} + \frac{-6}{(s+1)^2}$$

$* \; s = 1 \Rightarrow \frac{18}{4} = 12 + \frac{b}{2} - \frac{6}{4} \Rightarrow \frac{b}{2} = \frac{9}{2} - 12 + \frac{3}{2}$
$= -6$

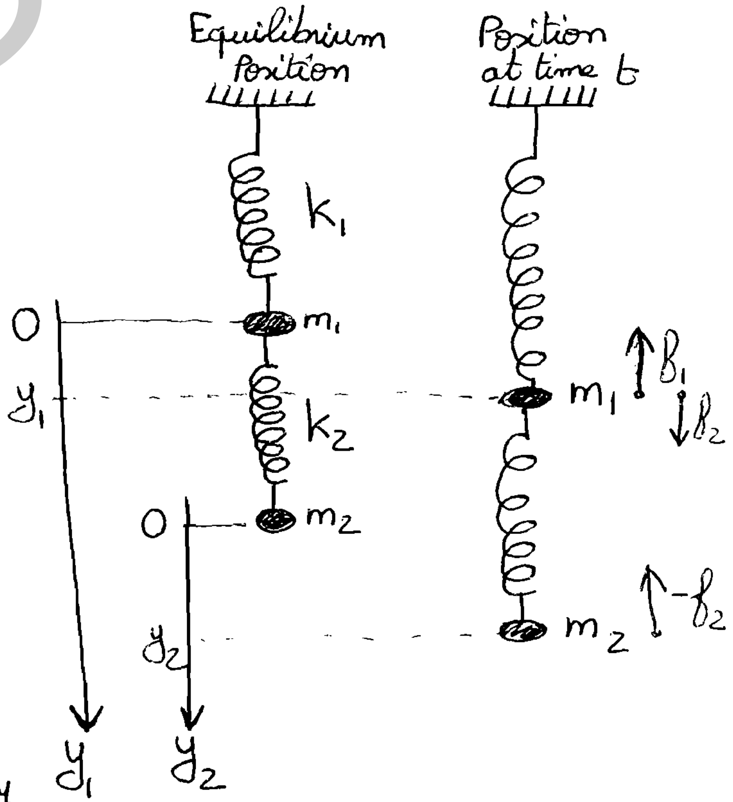
$\boxed{b = -12}$

$$I_1 = \mathcal{L}\{12\} - 12\,\mathcal{L}\{e^{-t}\} - 6\,\mathcal{L}\{te^{-t}\}$$

$$\boxed{i_1 = 12 - 12e^{-t} - 6te^{-t}}$$

$$i_1 - i_2 - i_3 = 0 \Rightarrow i_3 = i_1 - i_2 = 12 - 12e^{-t} - 12te^{-t}$$

B) Mechanical Systems



$m_1 = 2, \; m_2 = 1, \; k_1 = 4, \; k_2 = 2$

$y_1(0) = 3, \; y_2(0) = 0, \; y_1'(0) = 0, \; y_2'(0) = 0$

* Force from Spring 1 on m_1:

$f_1 = -k_1 \Delta L_1$ Where $\Delta L_1 = y_1$

because f_1 and ΔL_1 have opposite signs $\Rightarrow \boxed{f_1 = -k_1 y_1}$

* Force from Spring 2 on m_1:

$f_2 = +k_2 \Delta L_2$ Where $\Delta L_1 + \Delta L_2 = y_2$

because f_2 and ΔL_2 have the same sign

$\Delta L_2 = y_2 - y_1 \Rightarrow \boxed{f_2 = k_2(y_2 - y_1)}$

* Total force on m_1 : $F_1 = f_1 + f_2 = -k_1 y_1 + k_2(y_2 - y_1)$

By Newton's law: $F_1 = m_1 a(t) = m_1 y_1''$

$$\boxed{m_1 y_1'' = -k_1 y_1 + k_2(y_2 - y_1)}$$

$$2y_1'' = -4y_1 + 2(y_2 - y_1) \Rightarrow \boxed{y_1'' = -3y_1 + y_2}$$

* Total force on m_2: $F_2 = -f_2 = -k_2(y_2 - y_1)$

But $F_2 = m_2 y_2''$

$$\Rightarrow \quad m_2 y_2'' = -k_2(y_2 - y_1) \Rightarrow \boxed{y_2'' = -2(y_2 - y_1)}$$

$$* \begin{cases} \mathcal{L}\{y_1''\} = -3\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} \\ \mathcal{L}\{y_2''\} = -2\mathcal{L}\{y_2\} + 2\mathcal{L}\{y_1\}. \end{cases}$$

$$\begin{cases} s^2 Y_1 - 3s - 0 = -3Y_1 + Y_2 \\ s^2 Y_2 - 0 - 0 = -2Y_2 + 2Y_1 \end{cases}$$

$$\begin{cases} (s^2+3)Y_1 - Y_2 = 3s \\ -2Y_1 + (s^2+2)Y_2 = 0 \end{cases}$$

Cramer's Rule : $\Delta = \begin{vmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{vmatrix} = s^4 + 5s^2 + 6 - 2 = (s^2+1)(s^2+4)$

$$\Delta_1 = \begin{vmatrix} 3s & -1 \\ 0 & s^2+2 \end{vmatrix} = 3s(s^2+2)$$

$$\Delta_2 = \begin{vmatrix} s^2+3 & 3s \\ \end{vmatrix}$$

$$Y_1 = \frac{\Delta_1}{\Delta} = \frac{3s(s^2+2)}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+1} \text{ (very Complicated)}$$

If the powers of s are even, we let $S = s^2$

$$\frac{3(s^2+2)}{(s^2+1)(s^2+4)} = \frac{3(S+2)}{(S+1)(S+4)} = \frac{1}{S+1} + \frac{2}{S+4} = \frac{1}{s^2+1} + \frac{2}{s^2+4}$$

Multiply by s: $Y_1 = \frac{s}{s^2+1} + \frac{2s}{s^2+4} = \mathcal{L}\{\cos t\} + 2\mathcal{L}\{\cos 2t\}$.

$$\boxed{y_1 = \cos t + 2\cos 2t}$$

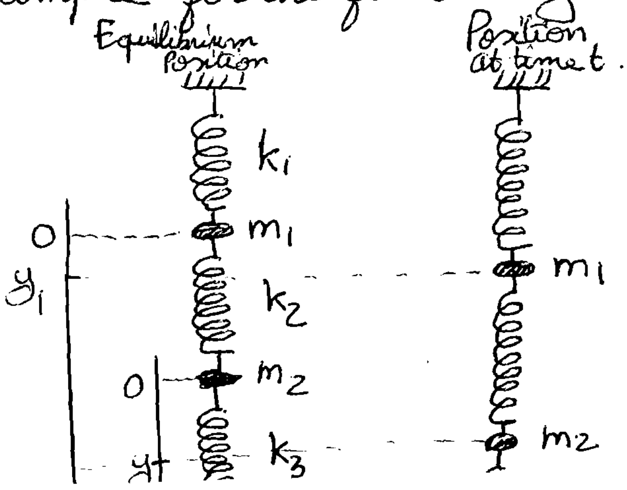
* $Y_2 = \frac{\Delta_2}{\Delta} = \frac{6s}{(s^2+1)(s^2+4)}$

$$\frac{6}{(s^2+1)(s^2+4)} = \frac{6}{(S+1)(S+4)} = \frac{2}{S+1} + \frac{-2}{S+4} = \frac{2}{s^2+1} - \frac{2}{s^2+4}$$

Multiply by s: $Y_2 = \frac{2s}{s^2+1} - \frac{2s}{s^2+4} = 2\mathcal{L}\{\cos t\} - 2\mathcal{L}\{\cos 2t\}$.

$$\boxed{y_2 = 2\cos t - 2\cos 2t}$$

HW Repeat the previous example for the following mechanical system where $K_3 = 2$.



Prob 6.7 (1 – 17) + 25 HW(3, 5, 25)

4) $\begin{cases} y_1' + y_2 = 0 & y_1(0) = 1 \\ y_1 + y_2' = 2\cos t & y_2(0) = 0 . \end{cases}$

$$\begin{cases} sY_1 - 1 + Y_2 = 0 \\ Y_1 + sY_2 - 0 = 2\dfrac{s}{s^2+1} \end{cases} \Rightarrow \begin{cases} sY_1 + Y_2 = 1 \\ Y_1 + sY_2 = \dfrac{2s}{s^2+1} \end{cases}$$

* $\Delta = \begin{vmatrix} s & 1 \\ 1 & s \end{vmatrix} = s^2 - 1$

* $\Delta_1 = \begin{vmatrix} 1 & 1 \\ \frac{2s}{s^2+1} & s \end{vmatrix} = s - \dfrac{2s}{s^2+1} = \dfrac{s^3 + s - 2s}{s^2+1} = \dfrac{s(s^2-1)}{s^2+1}$

$$Y_1 = \frac{\Delta_1}{\Delta} = \frac{s}{s^2+1} = \mathcal{L}\{\cos t\} \Rightarrow \boxed{y_1 = \cos t.}$$

* $\Delta_2 = \begin{vmatrix} s & 1 \\ 1 & \frac{2s}{s^2+1} \end{vmatrix} = \dfrac{2s^2}{s^2+1} - 1 = \dfrac{2s^2 - s^2 - 1}{s^2+1} = \dfrac{s^2-1}{s^2+1}$

$$Y_2 = \frac{\Delta_2}{\Delta} = \frac{1}{s^2+1} = \mathcal{L}\{\sin t\} \Rightarrow \boxed{y_2 = \sin t.}$$

Chapter 4 Systems of DE

4.0 Matrices - Eigenvalues

Part A:

Matrix: A 2×2 - matrix is a rectangular array of numbers

$\underset{\text{rows}}{2} \times \underset{\text{Columns}}{2}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$b = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \text{ is a } 2\times 1\text{-matrix}$$

* The identity 2×2 - matrix is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Matrix operations: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$, $v = \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1\text{-matrix}}$

1) Transpose: $A^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B^t = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, $v = \begin{pmatrix} x & y \end{pmatrix}_{1\times 2\text{-matrix}}$

2) Sum: $A + B = \begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix}$

$A + v$ is not defined bec. they don't have the same dimensions.

3) Multiplication by scalars: $2A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$, $-4B = \begin{pmatrix} -12 & -16 \\ 8 & 4 \end{pmatrix}$, $5v = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$

4) Product of matrices:

$$A.B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 3-4 & 4-2 \\ 9-8 & 12-4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 8 \end{pmatrix}$$

$2 \times 2 \qquad 2 \times 2 \qquad\qquad 2\times 2$

$$B.A = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 15 & 22 \\ -5 & -8 \end{pmatrix}$$

Notice that $AB \neq BA$.

$$A.v = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

5) Inverse: A square 2×2-matrix A is said to be invertible if $\exists$ a 2×2 matrix, denoted by A^{-1}, such that $\boxed{AA^{-1}=I}$

Formula: Let $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$. If $\det A = a_{11}a_{22}-a_{12}a_{21}\neq 0$

then $\boxed{A^{-1}=\dfrac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{pmatrix}}$

Example: Let $A=\begin{pmatrix} 2 & -3 \\ -4 & 6\end{pmatrix}$

$\det A = \begin{vmatrix} 2 & -3 \\ -4 & 6\end{vmatrix} = 12-12=0 \Rightarrow A$ is not invertible

Example: $A=\begin{pmatrix} 2 & 3 \\ 4 & 5\end{pmatrix}$

$\det A = 10-12 = -2 \neq 0 \Rightarrow A$ is invertible.

$$A^{-1}=\frac{1}{-2}\begin{pmatrix} 5 & -3 \\ -4 & 2\end{pmatrix}=\begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1\end{pmatrix}$$

* verfication $AA^{-1}=\begin{pmatrix} 2 & 3 \\ 4 & 5\end{pmatrix}\begin{pmatrix} -5/2 & 3/2 \\ 2 & -1\end{pmatrix}=\begin{pmatrix} -5+6 & 3-3 \\ -10+10 & 6-5\end{pmatrix}$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}=I.$$

Part B: Eigenvalues

Definition: A real (or complex) number λ is called an eigenvalue of A if $\exists$ a nonzero vector v such that $\boxed{\underset{2\times2}{A}\,\underset{2\times1}{v}=\lambda \underset{2\times1}{v}}$

* v is called an eigenvector corresponding to λ.

Theorem: The eigenvalues of A are the roots of the equation:

$\boxed{\det(A-\lambda I)=0}$ called characteristic equation

Example: Let $A=\begin{pmatrix} -3 & 1 \\ 1 & -3\end{pmatrix}$; Find the eigenvalues and eigenvectors of A.

