CHAPTER 8 TECHNIQUES OF INTEGRATION

8.1 INTEGRATION BY PARTS

1. $u = x$, $du = dx$; $dv = \sin \frac{x}{2}\,dx$, $v = -2\cos \frac{x}{2}$;
$\int x \sin \frac{x}{2}\,dx = -2x\cos \frac{x}{2} - \int \left(-2\cos \frac{x}{2}\right) dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$

2. $u = \theta$, $du = d\theta$; $dv = \cos \pi\theta\,d\theta$, $v = \frac{1}{\pi}\sin \pi\theta$;
$\int \theta \cos \pi\theta\,d\theta = \frac{\theta}{\pi}\sin \pi\theta - \int \frac{1}{\pi}\sin \pi\theta\,d\theta = \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi^2}\cos \pi\theta + C$

3.

		$\cos t$
t^2	$\xrightarrow{(+)}$	$\sin t$
$2t$	$\xrightarrow{(-)}$	$-\cos t$
2	$\xrightarrow{(+)}$	$-\sin t$
0		

$\int t^2 \cos t\,dt = t^2 \sin t + 2t\cos t - 2\sin t + C$

4.

		$\sin x$
x^2	$\xrightarrow{(+)}$	$-\cos x$
$2x$	$\xrightarrow{(-)}$	$-\sin x$
2	$\xrightarrow{(+)}$	$\cos x$
0		

$\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

5. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x\,dx$, $v = \frac{x^2}{2}$;
$\int_1^2 x \ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\,\frac{dx}{x} = 2\ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2\ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$

6. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^3\,dx$, $v = \frac{x^4}{4}$;
$\int_1^e x^3 \ln x\,dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\,\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$

7. $u = x$, $du = dx$; $dv = e^x dx$, $v = e^x$;
$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C$

8. $u = x$, $du = dx$; $dv = e^{3x} dx$, $v = \frac{1}{3}e^{3x}$;
$\int x e^{3x} dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x} dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$

9.

		e^{-x}
x^2	$\xrightarrow{(+)}$	$-e^{-x}$
$2x$	$\xrightarrow{(-)}$	e^{-x}
2	$\xrightarrow{(+)}$	$-e^{-x}$
0		

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} - 2x\,e^{-x} - 2\,e^{-x} + C$$

10.

		e^{2x}
$x^2 - 2x + 1$	$\xrightarrow{(+)}$	$\frac{1}{2}e^{2x}$
$2x - 2$	$\xrightarrow{(-)}$	$\frac{1}{4}e^{2x}$
2	$\xrightarrow{(+)}$	$\frac{1}{8}e^{2x}$
0		

$$\int (x^2 - 2x + 1)e^{2x}\,dx = \tfrac{1}{2}(x^2 - 2x + 1)e^{2x} - \tfrac{1}{4}(2x - 2)e^{2x} + \tfrac{1}{4}e^{2x} + C$$
$$= \left(\tfrac{1}{2}x^2 - \tfrac{3}{2}x + \tfrac{5}{4}\right)e^{2x} + C$$

11. $u = \tan^{-1} y$, $du = \frac{dy}{1+y^2}$; $dv = dy$, $v = y$;
$$\int \tan^{-1} y\,dy = y\tan^{-1} y - \int \frac{y\,dy}{(1+y^2)} = y\tan^{-1} y - \tfrac{1}{2}\ln(1+y^2) + C = y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

12. $u = \sin^{-1} y$, $du = \frac{dy}{\sqrt{1-y^2}}$; $dv = dy$, $v = y$;
$$\int \sin^{-1} y\,dy = y\sin^{-1} y - \int \frac{y\,dy}{\sqrt{1-y^2}} = y\sin^{-1} y + \sqrt{1-y^2} + C$$

13. $u = x$, $du = dx$; $dv = \sec^2 x\,dx$, $v = \tan x$;
$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

14. $\int 4x\sec^2 2x\,dx$; $[y = 2x] \to \int y\sec^2 y\,dy = y\tan y - \int \tan y\,dy = y\tan y - \ln|\sec y| + C$
$= 2x\tan 2x - \ln|\sec 2x| + C$

15.

		e^x
x^3	$\xrightarrow{(+)}$	e^x
$3x^2$	$\xrightarrow{(-)}$	e^x
$6x$	$\xrightarrow{(+)}$	e^x
6	$\xrightarrow{(-)}$	e^x
0		

$$\int x^3 e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)\,e^x + C$$

16.
$$\begin{array}{lcl} & & e^{-p} \\ p^4 & \xrightarrow{(+)} & -e^{-p} \\ 4p^3 & \xrightarrow{(-)} & e^{-p} \\ 12p^2 & \xrightarrow{(+)} & -e^{-p} \\ 24p & \xrightarrow{(-)} & e^{-p} \\ 24 & \xrightarrow{(+)} & -e^{-p} \\ 0 & & \end{array}$$

$$\int p^4 e^{-p}\,dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right) e^{-p} + C$$

17.
$$\begin{array}{lcl} & & e^{x} \\ x^2 - 5x & \xrightarrow{(+)} & e^{x} \\ 2x - 5 & \xrightarrow{(-)} & e^{x} \\ 2 & \xrightarrow{(+)} & e^{x} \\ 0 & & \end{array}$$

$$\int \left(x^2 - 5x\right) e^x\,dx = \left(x^2 - 5x\right) e^x - (2x - 5)e^x + 2e^x + C = x^2 e^x - 7xe^x + 7e^x + C$$
$$= \left(x^2 - 7x + 7\right) e^x + C$$

18.
$$\begin{array}{lcl} & & e^{r} \\ r^2 + r + 1 & \xrightarrow{(+)} & e^{r} \\ 2r + 1 & \xrightarrow{(-)} & e^{r} \\ 2 & \xrightarrow{(+)} & e^{r} \\ 0 & & \end{array}$$

$$\int \left(r^2 + r + 1\right) e^r\,dr = \left(r^2 + r + 1\right) e^r - (2r + 1)\, e^r + 2e^r + C$$
$$= \left[\left(r^2 + r + 1\right) - (2r + 1) + 2\right] e^r + C = \left(r^2 - r + 2\right) e^r + C$$