* Eigenvalues: $\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda\end{vmatrix} = (-3-\lambda)^2-1$

* Eigenvectors: $v = ?$ / $Av = \lambda v \Rightarrow (A - \lambda I)\, v = 0$

$\lambda_1 = -2$: $(A + 2I)\, v = 0 \Rightarrow \begin{pmatrix} -3+2 & 1 \\ 1 & -3+2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{cases} -x + y = 0 \\ x - y = 0. \end{cases} \Rightarrow y = x$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We need 1 eigenvector: Take $x = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -4$: $(A + 4I)\, v = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{cases} x + y = 0 \\ x + y = 0 \end{cases} \Rightarrow y = -x$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$x = 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is another eigenvector.

4.2 Basic Theory

Definition: A system of DE is said to be linear if it is of the form:

$$\begin{cases} y_1' = a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n + g_1(t). \\ y_2' = a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n + g_2(t). \\ \quad \vdots \\ y_n' = a_{n1} y_1 + a_{n2} y_2 + \cdots + a_{nn} y_n + g_n(t). \end{cases}$$

In matrix form: $\boxed{Y' = AY + G}$ (1)

Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

* If $G = 0$ then $\boxed{Y' = AY}$ (2) is called a Homogeneous system.

then, the general solution to (1) is $\boxed{Y = Y_h + Y_p}$

Definition: Let $Y_1, Y_2, \ldots, Y_n$ be n solutions to (2).
Their Wronskian is the determinant whose i^{th} column is Y_i

$$W = \begin{vmatrix} Y_1 & Y_2 & \cdots & Y_n \\ | & | & & | \\ | & | & & | \end{vmatrix}$$

Theorem: The solutions are linearly independent iff $W \neq 0$

Theorem: If $Y_1, Y_2, \ldots, Y_n$ are n linearly indep. solution to (2)
then the general solution is:

$$\boxed{Y_h = c_1 Y_1 + c_2 Y_2 + \ldots + c_n Y_n}.$$

4.3 Phase Portrait

A) Homogeneous Systems

Theorem: Consider the system $Y' = AY$. If v_1 is an eigenvector corresponding to an eigenvalue λ_1 of A then $\boxed{Y_1 = e^{\lambda_1 t} v_1}$ is a solution to the homogeneous system.

Proof: $*\ Y_1' = \lambda_1 e^{\lambda_1 t} v_1$

$*\ AY_1 = A e^{\lambda_1 t} v_1 = e^{\lambda_1 t}(A v_1) = e^{\lambda_1 t}(\lambda_1 v_1)$

$\therefore Y_1' = A Y_1$

Theorem: If A has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ then the general solution to $Y' = AY$ is:

$$Y = c_1 Y_1 + c_2 Y_2 + \ldots + c_n Y_n$$

B] Phase Portrait

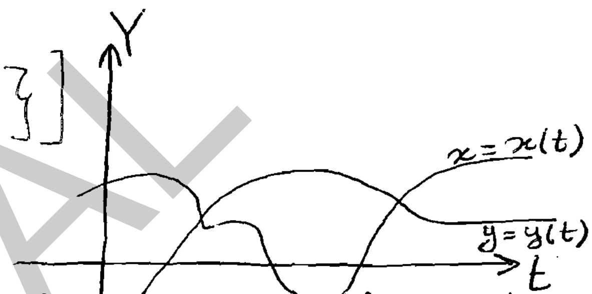
Consider the homogeneous system : $\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$.

$$Y' = AY \text{ Where } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Let $Y = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ be a solution to this system.

* We can plot the solution as 2 curves in the tY-plane.

Plot[{x[t], y[t]}, {t, -4, 20}]



* Another method to represent the solution graphically is to plot $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ as 1 curve in the xy-plane.

ParametricPlot[{x[t], y[t]}, {t, -4, 20}]

This curve is called an <u>orbit</u> (or path) (or trajectory)

The set of all orbits is called the Phase Portrait of the system

4.3 + 4.4 Critical Points and Stability

<u>Definition</u>: The slope of an orbit $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ at a point P is:

$$m = y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}$$

If $x' = y' = 0$ at P then $m = \frac{0}{0}$ indeterminate.

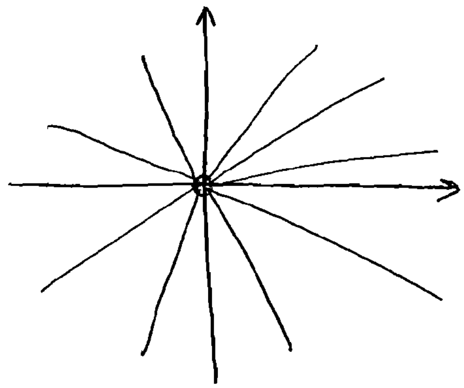
P is called a Critical Point. $(0,0)$ is a critical point for $\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$

There are 6 types of Critical Points.

$\longrightarrow$

Type 1 : Proper Node

<u>Definition</u>: The critical point O is called a proper node if the orbits are semi-straight lines passing through O.



<u>Example</u> $\begin{cases} x' = 2x \\ y' = 2y \end{cases} \Longleftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

* Eigenvalues: $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$

$$\Rightarrow \boxed{\lambda_1 = \lambda_2 = 2}$$

* Eigenvectors: $v \neq 0 \,/\, (A - \lambda I) v = 0$

$$(A - 2I) v = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{cases} 0x + 0y = 0 \\ 0x + 0y = 0 \end{cases}$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{v_1} + y \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{v_2}$$

$Y_1 = e^{\lambda_1 t} v_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y_2 = e^{\lambda_2 t} v_2 = e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

are lin. indep. solutions.

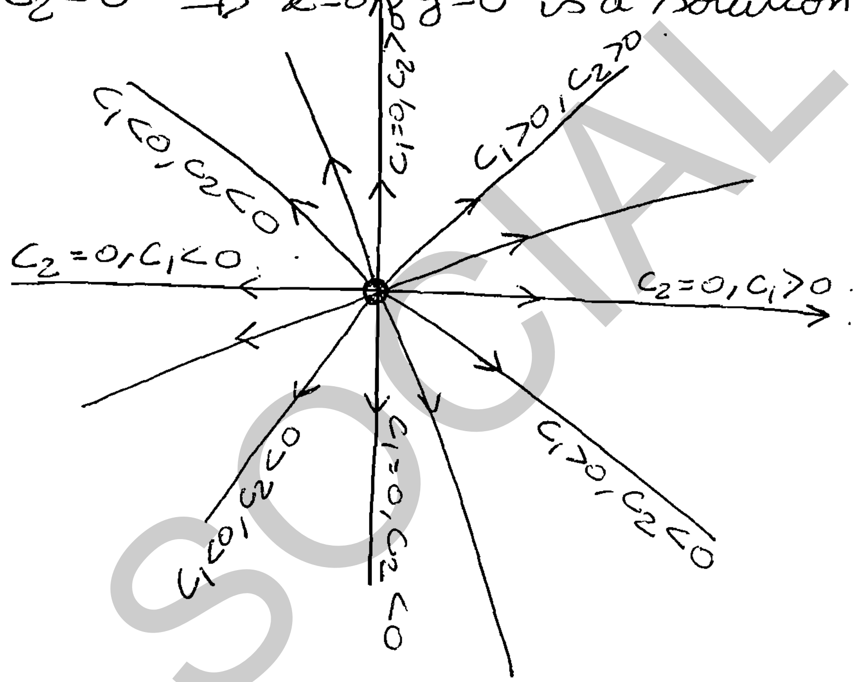
General Solution: $Y = C_1 Y_1 + C_2 Y_2$

$$\boxed{Y = c_1 e^{2t} v_1 + c_2 e^{2t} v_2}$$ vector form.

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} x = c_1 e^{2t} \\ y = c_2 e^{2t} \end{cases} \Rightarrow c_2 x = c_1 y \text{ (line)}$$

* $c_1 > 0$, $c_2 > 0$ $\Rightarrow$ $x, y > 0$ (semi-line in the 1st quadrant)
* $c_1 > 0$, $c_2 < 0$ $\Rightarrow$ $x > 0$, $y < 0$ (semi-line in the 4th quadrant)
* $c_1 < 0$, $c_2 > 0$ $\Rightarrow$ $x < 0$, $y > 0$ (semi-line in the 2nd quadrant)
* $c_1 < 0$, $c_2 < 0$ $\Rightarrow$ $x < 0$, $y < 0$ (semi-line in the 3rd quadrant)
* $c_1 = 0$, $c_2 = 0$ $\Rightarrow$ $x = 0$ & $y = 0$ is a solution (1 point).



<u>Stability</u>: $\lim_{t \to -\infty} x = 0$, $\lim_{t \to -\infty} y = 0$ $\Rightarrow$ The orbits are directed away from O.

We say that O is unstable.

* If $\lim x = 0$, $\lim y = 0$ as $t \to +\infty$.
The orbits will be directed toward O.
O is called a stable node.

<u>Rule 1</u>: If A has a double eigenvalue $\lambda_1 = \lambda_2$ and 2 linearly independent eigenvectors v_1, v_2 then O is a proper node.

* If $\lambda_1 > 0$ then O is unstable

Type 2 Improper Node

Definition: The critical point O is called an improper node if all orbits (except 2 of them) have the same slope at O and the remaining 2 orbits have a different slope.

Example: $\begin{cases} x' = -3x + y \\ y' = x - 3y \end{cases} \iff Y' = AY$ where $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$

* Eigenvalues : $\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0.$

  $\lambda_1 = -2$, $\lambda_2 = -4$ (see section 4.0)

* Eigenvectors : $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

* General Solution : $Y = c_1 Y_1 + c_2 Y_2$

$$Y = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

$$\boxed{Y = c_1 e^{-2t} v_1 + c_2 e^{-4t} v_2}$$ vector form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Test #2 $\begin{cases} x = c_1 e^{-2t} + c_2 e^{-4t} \\ y = c_1 e^{-2t} - c_2 e^{-4t} \end{cases}$ parametric form.

Stability: $\lim x = 0$ and $\lim y = 0$ as $t \to +\infty$

The orbits are directed toward O $\Rightarrow$ O is stable

* If $c_1 = 0$ then $Y = c_2 e^{-4t} v_2$ (2 semi-lines // v_2)
* If $c_2 = 0$ then $Y = c_1 e^{-2t} v_1$ (2 semi-lines // v_1)
* If $c_1 \neq 0$, $c_2 \neq 0$:

  Slope at O : $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'} = \frac{0}{\,}$

$$m = \lim_{t\to+\infty} \frac{-2c_1+4c_2e^{-2t}}{-2c_1-4c_2e^{-2t}} = \frac{-2c_1}{-2c_1} = 1 \text{ same slope as } v_1.$$

b) <u>Asymptotes</u> $x, y \to \pm\infty$ as $t \to -\infty$

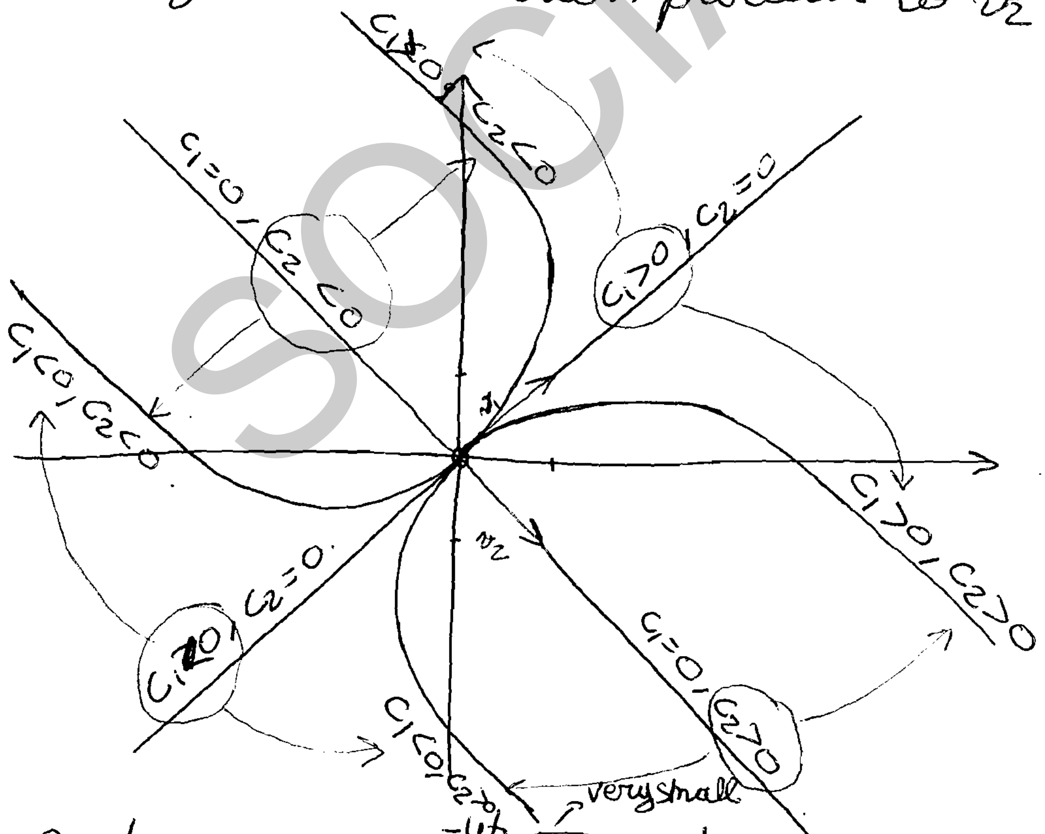
If $y = ax+b$ is the asymptote then $\begin{cases} a = \lim\limits_{t\to-\infty} \frac{y}{x} \\ b = \lim\limits_{t\to-\infty} y-ax. \end{cases}$

$$* \; a = \lim \frac{y}{x} = \lim_{t\to-\infty} \frac{c_1e^{-2t}-c_2e^{-4t}}{c_1e^{-2t}+c_2e^{-4t}} = \lim \frac{c_1e^{2t}-c_2}{c_1e^{2t}+c_2} = \frac{-c_2}{c_2} = -1$$

same slope as v_2

$$* \; b = \lim y-ax = \lim y+x = \lim_{t\to-\infty} 2c_1e^{-2t} = \pm\infty.$$

$\therefore$ The orbit have no asymptotes but they have an asymbtotic direction parallel to v_2.



$*$ near O, $t\to+\infty \Rightarrow e^{-4t} << e^{-2t}$

$$\Rightarrow x \simeq c_1e^{-2t} \text{ and } y \simeq c_1e^{-2t}.$$

If $c_1>0$ the orbit is in the 1[st] quadrant

If $c_1<0$ ~~th~~ $\Rightarrow x, y<0 \Rightarrow$ 3[rd] quadrant.

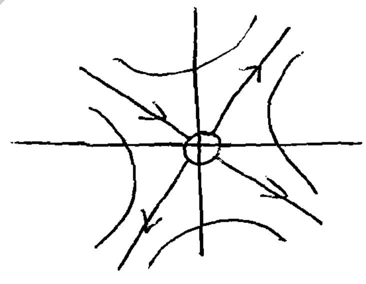
Rule 2 If A has 2 distinct real eigenvalues having the same sign then O an improper node.

* If $\lambda_1, \lambda_2 < 0$ then O is stable.
* If $\lambda_1, \lambda_2 > 0$ then O is unstable.

Type 3 Saddle Point

Definition: The critical point O is called a Saddle point if 2 orbits are directed toward O and 2 orbits are directed away from O and the remaining orbits don't pass through O.

Example: $\begin{cases} x' = x + 4y \\ y' = x + y \end{cases} \Rightarrow A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$



* Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix}$

$$= (1-\lambda)^2 - 4 = (1-\lambda-2)(1-\lambda+2)$$

$$= (-\lambda-1)(3-\lambda) = 0 \Rightarrow \boxed{\lambda_1 = -1,\ \lambda_2 = 3}$$

* Eigenvector: $v = ?\ /\ (A - \lambda I)v = 0$.

* $\lambda_1 = -1 \Rightarrow (A + I)v = 0 \Rightarrow \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$\Rightarrow \begin{cases} 2x + 4y = 0 \\ x + 2y = 0 \end{cases} \Rightarrow x = -2y.$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2y \\ y \end{pmatrix} = y\underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{v_1}. \quad v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

* $\lambda_2 = 3 \Rightarrow (A - 3I)v = 0 \Rightarrow \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$\Rightarrow \begin{cases} -2x + 4y = 0 \\ x - 2y \end{cases} \Rightarrow x = 2y.$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

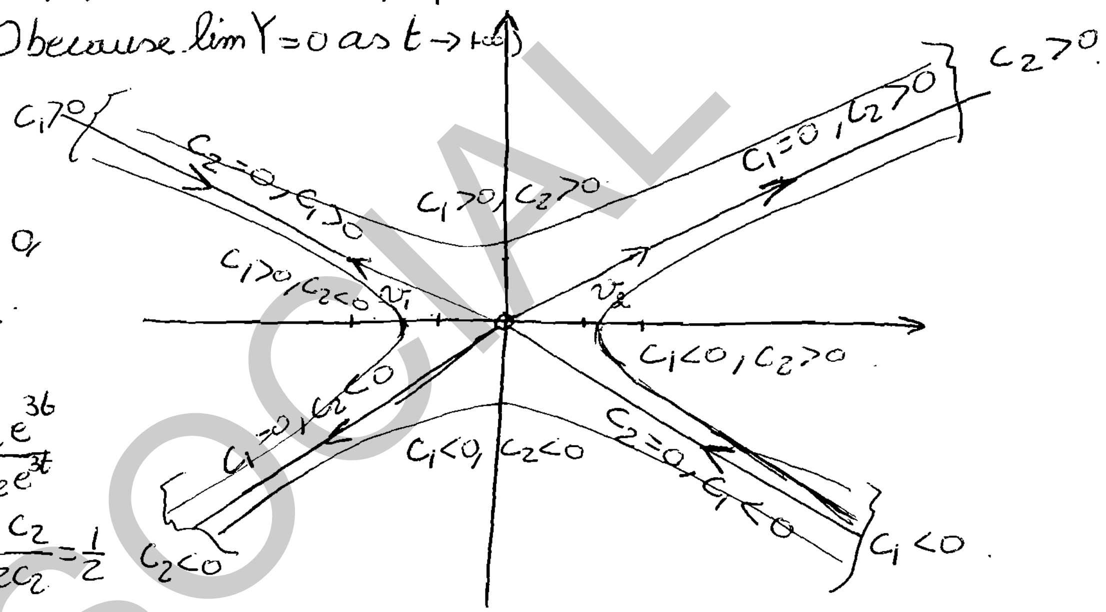
Gen. Sol: $Y = c_1 Y_1 + c_2 Y_2 \Rightarrow \boxed{Y = c_1 e^{-t} v_1 + c_2 e^{3t} v_2}$ vector form.

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} x = -2c_1 e^{-t} + 2c_2 e^{3t} \\ y = c_1 e^{-t} + c_2 e^{3t} \end{cases}$$

parametric equations

Orbits * $c_1 = 0 \Rightarrow Y = c_2 e^{3t} v_2$ (2 orbits // v_2 directed away from O because $\lim Y = 0$ as $t \to -\infty$)

* $c_2 = 0 \Rightarrow Y = c_1 e^{-t} v_1$ (2 semi lines // v_1 directed toward O because $\lim Y = 0$ as $t \to +\infty$)



* $c_1 \neq 0, c_2 \neq 0$

If $t \to +\infty$ then $\lim e^{-t} = 0$, $\lim e^{3t} = \infty \Rightarrow Y \simeq c_2 e^{3t} v_2$.

slope of the asymptote:

$$a = \lim \frac{y}{x} = \lim \frac{c_1 e^{-t} + c_2 e^{3t}}{-2c_1 e^{-t} + 2c_2 e^{3t}}$$

$$= \lim \frac{c_1 e^{-4t} + c_2}{-2c_1 e^{-4t} + 2c_2} = \frac{c_2}{2c_2} = \frac{1}{2}$$

Same slope as v_2

$$b = \lim(y - ax) = \lim\left(y - \tfrac{1}{2}x\right) = \lim 2c_1 e^{-t} = 0$$

$\Rightarrow y = \frac{1}{2}x$ is an asymptote // v_2

If $t \to -\infty$: $\lim e^{3t} = 0$ and $\lim e^{-t} = \infty \Rightarrow Y \simeq c_1 e^{-t} v_1$.

* $a = \lim \frac{y}{x} = \lim \frac{c_1 + c_2 e^{4t}}{-2c_1 + 2c_2 e^{4t}} = \frac{c_1}{-2c_1} = \frac{-1}{2}$ (same slope as v_1)

* $b = \lim(y - ax) = \lim\left(y + \tfrac{1}{2}x\right) = \lim\left(2c_2 e^{3t}\right) = 0$.

$\Rightarrow$ ~~b = lim~~ $y = -\frac{1}{2}x$ is an asymptote // v_1.

<u>Rule 3</u> If A has 2 distinct real eigenvalues having opposite signs then O is a Saddle.

In this case: 2 orbits are directed toward O

2 orbits are directed away from O.

Type 4 : Degenerate Node

Definition : The critical point O is called a Degenerate node if A has a double eigenvalue $\lambda_1 = \lambda_2$ and 1 eigenvector v_1.

Remark: $Y_1 = e^{\lambda_1 t} v_1$ is a solution to the system $Y' = AY$

Try to find a 2^{nd} solution of the form $Y_2 = t\,Y_1$

$* \; Y_2 = t\,Y_1 \Rightarrow Y_2' = Y_1 + t\,Y_1'$

Replace in the system $Y_2' = AY_2 \Rightarrow Y_1 + tY_1' = A(tY_1)$

$\Rightarrow Y_1 + t(AY_1) = At\,Y_1 \Rightarrow Y_1 = 0$ contradiction.

We can't find a 2^{nd} solution of the form $Y_2 = t\,Y_1 = te^{\lambda_1 t} v_1$

2^{nd} Solution: Try to find a 2^{nd} Solution of the form

$$Y_2 = e^{\lambda_1 t}(tv_1 + v_2)$$

$$Y_2' = \lambda_1 e^{\lambda_1 t}(tv_1 + v_2) + e^{\lambda_1 t} v_1$$

$$* \; Y_2' = AY_2 \Rightarrow \lambda_1 e^{\lambda_1 t}(tv_1 + v_2) + e^{\lambda_1 t} v_1 = Ae^{\lambda_1 t}(tv_1 + v_2)$$

$$\lambda_1 t v_1 + \lambda_1 v_2 + v_1 = A(tv_1) + A.v_2$$

$$t\lambda_1 v_1 + \lambda_1 v_2 + v_1 = t\,\lambda_1 v_1 + Av_2 \; (\text{bec } Av_1 = \lambda_1 v_1)$$

$$Av_2 - \lambda_1 v_2 = v_1 \Rightarrow Av_2 - \lambda_1 I v_2 = v_1$$

$$\Rightarrow (A - \lambda_1 I)v_2 = v_1 .$$

Theorem: The second solution is $Y_2 = e^{\lambda_1 t}(tv_1 + v_2)$

Where v_2 is a solution to the system $(A-\lambda_1 I)v_2$

$$(A - \lambda_1 I)v_2 = v_1$$

$*$ v_2 is not an eigenvector.

Example: $\begin{cases} x' = x - y \\ y' = x + 3y \end{cases} \Leftrightarrow Y' = AY$ Where $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

$*$ Eigenvalue : $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1$

*Eigenvector: $v = ? \,/\, (A-2I)v = 0 \Rightarrow \begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0$.

$$\Rightarrow \begin{cases} -x-y=0 \\ x+y=0 \end{cases} \Rightarrow y=-x$$

$$v = \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ -x\end{pmatrix} = x\underbrace{\begin{pmatrix}1\\ -1\end{pmatrix}}_{v_1}$$

$v_1 = \begin{pmatrix}1\\ -1\end{pmatrix}$ is the only eigenvector

$\therefore$ O is a degenerate node

$Y_1 = e^{2t}v_1$ is a solution.

<u>2^{nd} Solution:</u>

*Find $v_2 \,/\, (A-\lambda_1 I)v_2 = v_1 \Rightarrow (A-2I)v_2 = v_1$.

$$\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix} \Rightarrow \begin{cases} -x-y=1 \\ x+y=-1 \end{cases} \Rightarrow x=-y-1$$

$$v = \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-y-1\\ y\end{pmatrix}$$

We need one v_2: Take $y=0 \Rightarrow v_2 = \begin{pmatrix}-1\\ 0\end{pmatrix}$

* Find $Y_2 = e^{2t}(t v_1 + v_2)$

<u>General Solution:</u> $Y = c_1 Y_1 + c_2 Y_2$

$$\boxed{Y = c_1 e^{2t} v_1 + c_2 e^{2t}(t v_1 + v_2)}$$ solution in vector form.

$$\begin{pmatrix}x\\ y\end{pmatrix} = c_1 e^{2t}\begin{pmatrix}1\\ -1\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}t-1\\ -t\end{pmatrix}$$

$$\begin{cases} x = c_1 e^{2t} + c_2 e^{2t}(t-1) \\ y = -c_1 e^{2t} - c_2 t e^{2t} \end{cases}$$ solution in parametric form.

<u>Stability</u>: If $t \to -\infty$ then $\lim e^{2t} = 0$ and $\lim t e^{2t} = 0$.

$\Rightarrow \lim Y = 0$.

$\Rightarrow$ the orbits are directed away from O.

$\Rightarrow$ O is unstable.