19.
$$\begin{array}{lcl} & & e^{x} \\ x^5 & \xrightarrow{(+)} & e^{x} \\ 5x^4 & \xrightarrow{(-)} & e^{x} \\ 20x^3 & \xrightarrow{(+)} & e^{x} \\ 60x^2 & \xrightarrow{(-)} & e^{x} \\ 120x & \xrightarrow{(+)} & e^{x} \\ 120 & \xrightarrow{(-)} & e^{x} \\ 0 & & \end{array}$$

$$\int x^5 e^x\,dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + C$$
$$= \left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + C$$

20.

		e^{4t}
t^2	$\xrightarrow{(+)}$	$\frac{1}{4}e^{4t}$
$2t$	$\xrightarrow{(-)}$	$\frac{1}{16}e^{4t}$
2	$\xrightarrow{(+)}$	$\frac{1}{64}e^{4t}$
0		

$$\int t^2e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right)e^{4t} + C$$

21. $I = \int e^\theta \sin\theta\,d\theta$; $[u = \sin\theta,\ du = \cos\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I = e^\theta\sin\theta - \int e^\theta\cos\theta\,d\theta$;
$[u = \cos\theta,\ du = -\sin\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I = e^\theta\sin\theta - \left(e^\theta\cos\theta + \int e^\theta\sin\theta\,d\theta\right)$
$= e^\theta\sin\theta - e^\theta\cos\theta - I + C' \Rightarrow 2I = (e^\theta\sin\theta - e^\theta\cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^\theta\sin\theta - e^\theta\cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y}\cos y\,dy$; $[u = \cos y,\ du = -\sin y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow I = -e^{-y}\cos y - \int(-e^{-y})(-\sin y)\,dy = -e^{-y}\cos y - \int e^{-y}\sin y\,dy$; $[u = \sin y,\ du = \cos y\,dy;$
$dv = e^{-y}\,dy,\ v = -e^{-y}] \Rightarrow I = -e^{-y}\cos y - \left(-e^{-y}\sin y - \int(-e^{y})\cos y\,dy\right) = -e^{-y}\cos y + e^{-y}\sin y - I + C'$
$\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y}\sin y - e^{-y}\cos y) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x}\cos 3x\,dx$; $\left[u = \cos 3x;\ du = -3\sin 3x\,dx,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$
$\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$; $\left[u = \sin 3x,\ du = 3\cos 3x,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$
$\Rightarrow I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I + C'$
$\Rightarrow \frac{13}{4}I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C' \Rightarrow \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$, where $C = \frac{4}{13}C'$

24. $\int e^{-2x}\sin 2x\,dx$; $[y = 2x] \rightarrow \frac{1}{2}\int e^{-y}\sin y\,dy = I$; $[u = \sin y,\ du = \cos y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow I = \frac{1}{2}\left(-e^{-y}\sin y + \int e^{-y}\cos y\,dy\right)$ $[u = \cos y,\ du = -\sin y;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow I = -\frac{1}{2}e^{-y}\sin y + \frac{1}{2}\left(-e^{-y}\cos y - \int(-e^{-y})(-\sin y)\,dy\right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C'$
$\Rightarrow 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where $C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}}\,ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3}x\,dx \end{bmatrix} \rightarrow \int e^x \cdot \frac{2}{3}x\,dx = \frac{2}{3}\int xe^x\,dx$; $[u = x,\ du = dx;\ dv = e^x\,dx,\ v = e^x]$;
$\frac{2}{3}\int xe^x\,dx = \frac{2}{3}\left(xe^x - \int e^x\,dx\right) = \frac{2}{3}(xe^x - e^x) + C = \frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x,\ du = dx;\ dv = \sqrt{1-x}\,dx,\ v = -\frac{2}{3}\sqrt{(1-x)^3}$;
$\int_0^1 x\sqrt{1-x}\,dx = \left[-\frac{2}{3}\sqrt{(1-x)^3}\,x\right]_0^1 + \frac{2}{3}\int_0^1\sqrt{(1-x)^3}\,dx = \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$

27. $u = x,\ du = dx;\ dv = \tan^2 x\,dx,\ v = \int\tan^2 x\,dx = \int\frac{\sin^2 x}{\cos^2 x}\,dx = \int\frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\frac{dx}{\cos^2 x} - \int dx$
$= \tan x - x$; $\int_0^{\pi/3} x\tan^2 x\,dx = [x(\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3}(\tan x - x)\,dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$
$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2)$, $du = \frac{(2x+1)\,dx}{x+x^2}$; $dv = dx$, $v = x$; $\int \ln(x + x^2)\,dx = x\ln(x + x^2) - \int \frac{2x+1}{x(x+1)} \cdot x\,dx$
$= x\ln(x + x^2) - \int \frac{(2x+1)\,dx}{x+1} = x\ln(x + x^2) - \int \frac{2(x+1)-1}{x+1}\,dx = x\ln(x + x^2) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x)\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \rightarrow \int (\sin u)\,e^u\,du$. From Exercise 21, $\int (\sin u)\,e^u\,du = e^u\left(\frac{\sin u - \cos u}{2}\right) + C$
$= \frac{1}{2}\left[-x\cos(\ln x) + x\sin(\ln x)\right] + C$

30. $\int z(\ln z)^2\,dz$; $\begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\,dz \\ dz = e^u\,du \end{bmatrix} \rightarrow \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du$;

		e^{2u}
u^2	$\xrightarrow{(+)}$	$\frac{1}{2}e^{2u}$
$2u$	$\xrightarrow{(-)}$	$\frac{1}{4}e^{2u}$
2	$\xrightarrow{(+)}$	$\frac{1}{8}e^{2u}$
0		

$\int u^2e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$
$= \frac{z^2}{4}\left[2(\ln z)^2 - 2\ln z + 1\right] + C$

31. $\int x\sec x^2\,dx$ $\left[\text{Let } u = x^2,\ du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx\right] \rightarrow \int x\sec x^2\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$
$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C$

32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ $\left[\text{Let } u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx\right] \rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