Orbits:

* If $c_2 = 0$ then $Y = c_1 e^{2t} v_1$ (2 orbits $// v_1$)
* If $c_2 \neq 0$ then:

a) slope at O: $m = \lim\limits_{t \to -\infty} \dfrac{y'}{x'} = \lim \dfrac{-2c_1 e^{2t} - c_2 e^{2t} - 2c_2 t e^{2t}}{2c_1 e^{2t} + 2c_2 e^{2t}(t-1) + c_2 e^{2t}}$

because $\lim\limits_{t \to -\infty} Y = 0$

$$m = \lim \frac{-2c_1 - c_2 - 2c_2 t}{2c_1 + 2c_2(t-1) + c_2} = \frac{\infty}{\infty} \overset{HR}{=} \lim \frac{-2c_2}{2c_2} = -1$$

same slope as v_1

b) Asymptote: $y = ax + b$

$$a = \lim_{t \to +\infty} \frac{y}{x} = \lim \frac{-c_1 - c_2 t}{c_1 + c_2(t-1)} = \frac{\infty}{\infty} \overset{HR}{=} \frac{-c_2}{c_2} = -1$$

same slope as v_1

$$b = \lim(y - ax) = \lim(y + x) = \lim_{t \to +\infty} -c_2 e^{2t} = \pm\infty.$$

The orbit has an asymptotic direction $// v_1$

c) Points of intersection with coordinate axes:

x-axis: $y = 0 \Rightarrow -c_1 - c_2 t = 0 \Rightarrow t = -\dfrac{c_1}{c_2}$.

y-axis: $x = 0 \Rightarrow c_1 + c_2(t-1) = 0 \Rightarrow t - 1 = -\dfrac{c_1}{c_2}$

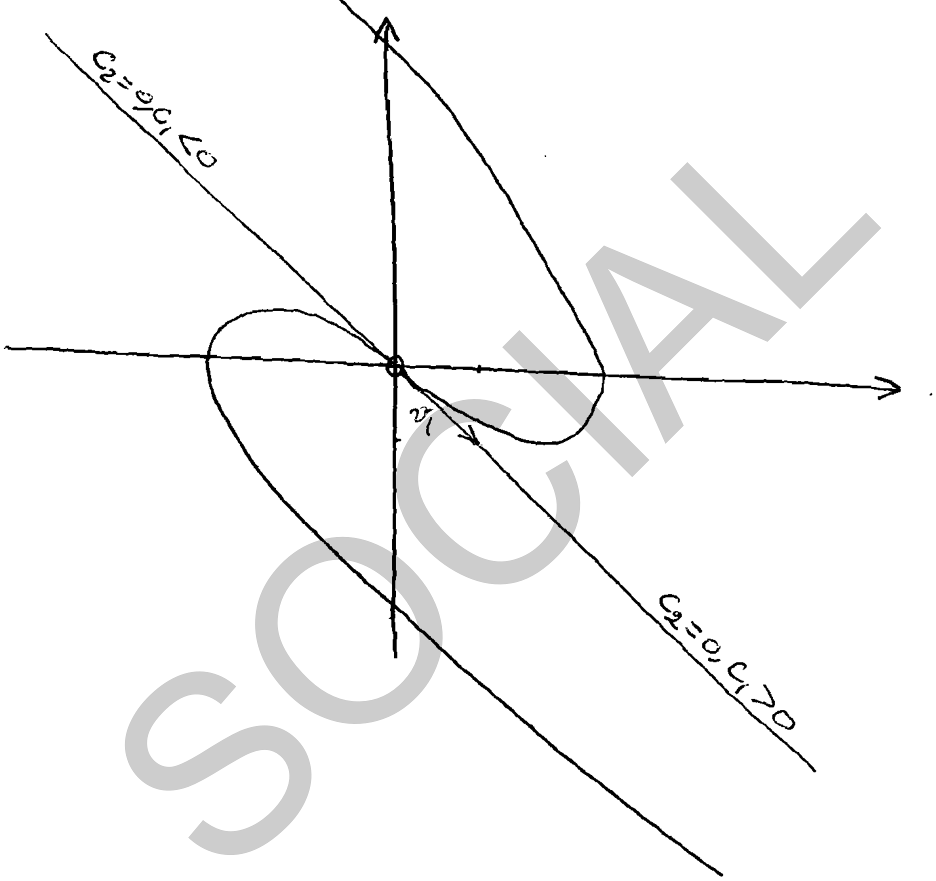
$\Rightarrow t = \dfrac{-c_1}{c_2} + 1$

$c_2 > 0$

t	$-\infty$	t_1	t_2	$+\infty$
x	0^-		0	$+\infty$
y	0^+	0		$-\infty$
	2nd quadrant	$\cap$ with x-axis	$\cap$ with y-axis	4th quadrant

$c_2 < 0$

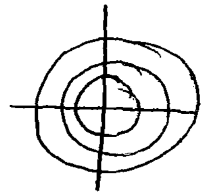
t	$-\infty$	t_1	t_2	$+\infty$
x	0^+		0	$-\infty$
y	0^-	0		$+\infty$
	4th quadrant	$\cap$ with x-axis	$\cap$ with y-axis	2nd quadrant.



<u>Rule 4</u>: If A has a double eigenvalue $\lambda_1=\lambda_2$ and 1 eigenvector v_1, then O is a degenerate node.
TWO orbits are // to v_1 and the remaining orbits are tangent to v_1 at O and have an asymptotic direction // v_1.

Type 5 Center

Definition: The critical point O is called a Center if the orbits are closed curves around O.



Example: $\begin{cases} x' = -2x + 8y \\ y' = -x + 2y \end{cases} \Leftrightarrow Y' = AY$

Where $A = \begin{pmatrix} -2 & 8 \\ -1 & 2 \end{pmatrix}$

Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 8 \\ -1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 8$

$= \lambda^2 - 4 + 8 = \lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4 = 4i^2$

$\Rightarrow \begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \pm 2i.$

If v_1 and v_2 are 2 eigenvectors then the solution is:

$$Y = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

$$Y = c_1 e^{2it} v_1 + c_2 e^{-2it} v_2$$

The solution is in complex form.

The real solution can be obtained as follows:

$$\begin{matrix} \dfrac{dx}{dt} = -2x + 8y \\ \dfrac{dy}{dt} = -x + 2y \end{matrix} \Rightarrow \begin{cases} dt = \dfrac{dx}{-2x+8y} \\ dt = \dfrac{dy}{-x+2y} \end{cases}$$

$$\frac{dx}{-2x+8y} = \frac{dy}{-x+2y} \Rightarrow (-x+2y)\,dx = (-2x+8y)\,dy$$

$$\underbrace{(-x+2y)}_{P}\,dx + \underbrace{(2x-8y)}_{Q}\,dy = 0.$$

$P_y = 2 \Big\} \Rightarrow P_y = Q_x \Rightarrow$ W is exact

* $f = ?\ /\ df = \omega$.

$f = ?\ /\ f_x = P,\ f_y = Q$

$f_x = P = -x + 2y \Rightarrow f = -\frac{x^2}{2} + 2xy + g(y)$

$$\left.\begin{array}{l} \Rightarrow f_y = 2x + g'(y) \\ \text{but } f_y = Q = 2x - 8y \end{array}\right\} \Rightarrow g'(y) = -8y \Rightarrow g(y) = -4y^2 + \cancel{C}^{\,0}.$$

$$f = \frac{-x^2}{2} + 2xy - 4y^2$$

<u>Solution:</u> $f = c \Rightarrow \boxed{\frac{-x^2}{2} + 2xy - 4y^2 = c}$

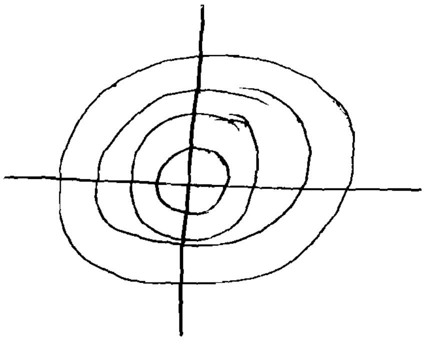
$ax^2 + bxy + cy^2 + dx + ey = c$

$\Delta = b^2 - 4ac = 4 - 4\left(\frac{-1}{2}\right)(-4) = 4 - 8 = -4 < 0$

$\Rightarrow$ The orbits are ellipses of center O

M $f[x\_, y\_] = \frac{-x^2}{2} + 2xy - 4y^2;$

```
ContourPlot[f[x,y], {x,-10,10}, {y,-10,10},
 ContourShading -> False, Contours -> {-1,-5,-10,
 -15,-20}, PlotPoints -> 50, AspectRatio ->
 Automatic];
```

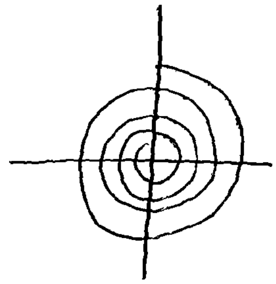


<u>Rule 5</u>: If the eigenvalues of A are of the form $\lambda_1, \lambda_2 = \pm i\beta$ then O is a center.

Type 6 : Spiral Points

Definition: The critical point O is called a Spiral point if the orbits spiral around O.

Example $\begin{cases} x' = -x + y \\ y' = -x - y \end{cases} \Rightarrow A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$



Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 + 1$

$$\left.\begin{array}{l} = \lambda^2 + 2\lambda + 2 = 0 \\ \Delta' = 1 - 2 = -1 = i^2 \end{array}\right\} \Rightarrow \begin{array}{l}\lambda_1 \\ \lambda_2\end{array} = -1 \pm i = \alpha \pm i\beta$$

The solution is of the form $Y = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ (in complex form)

If A is of the form $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ then the system can be solved using polar coordinates as follows:

$$x = r\cos\theta \Rightarrow x' = r'\cos\theta - r\sin\theta\,\theta'$$

$$y = r\sin\theta \Rightarrow y' = r'\sin\theta + r\cos\theta\,\theta'$$

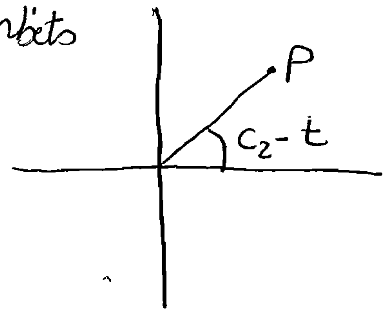
$$\begin{cases} x' = -x + y \\ y' = -x - y \end{cases} \Rightarrow \begin{cases} x' = r'\cos\theta - r\sin\theta\,\theta' = -r\cos\theta + r\sin\theta. \\ y' = r'\sin\theta + r\cos\theta\,\theta' = -r\cos\theta - r\sin\theta. \end{cases}$$

$$\begin{cases} (r'+r)\cos\theta - r\sin\theta\,(\theta'+1) = 0. \\ (r'+r)\sin\theta \underset{+}{-} r\cos\theta\,(\theta'+1) = 0 \end{cases} \Rightarrow \begin{cases} r' + r = 0 \\ \theta' + 1 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} r' = -r \\ \theta' = -1 \end{cases} \Rightarrow \begin{cases} r = c_1 e^{-t} \\ \theta = -t + c_2 \end{cases}$$

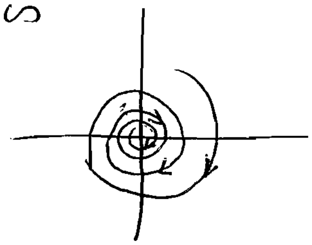
Solution: $\begin{cases} x = r\cos\theta \\ \ldots \end{cases} \Rightarrow \begin{cases} x = c_1 e^{-t}\cos(-t + c_2) \\ y = c_1 e^{-t}\sin(-t + c_2) \end{cases}$

* If t increases then $c_2 - t$ is decreasing. The orbits are traversed in the clockwise direction.



* If $t \to +\infty$ then $r = c_1 e^{-t} \to 0$ as we rotate in the clockwise direction the point P is approching O.

$\Rightarrow$ O is stable.



<u>Rule 6</u>: If A has 2 complex eigenvalues: $\lambda_1, \lambda_2 = \alpha \pm i\beta$ where $\alpha, \beta \neq 0$ then O is a spiral point.

* O is stable if $\alpha < 0$
* O is unstable if $\alpha > 0$.

<u>Prob 4.3</u> (1-8) HW (5, 6)

a) Solve the system
b) Type and Stability
c) Draw some of the orbits.

② $\begin{cases} x' = 5y \\ y' = 5x \end{cases} \Leftrightarrow Y' = AY$ where $A = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$

a) Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 5 \\ 5 & -\lambda \end{vmatrix} = \lambda^2 - 25 = 0 \Rightarrow \lambda^2 = 25$

$\Rightarrow \lambda_1, \lambda_2 = \pm 5$

$\lambda_1 = 5$, $\lambda_2 = -5$ have opposite signs $\Rightarrow$ O is a Saddle point.

Eigenvector: $v = ?$ / $(A - \lambda I) v = 0$

$\lambda_1 = 5$: $(A - 5I) v = 0 \Rightarrow \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{cases} -5x + 5y = 0 \\ 5x - 5y = 0. \end{cases}$

$\Rightarrow x = y.$

$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Solution : $\boxed{Y = c_1 e^{5t} v_1 + c_2 e^{-5t} v_2}$

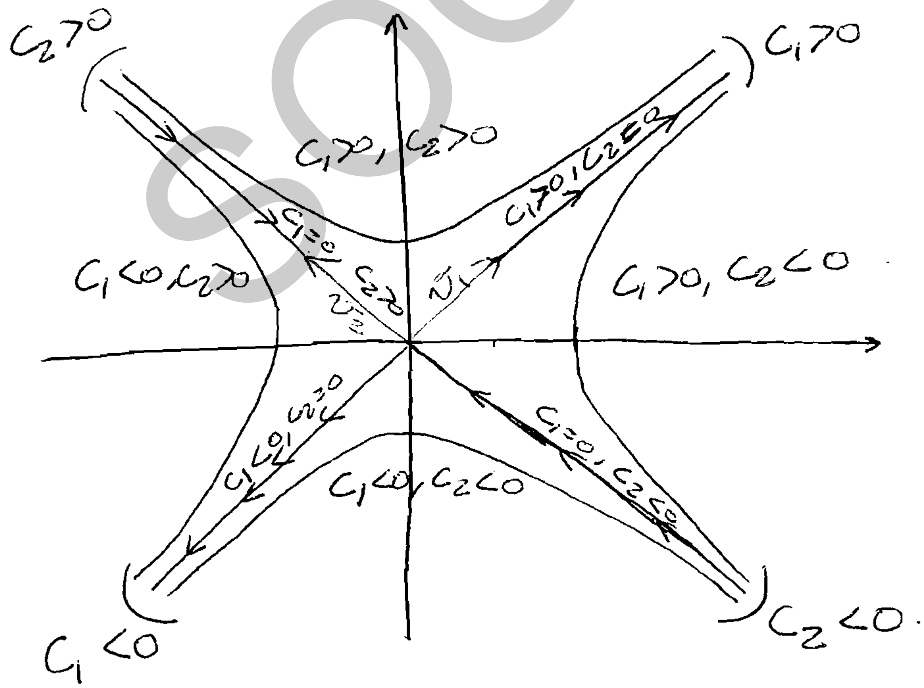
$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{cases} x = c_1 e^{5t} - c_2 e^{-5t} \\ y = c_1 e^{5t} + c_2 e^{-5t} \end{cases}$$

b) Stability is not defined.

c) * $c_1 = 0 \Rightarrow Y = c_2 e^{-5t} v_2$ (2 semi-lines // v_2 directed toward O be $\lim Y = 0$ as $t \to +\infty$)

* $c_2 = 0 \Rightarrow Y = c_1 e^{5t} v_2$ (2 orbits // v_1 directed away from O bec. $\lim Y = 0$ as $t \to -\infty$)



3) $\begin{cases} x' = 8x - y \\ y' = x + 10y \end{cases} \Longleftrightarrow Y' = AY$ Where $A = \begin{pmatrix} 8 & -1 \\ 1 & 10 \end{pmatrix}$

a) Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 8-\lambda & -1 \\ 1 & 10-\lambda \end{vmatrix} = 80 + \lambda^2 - 18\lambda + 1$$

$$\lambda^2 - 18\lambda + 81 = (\lambda - 9)^2 = 0 \Rightarrow \boxed{\lambda_1 = \lambda_2 = 9}$$

O is either a proper node or a degenerate node.

Eigenvectors: $v = ?\ / (A - 9I)v = 0$.

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{cases} -x - y = 0 \\ x + y = 0 \end{cases} \Rightarrow y = -x.$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{v_1} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Rightarrow$ O is a degenerate node

$Y_1 = e^{9t} v_1$ is a solution.

<u>2nd Solution</u>: $Y_2 = e^{9t}(t v_1 + v_2)$

Where $(A - 9I) v_2 = v_1 \Rightarrow \begin{cases} -x - y = 1 \\ x + y = -1 \end{cases}$

$\Rightarrow x = -y - 1$

$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y-1 \\ y \end{pmatrix}$; We need one v_2: Take $y = 0 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

General Solution: $Y = c_1 e^{9t} v_1 + c_2 e^{9t}(t v_1 + v_2)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{9t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{9t}\begin{pmatrix} t-1 \\ -t \end{pmatrix}.$$

$$\boxed{\begin{cases} x = c_1 e^{9t} + c_2 (t-1) e^{9t} \\ y = -c_1 e^{9t} - c_2 t e^{9t} \end{cases}}$$

b) $\lim Y = 0$ as $t \to -\infty$, The orbits are directed away from O
$\Rightarrow O$ is unstable.

c) Orbits:

$* \; c_1 = 0 \Rightarrow Y = c_2 e^{gt}(t v_1 + v_2)$ are not semi-lines XX

$* \; c_2 = 0 \Rightarrow Y = c_1 e^{gt} v_1$ (2 semi-lines $// v_1$)

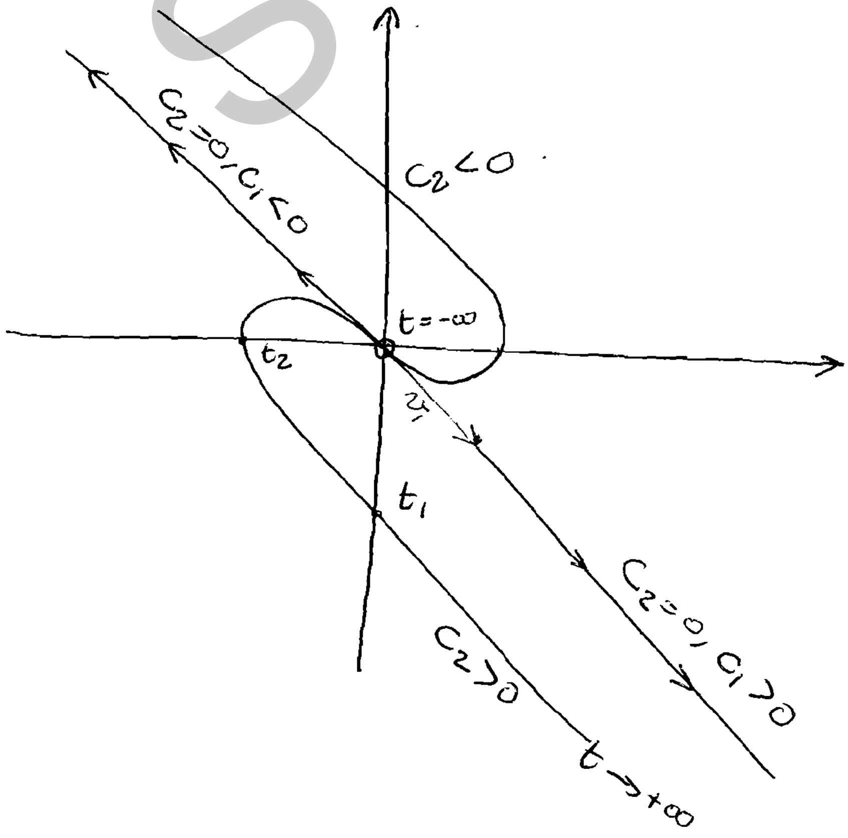
$* \; c_2 \neq 0$. The orbits has same slope as v_1 at O and has an asymptotic direction $// v_1$.

i) slope
ii) asymptote
iii) Table of values: $c_2 > 0$

t	$-\infty$	t_2	t_1	$+\infty$
x	0^-		0	$+\infty$
y	0^+	0		$-\infty$
	second quadrant	x-axis	y-axis	4th quadrant

$$x = 0 \Rightarrow c_1 + c_2(t-1) = 0 \Rightarrow t - 1 = -\frac{c_1}{c_2} \Rightarrow t_1 = 1 - \frac{c_1}{c_2}$$

$$y = 0 \Rightarrow -c_1 - c_2 t = 0 \Rightarrow t_2 = \frac{-c_1}{c_2}$$



Prob 4.4 (1-9) HW(5, 9, 9')

(9') $\begin{cases} x' = 3x - 2y \\ y' = 2x - y \end{cases}$

3) $\begin{cases} x' = 2x + y \\ y' = x + 2y \end{cases} \Rightarrow Y' = AY$ Where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

a) Eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$.

$\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0 \Rightarrow \lambda_1 = +1, \lambda_2 = +3$.

Same sign $\Rightarrow$ O is an improper node.

Eigenvectors: $v = ? \,/\, (A - \lambda I)v = 0$

$*\ \lambda_1 = 1 \Rightarrow (A - I)v = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x = -y$

$v = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$*\ \lambda_2 = 3 \Rightarrow (A - 3I)v = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x = y$.

$v = \begin{pmatrix} y \\ y \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boxed{Y = c_1 e^{t} v_1 + c_2 e^{3t} v_2}.$$

$\begin{cases} x = -c_1 e^{t} + c_2 e^{3t} \\ y = c_1 e^{t} + c_2 e^{3t} \end{cases}$

b) Stability: $\lim Y = 0$ if $t \to -\infty$

Orbits are directed away from O $\Rightarrow$ O is unstable.

c) Orbits:

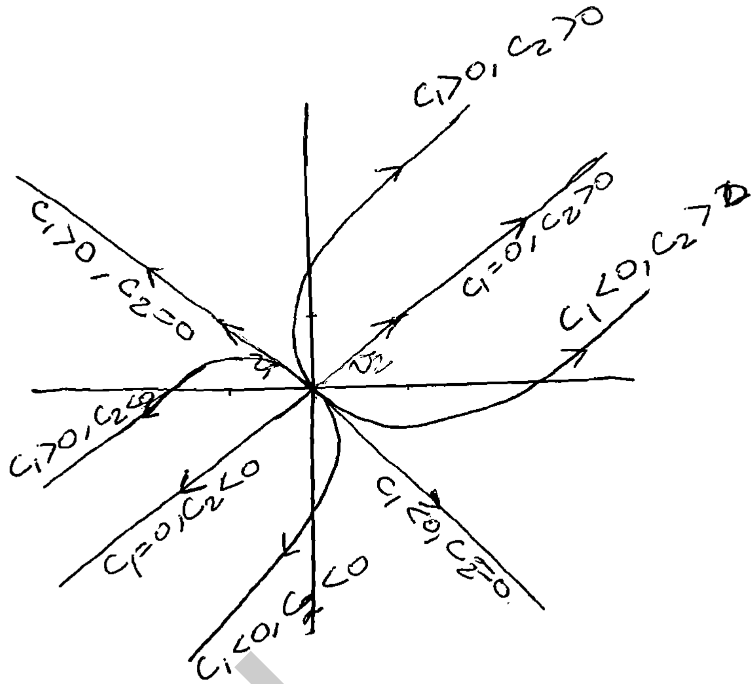
$*\ c_1 = 0 \Rightarrow Y = c_2 e^{3t} v_2$ (2 orbits $// v_2$)

$*\ c_2 = 0 \Rightarrow Y = c_1 e^{t} v_1$ (2 orbits $// v_1$)

<u>Slope at O:</u>

$$m=\frac{dy}{dx}=\frac{y'}{x'}=\lim_{t\to-\infty}\frac{c_1e^{t}+3c_2e^{3t}}{-c_1e^{t}+3c_2e^{3t}}$$
$$=\lim\frac{c_1+3c_2e^{2t}}{-c_1+3c_2e^{2t}}=-1$$

Same slope as v_1



<u>Asymptote</u>: $a=\lim\limits_{t\to+\infty}\frac{y}{x}=\lim\frac{c_1e^{t}+c_2e^{3t}}{-c_1e^{t}+c_2e^{3t}}$
$$=\lim\frac{c_1e^{-2t}+c_2}{-c_1e^{-2t}+c_2}=\frac{c_2}{c_2}=1$$

same slope as v_2.

$b=\lim(y-ax)=\dots=\pm\infty$; The orbits has an asymptotic direction $\parallel v_2$

4) $\begin{cases}x'=y\\ y'=-5x-2y\end{cases}\iff Y'=AY$ where $A=\begin{pmatrix}0&1\\-5&-2\end{pmatrix}$

a) Eigenvalues: $\det(A-\lambda I)=\begin{vmatrix}-\lambda & 1\\ -5 & -2-\lambda\end{vmatrix}=2\lambda+\lambda^2+5$

$\left.\begin{array}{l}\lambda^2+2\lambda+5=0\\ \Delta'=1-5=4i^2\end{array}\right]\Rightarrow\boxed{\begin{array}{l}\lambda_1\\ \lambda_2\end{array}=-1\pm 2i}\Rightarrow O$ is a spiral point.