33. $\int x(\ln x)^2\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^u\,du \end{bmatrix} \rightarrow \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du$;

		e^{2u}
u^2	$\xrightarrow{(+)}$	$\frac{1}{2}e^{2u}$
$2u$	$\xrightarrow{(-)}$	$\frac{1}{4}e^{2u}$
2	$\xrightarrow{(+)}$	$\frac{1}{8}e^{2u}$
0		

$\int u^2e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$
$= \frac{x^2}{4}\left[2(\ln x)^2 - 2\ln x + 1\right] + C = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$

34. $\int \frac{1}{x(\ln x)^2}\,dx$ $\left[\text{Let } u = \ln x,\ du = \frac{1}{x}\,dx\right] \rightarrow \int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$

35. $u = \ln x$, $du = \frac{1}{x}\,dx$; $dv = \frac{1}{x^2}\,dx$, $v = -\frac{1}{x}$;
$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$

36. $\int \frac{(\ln x)^3}{x}\,dx$ $\left[\text{Let } u = \ln x,\ du = \frac{1}{x}\,dx\right] \rightarrow \int \frac{(\ln x)^3}{x}\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln x)^4 + C$

37. $\int x^3 e^{x^4}\,dx$ $\left[\text{Let } u = x^4,\ du = 4x^3\,dx \Rightarrow \frac{1}{4}du = x^3\,dx\right] \to \int x^3 e^{x^4}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{x^4} + C$

38. $u = x^3,\ du = 3x^2\,dx;\ dv = x^2 e^{x^3}\,dx,\ v = \frac{1}{3}e^{x^3}$;
$\int x^5 e^{x^3}\,dx = \int x^3 e^{x^3} x^2 dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int e^{x^3} 3x^2 dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C$

39. $u = x^2,\ du = 2x\,dx;\ dv = \sqrt{x^2+1}\,x\,dx,\ v = \frac{1}{3}(x^2+1)^{3/2}$;
$\int x^3\sqrt{x^2+1}\,dx = \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{1}{3}\int (x^2+1)^{3/2} 2x\,dx = \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{2}{15}(x^2+1)^{5/2} + C$

40. $\int x^2 \sin x^3\,dx$ $\left[\text{Let } u = x^3,\ du = 3x^2\,dx \Rightarrow \frac{1}{3}du = x^2\,dx\right] \to \int x^2 \sin x^3\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C$
$= -\frac{1}{3}\cos x^3 + C$

41. $u = \sin 3x,\ du = 3\cos 3x\,dx;\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x$;
$\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\int \cos 3x \sin 2x\,dx$
$u = \cos 3x,\ du = -3\sin 3x\,dx;\ dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x$;
$\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x - \frac{3}{2}\left[-\frac{1}{2}\cos 3x \cos 2x - \frac{3}{2}\int \sin 3x \cos 2x\,dx\right]$
$= \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x + \frac{9}{4}\int \sin 3x \cos 2x\,dx \Rightarrow -\frac{5}{4}\int \sin 3x \cos 2x\,dx = \frac{1}{2}\sin 3x \sin 2x + \frac{3}{4}\cos 3x \cos 2x$
$\Rightarrow \int \sin 3x \cos 2x\,dx = -\frac{2}{5}\sin 3x \sin 2x - \frac{3}{5}\cos 3x \cos 2x + C$

42. $u = \sin 2x,\ du = 2\cos 2x\,dx;\ dv = \cos 4x\,dx,\ v = \frac{1}{4}\sin 4x$;
$\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\int \cos 2x \sin 4x\,dx$
$u = \cos 2x,\ du = -2\sin 2x\,dx;\ dv = \sin 4x\,dx,\ v = -\frac{1}{4}\cos 4x$;
$\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x - \frac{1}{2}\left[-\frac{1}{4}\cos 2x \cos 4x - \frac{1}{2}\int \sin 2x \cos 4x\,dx\right]$
$= \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x + \frac{1}{4}\int \sin 2x \cos 4x\,dx \Rightarrow \frac{3}{4}\int \sin 2x \cos 4x\,dx = \frac{1}{4}\sin 2x \sin 4x + \frac{1}{8}\cos 2x \cos 4x$
$\Rightarrow \int \sin 2x \cos 4x\,dx = \frac{1}{3}\sin 2x \sin 4x + \frac{1}{6}\cos 2x \cos 4x + C$

43. $\int e^x \sin e^x\,dx$ $\left[\text{Let } u = e^x,\ du = e^x\,dx\right] \to \int e^x \sin e^x\,dx = \int \sin u\,du = -\cos u + C = -\cos e^x + C$

44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ $\left[\text{Let } u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx\right] \to \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$

45. $\int \cos\sqrt{x}\,dx;\ \begin{bmatrix} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}}\,dx \\ dx = 2y\,dy \end{bmatrix} \to \int \cos y\, 2y\,dy = \int 2y \cos y\,dy;$
$u = 2y,\ du = 2\,dy;\ dv = \cos y\,dy,\ v = \sin y$;
$\int 2y \cos y\,dy = 2y \sin y - \int 2 \sin y\,dy = 2y \sin y + 2\cos y + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx;\ \begin{bmatrix} y = \sqrt{x} \\ dy = \frac{1}{2\sqrt{x}}\, dx \\ dx = 2y\, dy \end{bmatrix} \to \int y\, e^y\, 2y\, dy = \int 2y^2 e^y\, dy;$

		e^y
$2y^2$	$\xrightarrow{(+)}$	e^y
$4y$	$\xrightarrow{(-)}$	e^y
4	$\xrightarrow{(+)}$	e^y
0		

$$\int 2y^2 e^y\, dy = 2y^2 e^y - 4y\, e^y + 4\, e^y + C = 2x\, e^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$

47.

		$\sin 2\theta$
θ^2	$\xrightarrow{(+)}$	$-\frac{1}{2}\cos 2\theta$
2θ	$\xrightarrow{(-)}$	$-\frac{1}{4}\sin 2\theta$
2	$\xrightarrow{(+)}$	$\frac{1}{8}\cos 2\theta$
0		

$$\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$
$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

48.

		$\cos 2x$
x^3	$\xrightarrow{(+)}$	$\frac{1}{2}\sin 2x$
$3x^2$	$\xrightarrow{(-)}$	$-\frac{1}{4}\cos 2x$
$6x$	$\xrightarrow{(+)}$	$-\frac{1}{8}\sin 2x$
6	$\xrightarrow{(-)}$	$\frac{1}{16}\cos 2x$
0		

$$\int_0^{\pi/2} x^3 \cos 2x\, dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$
$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3\,(4 - \pi^2)}{16}$$

49. $u = \sec^{-1} t,\ du = \frac{dt}{t\sqrt{t^2-1}};\ dv = t\, dt,\ v = \frac{t^2}{2};$

$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2}\left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2}\frac{t\, dt}{2\sqrt{t^2-1}}$$
$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3} - 1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

50. $u = \sin^{-1}(x^2),\ du = \frac{2x\, dx}{\sqrt{1-x^4}};\ dv = 2x\, dx,\ v = x^2;$

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = \left[x^2 \sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2\cdot\frac{2x\, dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}}\frac{d\,(1-x^4)}{2\sqrt{1-x^4}}$$
$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

51. (a) $u = x,\ du = dx;\ dv = \sin x\, dx,\ v = -\cos x;$

$$S_1 = \int_0^{\pi} x \sin x\, dx = [-x\cos x]_0^{\pi} + \int_0^{\pi}\cos x\, dx = \pi + [\sin x]_0^{\pi} = \pi$$

(b) $S_2 = -\int_{\pi}^{2\pi} x \sin x\, dx = -\left[[-x\cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x\, dx\right] = -\left[-3\pi + [\sin x]_{\pi}^{2\pi}\right] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x \sin x\, dx = [-x\cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x\, dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1}\int_{n\pi}^{(n+1)\pi} x \sin x\, dx = (-1)^{n+1}\left[[-x\cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi}\right]$
$= (-1)^{n+1}\left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}\right] + 0 = (2n+1)\pi$

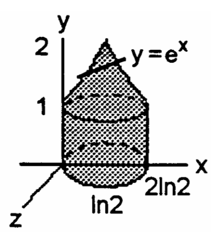
52. (a) $u = x,\ du = dx;\ dv = \cos x\, dx,\ v = \sin x;$
$S_1 = -\int_{\pi/2}^{3\pi/2} x\cos x\, dx = -\left[[x\sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x\, dx\right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x\cos x\, dx = [x\sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x\, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$

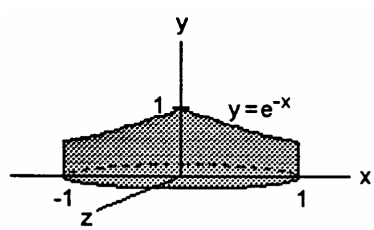
(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x\cos x\, dx = -\left[[x\sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x\, dx\right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$

(d) $S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x\cos x\, dx = (-1)^n\left[[x\sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - {}^n\int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x\, dx\right]$
$= (-1)^n\left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1}\right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

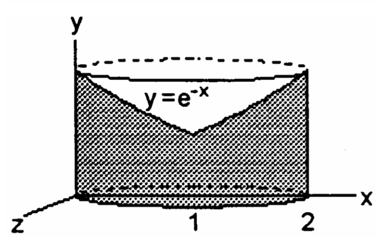
53. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)e^x\, dx = 2\pi \ln 2 \int_0^{\ln 2} e^x\, dx - 2\pi\int_0^{\ln 2} xe^x\, dx$
$= (2\pi \ln 2)\left[e^x\right]_0^{\ln 2} - 2\pi\left(\left[xe^x\right]_0^{\ln 2} - \int_0^{\ln 2} e^x\, dx\right)$
$= 2\pi \ln 2 - 2\pi\left(2\ln 2 - \left[e^x\right]_0^{\ln 2}\right) = -2\pi \ln 2 + 2\pi = 2\pi(1 - \ln 2)$



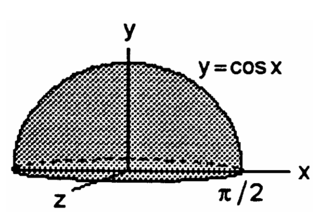
54. (a) $V = \int_0^1 2\pi xe^{-x}\, dx = 2\pi\left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\, dx\right)$
$= 2\pi\left(-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$
$= 2\pi - \frac{4\pi}{e}$



(b) $V = \int_0^1 2\pi(1-x)e^{-x}\, dx;\ u = 1 - x,\ du = -\, dx;\ dv = e^{-x}\, dx,$
$v = -e^{-x}\ ;\ V = 2\pi\left[\left[(1-x)\left(-e^{-x}\right)\right]_0^1 - \int_0^1 e^{-x}\, dx\right]$
$= 2\pi\left[[0 - 1(-1)] + \left[e^{-x}\right]_0^1\right] = 2\pi\left(1 + \frac{1}{e} - 1\right) = \frac{2\pi}{e}$



55. (a) $V = \int_0^{\pi/2} 2\pi x\cos x\, dx = 2\pi\left([x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx\right)$
$= 2\pi\left(\frac{\pi}{2} + [\cos x]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2} + 0 - 1\right) = \pi(\pi - 2)$



(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2} - x\right)\cos x\, dx;\ u = \frac{\pi}{2} - x,\ du = -\, dx;\ dv = \cos x\, dx,\ v = \sin x;$
$V = 2\pi\left[\left(\frac{\pi}{2} - x\right)\sin x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2} \sin x\, dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0+1) = 2\pi$

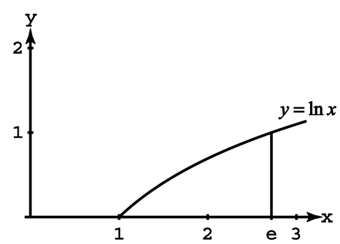
56. (a) $V = \int_0^{\pi} 2\pi x(x \sin x)\, dx;$

		$\sin x$
x^2	$\xrightarrow{(+)}$	$-\cos x$
$2x$	$\xrightarrow{(-)}$	$-\sin x$
2	$\xrightarrow{(+)}$	$\cos x$
0		