<u>Method of elimination:</u>

$$y=x'\Rightarrow y'=x''$$

equ 2: $x''=-5x-2x'\Rightarrow x''+2x'+5x=0$ HLDEC

Char. equ: $\lambda^2+2\lambda+5=0\Rightarrow\begin{array}{l}\lambda_1\\ \lambda_2\end{array}=-1\pm 2i$

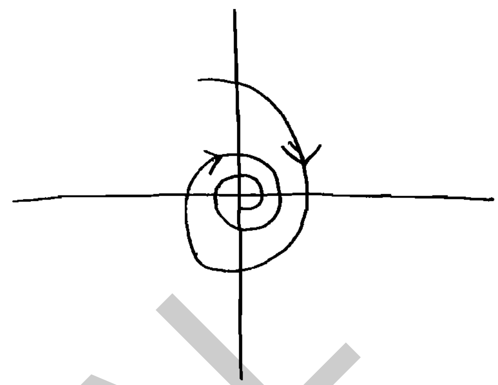
Solution: $x=e^{-t}(c_1\cos 2t+c_2\sin 2t)$

$y=x'=u'v+uv'=-e^{-t}(c_1\cos 2t+c_2\sin 2t)+e^{-t}(-2c_1\sin 2t+2c_2\cos 2t)$

b) Stability: $\lim x=0$ and $\lim y=0$ as $t\to+\infty$

* Sign of θ': Using polar coordinates; $\theta' = -\dfrac{(1+\tan\theta)^2+4}{\sec^2\theta} < 0.$

$\Rightarrow \theta$ is decreasing $\Rightarrow$ The orbits are directed clockwise.



⑧ $\begin{cases} x' = 3x+5y \\ y' = -5x-3y \end{cases} \Leftrightarrow Y' = AY$ Where $A = \begin{pmatrix} 3 & 5 \\ -5 & -3 \end{pmatrix}$

a) Eigenvalues: $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 5 \\ -5 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda)+25$

$\lambda^2+16=0 \Rightarrow \lambda^2 = 16i^2 \Rightarrow \lambda_{1,2} = \pm 4i = \pm i\beta$

$\Rightarrow O$ is a center.

To solve the system, eliminate t:

$$\frac{y'}{x'} = \frac{-5x-3y}{3x+5y} = \frac{dy}{dx} \Rightarrow (3x+5y)\,dy = (-5x-3y)\,dx.$$

$$(\underbrace{5x+3y}_{P})\,dx + (\underbrace{3x+5y}_{Q})\,dy = 0 \quad (\text{it must be exact})$$

$\left.\begin{array}{l} P_y = 3 \\ Q_x = 3 \end{array}\right\} \Rightarrow P_y = Q_x \Rightarrow w$ is exact.

$f = ?\ /\ df = w$

$f = ?\ /\ f_x = P,\ f_y = Q.$

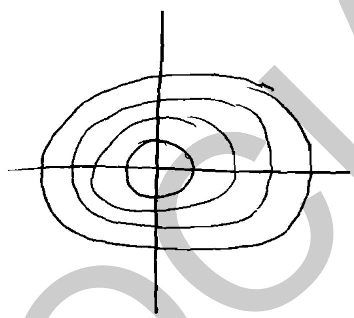
$* \; f_x = 5x + 3y \Rightarrow f = \frac{5x^2}{2} + 3xy + g(y)$

$$\left.\begin{array}{l} \Rightarrow f_y = 3x + g'(y) \\ \text{but } f_y = Q = 3x + 5y \end{array}\right\} \Rightarrow g'(y) = 5y \Rightarrow g = \frac{5y^2}{2} + \cancel{c}^{\,0}$$

Solution : $f = c \Rightarrow \frac{5x^2}{2} + 3xy + \frac{5y^2}{2} = c$.

$$ax^2 + bxy + cy^2 = cte$$

$\Delta = b^2 - 4ac = 9 - 4\left(\frac{5}{2}\right)\left(\frac{5}{2}\right) = -16 < 0 \Rightarrow$ The orbits are ellipses around O.



4.5 Nonlinear System:

Consider the nonlinear system $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$

If $P(x_0, y_0)$ is a critical point then $x' = 0$ and $y' = 0 \Rightarrow$

$f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$.

Calc 4 : The linear approximation (or linearization) of $f(x,y)$ at the point $P(x_0, y_0)$ is:

$$L_1(x,y) = f(x_0, y_0) + f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y.$$

Where $\Delta x = x - x_0$, $\Delta y = y - y_0$.

but at the critical point P, $f(x_0, y_0) = 0$.

Similarly, the linearization of g at the point P is:

$$L_2(x,y) = g_x \Delta x + g_y \Delta y$$

* If we replace f and g by their linearizations, we obtain the nonhomogeneous linear system.

$$\begin{cases} x' = f_x(x-x_0) + f_y(y-y_0) \\ y' = g_x(x-x_0) + g_y(y-y_0) \end{cases}$$

* To obtain a homogeneous system, we apply the change of variables: $\begin{cases} x_1 = \Delta x = x - x_0 \\ y_1 = \Delta y = y - y_0 \end{cases} \Rightarrow \begin{cases} x_1' = x' \\ y_1' = y' \end{cases}$.

$$\begin{cases} x_1' = f_x x_1 + f_y y_1 \\ y_1' = g_x x_1 + g_y y_1 \end{cases} \quad (3) \Leftrightarrow Y' = AY$$

<u>Theorem</u>: If $\det A \neq 0$ then the critical points of the nonlinear system (1) and the homogeneous linear system (3) are of the same type.

<u>Exceptions</u> If A has a double eigenvalues $\lambda_1 = \lambda_2$ or $\lambda_{1,2} = \pm i\beta$ then (1) might have an additional spiral point.

Example: $\rightarrow$

Example: Consider the nonlinear system $\begin{cases} x' = x^2 - y \\ y' = 2x - y + y^2 - x^4 . \end{cases}$

a) Critical points: $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} x^2 - y = 0 \\ 2x - y + y^2 - x^4 = 0 . \end{cases}$

$$\begin{cases} y = x^2 \\ 2x - x^2 + x^4 - x^4 = 0 \end{cases} \Rightarrow \begin{cases} y = x^2 \\ x(2-x) = 0 \Rightarrow x = 0 , x = 2 \end{cases}$$

* If $x = 0$ then $y = 0$.

* If $x = 2$ then $y = 4$.

$\therefore$ (1) has 2 critical points $O(0,0)$ and $P(2,4)$

b) Type:

Point O: The linearization of f and g at O are:

$$\begin{cases} L_1 = f_x(0,0)\,\Delta x + f_y(0,0)\,\Delta y . \\ L_2 = g_x(0,0)\,\Delta x + g_y(0,0)\,\Delta y . \end{cases}$$

* $f = x^2 - y \Rightarrow f_x = 2x$ and $f_y = -1$

$g = 2x - y + y^2 - x^4 \Rightarrow g_x = 2 - 4x^3$ and $g_y = -1 + 2y$

$$(1): \begin{cases} x' = L_1 \\ y' = L_2 \end{cases} \Rightarrow \begin{cases} x' = 0\,\Delta x - 1\,\Delta y . \\ y' = 2\Delta x - \Delta y \end{cases} \Rightarrow \begin{cases} x' = -y . \\ y' = 2x - y \end{cases} \quad (3)$$

homogeneous

$$A = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda + 2 = 0$$

$$\Delta = 1 - 8 = -7 = 7i^2$$

$\det A = \begin{vmatrix} 0 & -1 \\ 2 & -1 \end{vmatrix} = 2 \neq 0 \Rightarrow O$ is a spiral point for (1)

<u>At the point $P(2,4)$:</u> $f_x = 4$, $f_y = -1$

$g_x = -30$, $g_y = 7$

$$\begin{cases} x' = f_x \Delta x + f_y \Delta y \\ y' = g_x \Delta x + g_y \Delta y \end{cases} \Rightarrow \begin{cases} x' = 4\Delta x - \Delta y \\ y' = -30\Delta x + 7\Delta y. \end{cases} \quad (2) \text{ non homogeneous.}$$

Change of variables: $x_1 = \Delta x$, $y_1 = \Delta y$

$$\begin{cases} x_1' = 4x_1 - y_1 \\ y_1' = -30x_1 + 7y_1 \end{cases} \quad (3) \Rightarrow Y' = AY \text{ where } A = \begin{pmatrix} 4 & -1 \\ -30 & 7 \end{pmatrix}$$

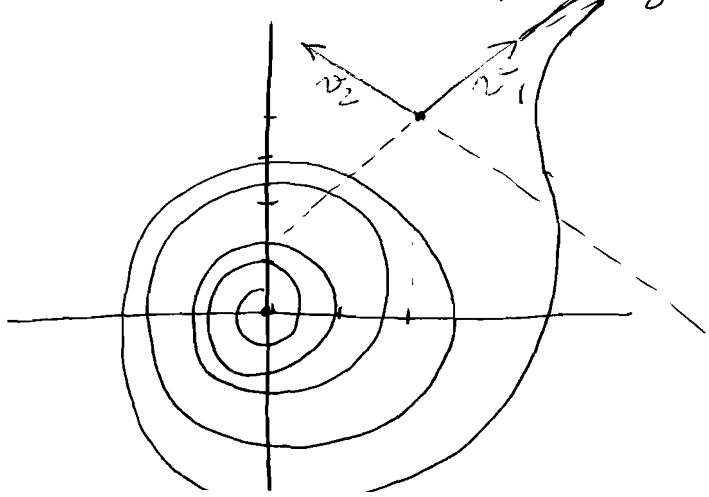
$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ -30 & 7-\lambda \end{vmatrix} = 28 + \lambda^2 - 11\lambda - 30 = \lambda^2 - 11\lambda - 2 = 0.$$

$$\Delta = 121 + 8 = 129 \Rightarrow \lambda_{1,2} = \frac{11 \pm \sqrt{129}}{2}$$

$$\lambda_1 = \frac{11 + \sqrt{129}}{2} > 0, \quad \lambda_2 = \frac{11 - \sqrt{129}}{2} < 0$$

$\Rightarrow O'$ is a saddle point for (3).

$*$ $\det A = \begin{vmatrix} 4 & -1 \\ -30 & 7 \end{vmatrix} = -2 \neq 0 \Rightarrow P(2,4)$ is a saddle point for (1).



Prob 4.5 (1-6) HW (2, 3, 5) $\rightarrow$ 2 critical points.

① $\begin{cases} x' = y + y^2 \\ y' = 3x \end{cases}$ (1) nonlinear system

Type of critical points

* Critical points: $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} y + y^2 = 0 \\ 3x = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \text{ or } y = -1 \\ x = 0 \end{cases}$

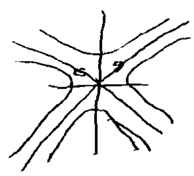
There are 2 critical points: $O(0,0)$ and $P(0,-1)$

* Linearization: $f = y + y^2 \Rightarrow f_x = 0$ and $f_y = 1 + 2y$

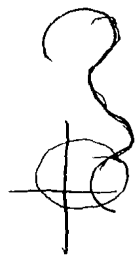
$g = 3x \Rightarrow g_x = 3$ and $g_y = 0$.

at the point $O(0,0)$: $f = 0,\ f_x = 0,\ f_y = 1$

$g = 0,\ g_x = 3,\ g_y = 0$



(3)



(1)

$\begin{cases} x' = f_x \Delta x + f_y \Delta y \\ y' = g_x \Delta x + g_y \Delta y \end{cases} \Rightarrow \begin{cases} x' = y \\ y' = 3x \end{cases}$ (3) Homogeneous Linear System

$Y' = AY$ Where $A = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$

* eigenvalues: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 - 3 \Rightarrow \lambda_1, \lambda_2 = \pm\sqrt{3}$

$\lambda_1 > 0,\ \lambda_2 < 0 \Rightarrow O$ is a saddle point for (3)

$\det A = \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = -3 \neq 0 \Rightarrow O$ is a saddle point for (1)

* at the point $P(0,-1)$: $f = 0,\ f_x = 0,\ f_y = -1$

$g = 0,\ g_x = 3,\ g_y = 0$.

* Let $x_1 = \Delta x = x$, $y_1 = \Delta y = y+1 \Rightarrow x_1' = x'$ and $y_1' = y'$

$$\begin{cases} x_1' = -y_1 \\ y_1' = 3x_1 \end{cases} \quad (3) \quad \text{Homogeneous linear}$$

$Y' = AY$ Where $A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 + 3 = 0 \Rightarrow \lambda_1, \lambda_2 = \pm i\sqrt{3} = \pm i\beta$

O' is a center.

$\det A = 3 \neq 0 \Rightarrow P(0,-1)$ is a center for (1)

* <u>Exception</u>: System (1) might have an additional spiral point.

(4) $$\begin{cases} x' = -3x + y - y^2 \\ y' = x - 3y \end{cases} \quad (1) \quad \text{nonlinear}$$

a) Critical points: $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} -3x + y - y^2 = 0 \\ x - 3y = 0 \end{cases} \Rightarrow \begin{cases} -9y + y - y^2 = 0. \\ x = 3y \end{cases}$

$\begin{cases} 8y + y^2 = 0 \Rightarrow y = 0 \text{ or } y = -8. \\ x = 3y \end{cases}$

There are 2 critical points: $O(0,0)$ and $P(-24,-8)$.

b) Linear System:

$f = -3x + y - y^2 \Rightarrow f_x = -3$, $f_y = 1 - 2y$.

$g = x - 3y \Rightarrow g_x = 1$, $g_y = -3$.

* At $O(0,0)$: $f_x = -3$, $f_y = 1$

$\{ x' = -3\Delta x + \Delta y$

$$\begin{cases} x' = -3\Delta x + \Delta y \\ y' = \Delta x - 3\Delta y \end{cases} \Rightarrow \begin{cases} x' = -3x + y \\ y' = x - 3y \end{cases} \quad (3)$$

$$\Rightarrow Y' = AY \text{ Where } A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)^2 - 1 = \lambda^2 + 6\lambda + 8$$
$$= (\lambda + 2)(\lambda + 4) = 0.$$

$$\Rightarrow \lambda_1 = -2, \ \lambda_2 = -4$$
$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

λ_1, λ_2 have the same sign $\Rightarrow$ O is an improper node for (3)

$\det A = 9 - 1 = 8 \neq 0 \Rightarrow$ O is an improper node for (1)

* at $P(-24, -8)$: $f_x = -3$, $f_y = 17$

$g_x = 1$, $g_y = -3$

$$\begin{cases} x' = -3\Delta x + 17\Delta y \\ y' = \Delta x - 3\Delta y \end{cases} \quad (2) \quad \text{Where} \quad \Delta x = x + 24, \ \Delta y = y + 8$$

non homogeneous linear

* change of variables : $\begin{cases} x_1 = \Delta x = x + 24 \\ y_1 = \Delta y = y + 8 \end{cases} \Rightarrow \begin{cases} x_1' = x' \\ y_1' = y' \end{cases}$

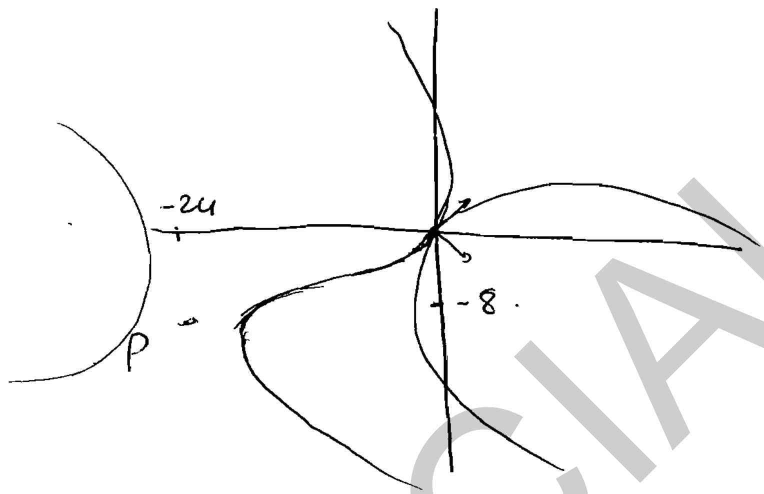
$$\begin{cases} x_1' = -3x_1 + 17y_1 \\ y_1' = x_1 - 3y_1 \end{cases} \quad (3) \Rightarrow Y' = AY \text{ where } A = \begin{pmatrix} -3 & 17 \\ 1 & -3 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 17 \\ \ldots \end{vmatrix} = (-3-\lambda)^2 - 17 = (\lambda + 3)^2 - 17$$

$$=(\lambda+3-\sqrt{17})(\lambda+3+\sqrt{17})=0 \Rightarrow \begin{cases} \lambda_1 = -3+\sqrt{17} > 0 \\ \lambda_2 = -3-\sqrt{17} < 0 \end{cases}$$

$\Rightarrow$ P is a saddle point for (3)

$\det A = 9-17 = -8 \neq 0 \Rightarrow$ P is a saddle point for (1)



<u>Final Exam</u>

Chap 4 (2 questions) (20%-25%)

Chap 5 (1 question) 10%

Chap 21 (1 question) 10%.

Chapter 5 Power Series Method

5.2 Power series Solutions

Definition: A function $f(x)$ is said to be analytic at x_0 if the Taylor series of $f(x)$ at x_0 has a nonzero radius of convergence.

Example: $\frac{1}{x}$ is not analytic at 0

$\cos x, \sin x, e^x, \ldots$ are analytic at 0

$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$ is analytic at 0

Theorem Consider the LDE: $y'' + a(x)y' + b(x)y = 0$.

If $a(x)$ and $b(x)$ are analytic at x_0 then the DE has at least 1 solution of the form

$$y = \sum_{n=0}^{+\infty} C_n (x - x_0)^n$$

Example: Solve the HLDE: $y'' + 2xy' + 2y = 0$

$a(x) = 2x$ and $b(x) = 2$ are analytic at 0

$\Rightarrow$ the differential equation has at least 1 solution of the form $y = \sum_{n=0}^{+\infty} C_n x^n$

$$y' = \sum_{n=0}^{+\infty} n C_n x^{n-1} \Rightarrow y'' = \sum_{n=0}^{+\infty} n(n-1) C_n x^{n-2}$$

Replace in the DE.

$$\sum_{n=0}^{+\infty} n(n-1) C_n x^{n-2} + 2 \sum_{n=0}^{+\infty} n C_n x^n + 2 \sum_{n=0}^{+\infty} C_n x^n = 0$$

$$\sum_{n=0}^{+\infty} n(n-1) C_n x^{n-2} + 2 \sum_{n=0}^{+\infty} (n+1) C_n x^n = 0.$$

$$\sum_{m=-2}^{+\infty} (m+2)(m+1) C_{m+2}\, x^m + 2 \sum_{m=0}^{+\infty} (m+1) C_m\, x^m = 0.$$

Fix the starting values:

$$0x^{-2} + 0x^{-1} + \sum_{m=0}^{\infty} (m+2)(m+1) C_{m+2}\, x^m + 2 \sum_{m=0}^{\infty} (m+1) C_m\, x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+1)\left[(m+2) C_{m+2} + 2 C_m\right] x^m = 0.$$

It is of the form $\sum \alpha_m x^m = 0 \Rightarrow \alpha_m = 0 \;\forall m \geq 0$

$$\Rightarrow \underbrace{(m+1)}_{\neq 0 \;\forall m \geq 0}\left[(m+2) C_{m+2} + 2 C_m\right] = 0 \;\forall m \geq 0.$$

$$\underbrace{(m+2)}_{\neq 0} C_{m+2} = -2 C_m \Rightarrow \boxed{C_{m+2} = \frac{-2 C_m}{m+2}} \;\forall m \geq 0$$

* $(m+2) - m = 2$ groups of equations.

$m=0: \; C_2 = -C_0$	$m=1: \; C_3 = -\frac{2}{3} C_1$
$m=2: \; C_4 = -\frac{C_2}{2} = \frac{C_0}{2}$	$m=3: \; C_5 = \frac{-2}{5} C_3 = \frac{4}{3 \times 5} C_1$
$m=4: \; C_6 = -\frac{C_4}{3} = -\frac{C_0}{6} = -\frac{C_0}{3!}$	$m=5: \; C_7 = \frac{-2}{7} C_5 = \frac{-8}{3 \times 5 \times 7} C_1$
$C_{2k} = (-1)^k \frac{C_0}{k!}$	$C_{2k+1} = (-1)^k \frac{2^k}{3.5.7 \ldots (2k+1)}$

Solution: $y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots$

$$y = C_0\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) + C_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots\right)$$

This is the general solution because it contains 2 constants.

(Stop)

$$y = C_0 e^{-x^2} + C_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots\right).$$

Can you simplify y_2?

$$y_1 = e^{-x^2} \Rightarrow y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int a(x)\,dx}\,dx$$

$$y_2 = e^{-x^2}\int \frac{1}{e^{-2x^2}} e^{-\int 2x\,dx}\,dx = e^{-x^2}\int e^{2x^2} e^{-x^2}\,dx.$$

$$y_2 = e^{-x^2}\int e^{x^2}\,dx \text{ can't be evaluated.}$$

Prob 5.2 (16–23) HW (17, 19, 21)

(22) $y'' - 4xy' + (4x^2 - 2)y = 0.$

$a(x) = -4x$ and $b(x) = 4x^2 - 2$ are analytic at 0

$\Rightarrow \exists$ at least 1 solution of the form $y = \sum_{n=0}^{\infty} C_n x^n$.

$$y' = \sum_0 n C_n x^{n-1} \Rightarrow y'' = \sum_0 n(n-1) C_n x^{n-2}$$

$$y'' - 4xy' + 4x^2 y - 2y = 0 \Rightarrow \sum_0 n(n-1)C_n x^{n-2} - 4\sum_0 nC_n x^n + 4\sum_0 C_n x^{n+2} - 2\sum_0 C_n x^n = 0.$$

$$\Rightarrow \sum_0 n(n-1)C_n x^{n-2} + 4\sum_0 C_n x^{n+2} - \sum_0 (4n+2)C_n x^n = 0.$$

$m = n-2 \qquad m = n+2 \qquad m = n.$

Fix the powers:

$$\sum^{\infty} (m+2)(m+1)C_{m+2}\, x^m + 4\sum^{\infty} C_{m-2}\, x^m - \sum (4m+2)C\ x^m - -$$

Fix the starting Values

$$0x^{-2}+0x^{-1}+2C_2x^0+6C_3x+\sum_{m=2}^{\infty}(m+2)(m+1)C_{m+2}x^m+4\sum_{m=2}^{\infty}C_{m-2}x^m$$
$$-2C_0x^0-6C_1x-\sum_{m=2}^{\infty}(4m+2)C_mx^m=0.$$

$$\Rightarrow (2C_2-2C_0)+(6C_3-6C_1)x$$
$$+\sum_{m=2}^{\infty}\left[(m+2)(m+1)C_{m+2}+4C_{m-2}-(4m+2)C_m\right]x^m=0$$

$$\Rightarrow \begin{cases} 2C_2-2C_0=0 \Rightarrow C_2=C_0 \\ 6C_3-6C_1=0 \Rightarrow C_3=C_1 \\ \underbrace{(m+2)(m+1)}_{\neq 0}C_{m+2}=-4C_{m-2}+(4m+2)C_m \quad \forall m\geq 2. \end{cases}$$

$$C_{m+2}=\frac{(4m+2)C_m-4C_{m-2}}{(m+2)(m+1)} \quad (3\text{coef.} \Rightarrow 1\text{group})$$

$m=2 \Rightarrow C_4=\frac{10C_2-4C_0}{12}=\frac{10C_0-4C_0}{12}=\frac{C_0}{2}.$

$m=3 \Rightarrow C_5=\frac{22C_3-4C_1}{20}=\frac{18}{20}C_1=\frac{9}{10}C_1$

$m=4 \Rightarrow C_6=\frac{26C_4-4C_2}{30}=\frac{13C_0-4C_0}{30}=\frac{9}{30}C_0.$

$m=5 \Rightarrow C_7=\frac{30C_5-4C_3}{42}=\frac{27C_1-4C_1}{42}=\frac{23}{42}C_1.$

<u>Solution</u>: $y=\sum_{n=0}^{\infty}C_nx^n=C_0+C_1x+C_2x^2+C_3x^3+\cdots$

$$y=C_0+C_1x+C_0x^2+C_1x^3+\frac{C_0}{2}x^4+\frac{9}{10}C_1x^5+\frac{9}{30}C_0x^6+\cdots.$$

$$y=C_0\left(1+x^2+\frac{x^4}{2}+\frac{9}{30}x^6+\cdots\right)+C_1\left(x+x^3+\frac{9}{10}x^5+\frac{23}{42}x^7+\cdots\right)$$

5.3 Legendre DE

Definition The Legendre differential equation is of the form

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0.$$

Where p is a real constant.