$\Rightarrow V = 2\pi \int_0^{\pi} x^2 \sin x\, dx = 2\pi \left[-x^2 \cos x + 2x \sin x + 2 \cos x\right]_0^{\pi} = 2\pi\left(\pi^2 - 4\right)$

(b) $V = \int_0^{\pi} 2\pi(\pi - x)x \sin x\, dx = 2\pi^2 \int_0^{\pi} x \sin x\, dx - 2\pi \int_0^{\pi} x^2 \sin x\, dx = 2\pi^2 \left[-x \cos x + \sin x\right]_0^{\pi} - \left(2\pi^3 - 8\pi\right)$
$= 8\pi$

57. (a) $A = \int_1^e \ln x\, dx = \left[x \ln x\right]_1^e - \int_1^e dx$
$= (e \ln e - 1 \ln 1) - \left[x\right]_1^e = e - (e - 1) = 1$

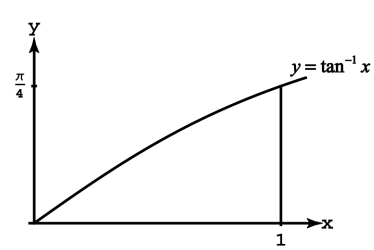


(b) $V = \int_1^e \pi (\ln x)^2\, dx = \pi\left(\left[x (\ln x)^2\right]_1^e - \int_1^e 2 \ln x\, dx\right)$
$= \pi\left[\left(e(\ln e)^2 - 1(\ln 1)^2\right) - \left(\left[2x \ln x\right]_1^e - \int_1^e 2\, dx\right)\right]$
$= \pi\left[e - \left((2e \ln e - 2(1) \ln 1) - \left[2x\right]_1^e\right)\right]$
$= \pi\left[e - (2e - (2e - 2))\right] = \pi(e - 2)$

(c) $V = \int_1^e 2\pi(x + 2) \ln x\, dx = 2\pi \int_1^e (x + 2) \ln x\, dx = 2\pi\left(\left[\left(\tfrac{1}{2}x^2 + 2x\right)\ln x\right]_1^e - \int_1^e \left(\tfrac{1}{2}x + 2\right) dx\right)$
$= 2\pi\left(\left(\tfrac{1}{2}e^2 + 2e\right)\ln e - \left(\tfrac{1}{2} + 2\right)\ln 1 - \left[\left(\tfrac{1}{4}x^2 + 2x\right)\right]_1^e\right) = 2\pi\left(\left(\tfrac{1}{2}e^2 + 2e\right) - \left(\left(\tfrac{1}{4}e^2 + 2e\right) - \tfrac{9}{4}\right)\right) = \tfrac{\pi}{2}\left(e^2 + 9\right)$

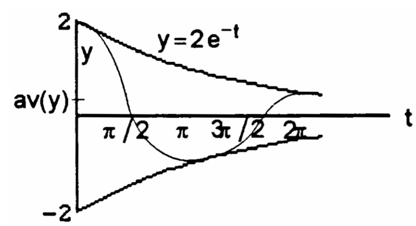
(d) $M = \int_1^e \ln x\, dx = 1$ (from part (a)); $\bar{x} = \tfrac{1}{1}\int_1^e x \ln x\, dx = \left[\tfrac{1}{2}x^2 \ln x\right]_1^e - \int_1^e \tfrac{1}{2}x\, dx = \left(\tfrac{1}{2}e^2 \ln e - \tfrac{1}{2}(1)^2 \ln 1\right) - \left[\tfrac{1}{4}x^2\right]_1^e$
$= \tfrac{1}{2}e^2 - \left(\tfrac{1}{4}e^2 - \tfrac{1}{4}(1)^2\right) = \tfrac{1}{4}\left(e^2 + 1\right); \bar{y} = \tfrac{1}{1}\int_1^e \tfrac{1}{2}(\ln x)^2\, dx = \tfrac{1}{2}\left(\left[x (\ln x)^2\right]_1^e - \int_1^e 2 \ln x\, dx\right)$
$= \tfrac{1}{2}\left(\left(e (\ln e)^2 - 1 \cdot (\ln 1)^2\right) - \left(\left[2x \ln x\right]_1^e - \int_1^e 2\, dx\right)\right) = \tfrac{1}{2}\left(e - \left((2e \ln e - 2(1) \ln 1) - \left[2x\right]_1^e\right)\right)$
$= \tfrac{1}{2}(e - 2e + 2e - 2) = \tfrac{1}{2}(e - 2) \Rightarrow (\bar{x}, \bar{y}) = \left(\tfrac{e^2 + 1}{4}, \tfrac{e - 2}{2}\right)$ is the centroid.

58. (a) $A = \int_0^1 \tan^{-1} x\, dx = \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \tfrac{x}{1 + x^2} dx$
$= \left(\tan^{-1} 1 - 0\right) - \tfrac{1}{2}\left[\ln\left(1 + x^2\right)\right]_0^1$
$= \tfrac{\pi}{4} - \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$

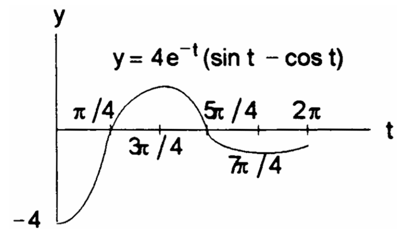


(b) $V = \int_0^1 2\pi x \tan^{-1} x\, dx$
$= 2\pi\left(\left[\tfrac{x^2}{2} \tan^{-1} x\right]_0^1 - \tfrac{1}{2}\int_0^1 \tfrac{x^2}{1 + x^2} dx\right)$
$= 2\pi\left(\tfrac{1}{2} \tan^{-1} 1 - 0 - \tfrac{1}{2}\int_0^1 \left(1 - \tfrac{1}{1 + x^2}\right) dx\right) = 2\pi\left(\tfrac{\pi}{8} - \tfrac{1}{2}\left[x - \tan^{-1} x\right]_0^1\right) = 2\pi\left(\tfrac{\pi}{8} - \tfrac{1}{2}\left(1 - \tan^{-1} 1 - (0 - 0)\right)\right)$
$= 2\pi\left(\tfrac{\pi}{8} - \tfrac{1}{2}\left(1 - \tfrac{\pi}{4}\right)\right) = \tfrac{\pi(\pi - 2)}{2}$

59. $\text{av(y)} = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt$
$= \frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
(see Exercise 22) $\Rightarrow \text{av(y)} = \frac{1}{2\pi}\left(1 - e^{-2\pi}\right)$



60. $\text{av(y)} = \frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t - \cos t)\,dt$
$= \frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt - \frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$
$= \frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t - \cos t}{2}\right) - e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
$= \frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi} = 0$



61. $I = \int x^n \cos x\,dx;\ [u = x^n,\ du = nx^{n-1}\,dx;\ dv = \cos x\,dx,\ v = \sin x]$
$\Rightarrow I = x^n \sin x - \int nx^{n-1}\sin x\,dx$

62. $I = \int x^n \sin x\,dx;\ [u = x^n,\ du = nx^{n-1}\,dx;\ dv = \sin x\,dx,\ v = -\cos x]$
$\Rightarrow I = -x^n \cos x + \int nx^{n-1}\cos x\,dx$

63. $I = \int x^n e^{ax}\,dx;\ \left[u = x^n,\ du = nx^{n-1}\,dx;\ dv = e^{ax}\,dx,\ v = \frac{1}{a}e^{ax}\right]$
$\Rightarrow I = \frac{x^n e^{ax}}{a}e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx,\ a \neq 0$

64. $I = \int (\ln x)^n\,dx;\ \left[u = (\ln x)^n,\ du = \frac{n(\ln x)^{n-1}}{x}\,dx;\ dv = 1\,dx,\ v = x\right]$
$\Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1}\,dx$

65. $\int_a^b (x-a)\,f(x)\,dx;\ \left[u = x - a,\ du = dx;\ dv = f(x)\,dx,\ v = \int_b^x f(t)\,dt = -\int_x^b f(t)\,dt\right]$
$= \left[(x-a)\int_b^x f(t)\,dt\right]_a^b - \int_a^b\left(\int_b^x f(t)\,dt\right)dx = \left((b-a)\int_b^b f(t)\,dt - (a-a)\int_b^a f(t)\,dt\right) - \int_a^b\left(-\int_x^b f(t)\,dt\right)dx$
$= 0 + \int_a^b\left(\int_x^b f(t)\,dt\right)dx = \int_a^b\left(\int_x^b f(t)\,dt\right)dx$

66. $\int \sqrt{1-x^2}\,dx;\ \left[u = \sqrt{1-x^2},\ du = \frac{-x}{\sqrt{1-x^2}}dx;\ dv = dx,\ v = x\right]$
$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = x\sqrt{1-x^2} - \left(\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx\right)$
$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$
$\Rightarrow \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \sqrt{1-x^2}\,dx \Rightarrow 2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{1}{\sqrt{1-x^2}}\,dx$
$\Rightarrow \int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx + C$

67. $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \sin y\,dy = x\sin^{-1} x + \cos y + C = x\sin^{-1} x + \cos\left(\sin^{-1} x\right) + C$

68. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \tan y\,dy = x\tan^{-1} x + \ln|\cos y| + C = x\tan^{-1} x + \ln\left|\cos\left(\tan^{-1} x\right)\right| + C$

69. $\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \sec y\, dy = x \sec^{-1} x - \ln|\sec y + \tan y| + C$
$= x \sec^{-1} x - \ln|\sec(\sec^{-1} x) + \tan(\sec^{-1} x)| + C = x \sec^{-1} x - \ln\left|x + \sqrt{x^2 - 1}\right| + C$

70. $\int \log_2 x\, dx = x \log_2 x - \int 2^y\, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{x}{\ln 2} + C$

71. Yes, $\cos^{-1} x$ is the angle whose cosine is x which implies $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$.

72. Yes, $\tan^{-1} x$ is the angle whose tangent is x which implies $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$.

73. (a) $\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int \sinh y\, dy = x \sinh^{-1} x - \cosh y + C = x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C$;
check: $d[x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh(\sinh^{-1} x) \frac{1}{\sqrt{1+x^2}}\right] dx$
$= \sinh^{-1} x\, dx$
(b) $\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int x \left(\frac{1}{\sqrt{1+x^2}}\right) dx = x \sinh^{-1} x - \frac{1}{2} \int (1 + x^2)^{-1/2} 2x\, dx$
$= x \sinh^{-1} x - (1 + x^2)^{1/2} + C$
check: $d\left[x \sinh^{-1} x - (1 + x^2)^{1/2} + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right] dx = \sinh^{-1} x\, dx$

74. (a) $\int \tanh^{-1} x\, dx = x \tanh^{-1} x - \int \tanh y\, dy = x \tanh^{-1} x - \ln|\cosh y| + C = x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C$;
check: $d[x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{\sinh(\tanh^{-1} x)}{\cosh(\tanh^{-1} x)} \frac{1}{1-x^2}\right] dx$
$= \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\, dx$
(b) $\int \tanh^{-1} x\, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2}\, dx = x \tanh^{-1} x - \frac{1}{2} \int \frac{2x}{1-x^2}\, dx = x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C$
check: $d\left[x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\, dx$

8.2 TRIGONOMETRIC INTEGRALS

1. $\int \cos 2x\, dx = \frac{1}{2} \int \cos 2x \cdot 2dx = \frac{1}{2} \sin 2x + C$

2. $\int_0^{\pi} 3 \sin \frac{x}{3}\, dx = 9 \int_0^{\pi} \sin \frac{x}{3} \cdot \frac{1}{3} dx = 9\left[-\cos \frac{x}{3}\right]_0^{\pi} = 9\left(-\cos \frac{\pi}{3} + \cos 0\right) = 9\left(-\frac{1}{2} + 1\right) = \frac{9}{2}$

3. $\int \cos^3 x \sin x\, dx = -\int \cos^3 x (-\sin x) dx = -\frac{1}{4} \cos^4 x + C$