Theorem: If $p>0$ is an integer then one of the solutions is a polynomial called Legendre polynomial

Example: $p=2 \Rightarrow (1-x^2)y'' - 2xy' + 6y = 0$.

$$a(x) = \frac{-2x}{1-x^2} \text{ and } b(x) = \frac{6}{1-x^2} \text{ are analytic at } 0$$

$$\Rightarrow \exists \text{ at least 1 solution of the form } y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_0 n C_n x^{n-1} \Rightarrow y'' = \sum_0 n(n-1) C_n x^{n-2}.$$

$$(1) \Rightarrow y'' - x^2 y'' - 2xy' + 6y = 0.$$

$$\Rightarrow \sum_0 n(n-1) C_n x^{n-2} - \sum_0 n(n-1) C_n x^n - 2\sum n C_n x^n + 6 \sum_0 C_n x^n = 0$$

$$\Rightarrow \sum_0 n(n-1) C_n x^{\underset{m=n-2}{n-2}} - \sum_0 \left[n(n-1) + 2n - 6\right] C_n x^{\underset{m=n}{n}} = 0$$

$$\Rightarrow \sum_{m=-2}^{\infty} (m+2)(m+1) C_{m+2} x^m - \sum_{m=0}^{\infty} (m^2+m-6) C_m x^m = 0$$

$$0x^{-2} + 0x^{-1} + \sum_{m=0}^{\infty} \left[(m+2)(m+1) C_{m+2} - (m-2)(m+3) C_m\right] x^m = 0.$$

$$\Rightarrow \underbrace{(m+2)(m+1)}_{\neq 0} C_{m+2} = (m-2)(m+3) C_m \quad \forall m \geq 0.$$

$$\Rightarrow C_{m+2} = \frac{(m-2)(m+3)}{(m+2)(m+1)} C_m \quad \forall m \geq 0.$$

$*(m+2) - m = 2$ groups of equations

$m=0: \; C_2 = \frac{-6}{2} C_0 = -3C_0$

$m=2: \; C_4 = 0\, C_2 = 0$

$m=4: \; C_6 = --- C_4 = 0$

$m=6: \; C_8 = \alpha C_6 = 0.$

$\vdots$

$m=1: \; C_3 = \frac{-4}{6} C_1 = \frac{-2}{3} C_1$

$m=3: \; C_5 = \frac{6}{20} C_3 = \frac{3}{10}\left(\frac{-4}{6} C_1\right) = \frac{-1}{5} C_1.$

$m=5: \; C_7 = \frac{24}{42} C_5 = \frac{4}{7}\left(-\frac{1}{5} C_1\right) = \frac{-4}{35} C_1$

$\vdots$

<u>Solution</u>: $y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^m$

$$y = C_0 + C_1 x - 3C_0 x^2 - \frac{2}{3} C_1 x^3 + 0x^4 - \frac{1}{5} C_1 x^5 + 0x^6 - \frac{4}{35} C_1 x^7 + \cdots$$

$$y = C_0 (1 - 3x^2) + C_1 \left(x - \frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{4}{35}x^7 + \cdots\right)$$

$P_2(x) = 1 - 3x^2$ is called a Legendre polynomial of degree 2

The 2nd Solution $y_2 = x - \frac{2}{3}x^3 - \cdots$ is called a Legendre function.

Ch

Chapter 21 Numerical Methods to solve a DE.

21.1 First-Order DE

A) Euler Method

Consider the 1st-order DE: $y' = f(x,y)$. Let $y = y(x)$ be a solution satisfying the initial condition $y(x_0) = y_0$.

The solution can be estimated numerically as follows:

Step 1: Estimate (→ DE) the values y_i for selected values of x_i.

Step 2: Find the interpolating polynomial passing through the points (x_i, y_i)

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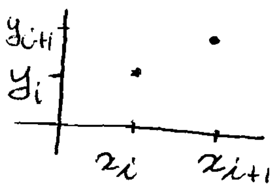
This polynomial can be used to estimate the solution.

Consider the sequence x_i defined by: $x_i = x_{i-1} + h$

Where h is a constant, called the Step size.



The Taylor series of $y(x)$ at the point y_i is:

$$y(x) = y(x_i) + y'(x_i)(x - x_i) + \frac{y''(x_i)}{2!}(x-x_i)^2 + \frac{y'''(x_i)}{3!}(x-x_i)^3 + \dots$$

Let $x = x_{i+1} = x_i + h$

$$y(x_{i+1}) = y(x_i) + y'(x_i)\,h + \frac{y''(x_i)}{2!}h^2 + \frac{y'''(x_i)}{3!}h^3 + \dots$$

If h is small then $h^2, h^3, \dots$ are very small

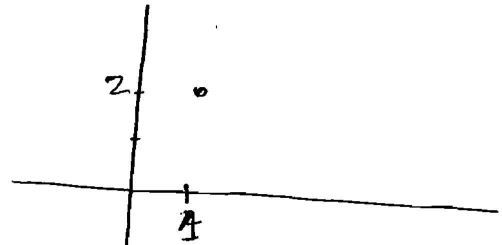
$$\Rightarrow y(x_{i+1}) \simeq y(x_i) + h\,y'(x_i)$$

$$\Rightarrow y(x_{i+1}) \simeq y(x_i) + h\,f(x_i, y_i)$$

Formula: $\boxed{y_{i+1} \simeq y_i + h\,f(x_i, y_i)}$

<u>Example</u>: Estimate the solution to the IVP: $\begin{cases} y' = 5\sin(x+y) \\ y(1) = 2 \end{cases}$

Do 10 steps with $h = 0.1$



* $x_0 = 1$, $y_0 = 2$

* $x_1 = x_0 + h = 1.1 \Rightarrow y_1 \simeq y_0 + 0.1\, f(1,2)$

$$y_1 = 2 + (0.1)5\sin(\underbrace{1+2}_{\text{radians}}) = 2.07056 \text{ (6 digits)}$$

* $x_2 = x_1 + h = 1.2 \Rightarrow y_2 \simeq y_1 + 0.5\,(x_1 + y_1) = 2.05608$

* $x_3 = x_2 + h = 1.3 \Rightarrow y_3 \simeq y_2 + 0.5\sin(x_2 + y_2) = 1.99896$.

* $x_4 = 1.4$, $y_4 = 1.9206$

* $x_5 = 1.5$, $y_5 = 1.83157$.

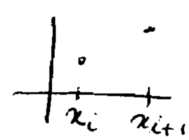
$\vdots$

* $x_{10} = 2$, $y_{10} = 1.34256$.

<u>B) Improved – Euler Method</u>

<u>Remark</u>: The average value of y' over the interval $[x_i, x_{i+1}]$ is:

$$\overline{y'} = Av(y') = \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} y'\,dx = \frac{1}{h}(y)\Big|_{x_i}^{x_{i+1}}$$



$$\Rightarrow \overline{y'} = \frac{1}{h}\left(y(x_{i+1}) - y(x_i)\right) \Rightarrow y_{i+1} + y_i = h\overline{y'}$$

$$\Rightarrow \boxed{\underbrace{y_{i+1}}_{\text{exact value.}} = y_i + h\overline{y'}}$$

*In Euler's method we used $y'(x_i)$ to estimate $\overline{y'}$ but the average of $y'(x_i)$ and $y'(x_{i+1})$ gives a better approximation to y'.

$$\overline{y'} \simeq \frac{y'(x_i) + y'(x_{i+1})}{2} = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}$$

(2) $\Rightarrow$ $$y_{i+1} \simeq y_i + \frac{h}{2}\Big[\underbrace{f(x_i, y_i)}_{d_1} + \underbrace{f(x_{i+1}, y_{i+1})}_{d_2}\Big].$$

$*\, d_2 = f(x_{i+1}, y_{i+1})$ where $y_{i+1} = y_i + h f(x_i, y_i)$

$y_{i+1} = y_i + h d_1$

$$d_2 = f(x_{i+1}, y_i + h d_1)$$

Formula: $\boxed{y_{i+1} \simeq y_i + \frac{h}{2}(d_1 + d_2)}$

where $\boxed{d_1 = f(x_i, y_i)}$ and $\boxed{d_2 = f(x_{i+1}, y_i + h d_1)}$

Example: Solve the IVP: $\begin{cases} y' = 5\sin(x+y) \\ y(1) = 2. \end{cases}$

Do 3 steps with $h = 0.1$

$x_0 = \phi$, $y_0 = 2$.

Step1

$x_1 = x_0 + h = 1.1$

$d_1 = f(x_0, y_0) = 5\sin(3) = 0.7056$.

$d_2 = f(x_1, y_0 + h d_1) = f(1.1, \underbrace{2 + 0.1 \times (0.7056)}_{2 + 0.07056}) = -0.144816$

$y_1 = y_0 + \frac{h}{2}(d_1 + d_2) = 2 + \frac{0.1}{2}(0.7056 - 0.144816)$

$= 2.02804$.

Step2

$x_2 = x_1 + h = 1.2$

$d_1 = f(x_1, y_1) = 0.0677653$.

$d_2 = f(x_2, y_1 + h d_1) = -0.46544$

$y_2 = y_1 + \frac{h}{2}(d_1 + d_2) = 2.00816$.

Step3 $\quad x_3 = x_2 + h = 1.3$

$$d_1 = f(x_2, y_2) = -0.332568$$

$$d_2 = f(x_3, y_2 + h\,d_1) = -0.664557$$

$$y_3 = y_2 + \frac{h}{2}(d_1 + d_2) = 1.9583$$

C) Runge - Kutta Method

Recall that $y_{i+1} = y_i + h\bar{y}'$

$\bar{y}'$ can be estimated using the average of 6 derivatives

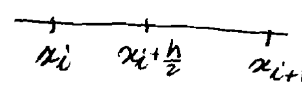
Formula: $\quad y_{i+1} \simeq y_i + \frac{h}{6}(d_1 + 2d_2 + 2d_3 + d_4)$

$$d_1 = y'(x_i) = \boxed{f(x_i, y_i) = d_1}$$

$$d_2 = y'(x_i + \tfrac{h}{2}) = f(x_i + \tfrac{h}{2}, y(x_i + \tfrac{h}{2})) = \boxed{f(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2}d_1) = d_2}$$

$$d_3 = y'(x_i + \tfrac{h}{2}) = f(x_i + \tfrac{h}{2}, y(x_i + \tfrac{h}{2})) = \boxed{f(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2}d_2) = d_3}$$

$$d_4 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1}) = \boxed{f(x_{i+1}, y_i + h\,d_3) = d_4}$$



Example: Using Runge-Kutta method (2 steps), solve the

$$\text{IVP: } \begin{cases} y' = 4\cdot\cos(x+y) \\ y(2) = -3 \end{cases}, \quad h = 0.5$$

$x_0 = 2,\ y_0 = -3$

1st iteration: $\quad x_1 = x_0 + h = 2.5$

$$d_1 = f(x_0, y_0) = f(2, -3) = 4\cos(2-3) = 2.1621$$

$$d_2 = f(x_0 + 0.25, y_0 + 0.25d_1) = f(2.25, -2.4597) = 3.91238$$

$$d_3 = f(x_0 + 0.25, y_0 + 0.25d_2) = f(2.25, -2.0219) = 3.8964$$

2^{nd} step

$x_2 = x_1 + h = 3$

$d_1 = f(x_1, y_1) = 2.08553$

$d_2 = f(x_1 + 0.25, y_1 + 0.25 d_1) = -0.884283$

$d_3 = f(x_1 + 0.25, y_1 + 0.25 d_2) = 1.98593$

$d_4 = f(x_2, y_1 + h d_3) = -3.2408$

$y_2 = -1.47767 + \frac{0.5}{6}(d_1 + 2d_2 + 2d_3 + d_4) = -1.39034$

step 3 $x_3 = 3.5$, $y_3 = -1.74264$

step 4 $x_4 = 4$, $y_4 = -2.19727$

Prob 21.1 (1−17) HW (3 Euler, 7 Improved Euler, 13, 15 Runge-Kutta) (3 iterations)

21.3 System of DE

A) Euler's Method

Consider the system of DE: $\begin{cases} x' = f(t, x, y) \\ y' = g(t, x, y) \end{cases}$

It can be written in Matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \Leftrightarrow X' = F(t, X)$$

Formula: $\boxed{X_{i+1} = X_i + h\, F(t_i, X_i)}$

$$\Leftrightarrow \begin{cases} x_{i+1} = x_i + h f(t_i, x_i, y_i) \\ y_{i+1} = y_i + h g(t_i, x_i, y_i) \end{cases}$$

Example. Solve the IVP: $\begin{cases} x' = y \\ y' = tx \end{cases}$ $\quad x(0) = 1 \to x(t_0) = x_0$ $\quad h = 0.2$

$y(0) = 0 \to y(t_0) = y_0$

(3 steps).

Initial values: $t_0 = 0$, $x_0 = 1$, $y_0 = 0$, $h = 0.2$, $f = y$ and $g = tx$

Step1 $x_1 = x_0 + h f(t_0, x_0, y_0) = 1 + 0.2\, f(0,1,0) = 1 + 0.2(0) = 1$

$y_1 = y_0 + h\,(t_0, x_0, y_0) = 0 + 0.2\, g(0,1,0) = 0 + 0 = 0.$

Step2 $t_1 = t_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_1 + h f(t_1, x_1, y_1) = 1 + 0.2\, y_1 = 1$

$y_2 = y_1 + h g(t_1, x_1 + y_1) = 0 + 0.2\, t_1 x_1 = 0.04.$

Step3 $t_2 = t_1 + h = 0.4.$

$x_3 = x_2 + h f(t_2, x_2, y_2) = 1 + 0.2 y_2 = 1.008$

$y_3 = y_2 + h g(t_2, x_2 y_2) = 0.04 + 0.2\, t_2 x_2 = 0.12$

End.