4. $\int \sin^4 2x \cos 2x\, dx = \frac{1}{2} \int \sin^4 2x \cos 2x \cdot 2dx = \frac{1}{10} \sin^5 2x + C$

5. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int \left(1 - \cos^2 x\right) \sin x\, dx = \int \sin x\, dx - \int \cos^2 x \sin x\, dx = -\cos x + \frac{1}{3} \cos^3 x + C$

6. $\int \cos^3 4x\, dx = \int \cos^2 4x \cos 4x\, dx = \frac{1}{4} \int \left(1 - \sin^2 4x\right) \cos 4x \cdot 4dx = \frac{1}{4} \int \cos 4x \cdot 4dx - \frac{1}{4} \int \sin^2 4x \cos 4x \cdot 4dx$
$= \frac{1}{4} \sin 4x - \frac{1}{12} \sin^3 4x + C$

7. $\int \sin^5 x\, dx = \int (\sin^2 x)^2 \sin x\, dx = \int (1 - \cos^2 x)^2 \sin x\, dx = \int (1 - 2\cos^2 x + \cos^4 x) \sin x\, dx$
$= \int \sin x\, dx - \int 2\cos^2 x \sin x\, dx + \int \cos^4 x \sin x\, dx = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$

8. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right) dx$ (using Exercise 7) $= \int_0^{\pi} \sin\left(\frac{x}{2}\right) dx - \int_0^{\pi} 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx + \int_0^{\pi} \cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx$
$= \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)\right]_0^{\pi} = (0) - \left(-2 + \frac{4}{3} - \frac{2}{5}\right) = \frac{16}{15}$

9. $\int \cos^3 x\, dx = \int (\cos^2 x)\cos x\, dx = \int (1 - \sin^2 x)\cos x\, dx = \int \cos x\, dx - \int \sin^2 x \cos x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$

10. $\int_0^{\pi/6} 3\cos^5 3x\, dx = \int_0^{\pi/6} (\cos^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - \sin^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - 2\sin^2 3x + \sin^4 3x)\cos 3x \cdot 3dx$
$= \int_0^{\pi/6} \cos 3x \cdot 3dx - 2\int_0^{\pi/6} \sin^2 3x \cos 3x \cdot 3dx + \int_0^{\pi/6} \sin^4 3x \cos 3x \cdot 3dx = \left[\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5}\right]_0^{\pi/6}$
$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0) = \frac{8}{15}$

11. $\int \sin^3 x \cos^3 x\, dx = \int \sin^3 x \cos^2 x \cos x dx = \int \sin^3 x\,(1 - \sin^2 x)\cos x\, dx = \int \sin^3 x \cos x\, dx - \int \sin^5 x \cos x\, dx$
$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

12. $\int \cos^3 2x \sin^5 2x\, dx = \frac{1}{2}\int \cos^3 2x \sin^5 2x \cdot 2dx = \frac{1}{2}\int \cos 2x \cos^2 2x \sin^5 2x \cdot 2dx = \frac{1}{2}\int (1 - \sin^2 2x) \sin^5 2x \cos 2x \cdot 2dx$
$= \frac{1}{2}\int \sin^5 2x \cos 2x \cdot 2dx - \frac{1}{2}\int \sin^7 2x \cos 2x \cdot 2dx = \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C$

13. $\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2}\, dx = \frac{1}{2}\int \left(1 + \cos 2x\right) dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x \cdot 2dx$
$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

14. $\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}\int_0^{\pi/2} \left(1 - \cos 2x\right) dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2} \cos 2x\, dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{4}\int_0^{\pi/2} \cos 2x \cdot 2dx$
$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \left(\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin 2(0)\right) = \left(\frac{\pi}{4} - 0\right) - (0 - 0) = \frac{\pi}{4}$

15. $\int_0^{\pi/2} \sin^7 y\, dy = \int_0^{\pi/2} \sin^6 y \sin y\, dy = \int_0^{\pi/2} (1 - \cos^2 y)^3 \sin y\, dy = \int_0^{\pi/2} \sin y\, dy - 3\int_0^{\pi/2} \cos^2 y \sin y\, dy$
$+ 3\int_0^{\pi/2} \cos^4 y \sin y\, dy - \int_0^{\pi/2} \cos^6 y \sin y\, dy = \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7}\right) = \frac{16}{35}$

16. $\int 7\cos^7 t\, dt$ (using Exercise 15) $= 7\left[\int \cos t\, dt - 3\int \sin^2 t \cos t\, dt + 3\int \sin^4 t \cos t\, dt - \int \sin^6 t \cos t\, dt\right]$
$= 7\left(\sin t - 3\frac{\sin^3 t}{3} + 3\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right) + C = 7\sin t - 7\sin^3 t + \frac{21}{5}\sin^5 t - \sin^7 t + C$

17. $\int_0^{\pi} 8\sin^4 x\, dx = 8\int_0^{\pi} \left(\frac{1 - \cos 2x}{2}\right)^2 dx = 2\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) dx = 2\int_0^{\pi} dx - 2\int_0^{\pi} \cos 2x \cdot 2dx + 2\int_0^{\pi} \frac{1 + \cos 4x}{2}\, dx$
$= [2x - 2\sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x\, dx = 2\pi + \left[x + \frac{1}{2}\sin 4x\right]_0^{\pi} = 2\pi + \pi = 3\pi$

18. $\int 8\cos^4 2\pi x\, dx = 8\int \left(\frac{1 + \cos 4\pi x}{2}\right)^2 dx = 2\int (1 + 2\cos 4\pi x + \cos^2 4\pi x) dx = 2\int dx + 4\int \cos 4\pi x\, dx + 2\int \frac{1 + \cos 8\pi x}{2}\, dx$
$= 3\int dx + 4\int \cos 4\pi x\, dx + \int \cos 8\pi x\, dx = 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C$

19. $\int 16 \sin^2 x \cos^2 x\, dx = 16\int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) dx = 4\int (1 - \cos^2 2x) dx = 4\int dx - 4\int \left(\frac{1 + \cos 4x}{2}\right) dx$
$= 4x - 2\int dx - 2\int \cos 4x\, dx = 4x - 2x - \frac{1}{2}\sin 4x + C = 2x - \frac{1}{2}\sin 4x + C = 2x - \sin 2x \cos 2x + C$
$= 2x - 2\sin x \cos x\,(2\cos^2 x - 1) + C = 2x - 4\sin x \cos^3 x + 2\sin x \cos x + C$

20. $\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy = 8\int_0^{\pi}\left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy = \int_0^{\pi} dy - \int_0^{\pi}\cos 2y\,dy - \int_0^{\pi}\cos^2 2y\,dy + \int_0^{\pi}\cos^3 2y\,dy$
$= \left[y - \frac{1}{2}\sin 2y\right]_0^{\pi} - \int_0^{\pi}\left(\frac{1+\cos 4y}{2}\right)dy + \int_0^{\pi}(1-\sin^2 2y)\cos 2y\,dy = \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi}\cos 4y\,dy + \int_0^{\pi}\cos 2y\,dy$
$- \int_0^{\pi}\sin^2 2y\cos 2y\,dy = \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2}\cdot\frac{\sin^3 2y}{3}\right]_0^{\pi} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

21. $\int 8\cos^3 2\theta\sin 2\theta\,d\theta = 8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4} + C = -\cos^4 2\theta + C$

22. $\int_0^{\pi/2}\sin^2 2\theta\cos^3 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta\cos 2\theta\,d\theta - \int_0^{\pi/2}\sin^4 2\theta\cos 2\theta\,d\theta$
$= \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

23. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi}\left|\sin\frac{x}{2}\right|dx = \int_0^{2\pi}\sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = 2+2 = 4$

24. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{\pi}\sqrt{2}\,|\sin 2x|\,dx = \int_0^{\pi}\sqrt{2}\sin 2x\,dx = \left[-\sqrt{2}\cos 2x\right]_0^{\pi} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

25. $\int_0^{\pi}\sqrt{1-\sin^2 t}\,dt = \int_0^{\pi}|\cos t|\,dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{\pi}\cos t\,dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{\pi} = 1 - 0 - 0 + 1 = 2$

26. $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}|\sin\theta|\,d\theta = \int_0^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = 1+1 = 2$

27. $\int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x}{\sqrt{1-\cos x}}\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}}\,dx = \int_{\pi/3}^{\pi/2}\frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{\sin^2 x}}\,dx$
$= \int_{\pi/3}^{\pi/2}\sin x\sqrt{1+\cos x}\,dx = \left[-\frac{2}{3}(1+\cos x)^{3/2}\right]_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(1+\cos\left(\frac{\pi}{2}\right)\right)^{3/2} + \frac{2}{3}\left(1+\cos\left(\frac{\pi}{3}\right)\right)^{3/2} = -\frac{2}{3} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}$
$= \sqrt{\frac{3}{2}} - \frac{2}{3}$

28. $\int_0^{\pi/6}\sqrt{1+\sin x}\,dx = \int_0^{\pi/6}\frac{\sqrt{1+\sin x}}{1}\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6}\frac{\cos x}{\sqrt{1-\sin x}}\,dx$
$= \left[-2(1-\sin x)^{1/2}\right]_0^{\pi/6} = -2\sqrt{1-\sin\left(\frac{\pi}{6}\right)} + 2\sqrt{1-\sin 0} = -2\sqrt{\frac{1}{2}} + 2\sqrt{1} = 2-\sqrt{2}$

29. $\int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x}{\sqrt{1-\sin x}}\frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{1-\sin^2 x}}\,dx = \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{\cos^2 x}}\,dx$
$= \int_{5\pi/6}^{\pi}\frac{\cos^4 x\sqrt{1+\sin x}}{-\cos x}\,dx = -\int_{5\pi/6}^{\pi}\cos^3 x\sqrt{1+\sin x}\,dx = -\int_{5\pi/6}^{\pi}\cos x(1-\sin^2 x)\sqrt{1+\sin x}\,dx$
$= -\int_{5\pi/6}^{\pi}\cos x\sqrt{1+\sin x}\,dx + \int_{5\pi/6}^{\pi}\cos x\sin^2 x\sqrt{1+\sin x}\,dx;\ u^2\sqrt{u}\,du$
$\left[\text{Let } u = 1+\sin x \Rightarrow u-1 = \sin x \Rightarrow du = \cos x\,dx,\ x = \frac{5\pi}{6} \Rightarrow u = 1+\sin\left(\frac{5\pi}{6}\right) = \frac{3}{2},\ x = \pi \Rightarrow u = 1+\sin\pi = 1\right]$
$= \left[-\frac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}(u-1)^2\sqrt{u}\,du = \left[-\frac{2}{3}(1+\sin x)^{3/2}\right]_{5\pi/6}^{\pi} + \int_{3/2}^{1}\left(u^{5/2} - 2u^{3/2} + \sqrt{u}\right)du$
$= \left(-\frac{2}{3}(1+\sin\pi)^{3/2} + \frac{2}{3}\left(1+\sin\left(\frac{5\pi}{6}\right)\right)^{3/2}\right) + \left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_{3/2}^{1}$
$= \left(-\frac{2}{3} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}\right) + \left(\frac{2}{7} - \frac{4}{5} + \frac{2}{3}\right) - \left(\frac{2}{7}\left(\frac{3}{2}\right)^{7/2} - \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}\right) = \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} - \frac{2}{7}\left(\frac{3}{2}\right)^{7/2} - \frac{18}{35}$

30. $\int_{\pi/2}^{7\pi/12}\sqrt{1-\sin 2x}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{1-\sin 2x}}{1}\frac{\sqrt{1+\sin 2x}}{\sqrt{1+\sin 2x}}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{1-\sin^2 2x}}{\sqrt{1+\sin 2x}}\,dx = \int_{\pi/2}^{7\pi/12}\frac{\sqrt{\cos^2 2x}}{\sqrt{1+\sin 2x}}\,dx$
$= \int_{\pi/2}^{7\pi/12}\frac{-\cos 2x}{\sqrt{1+\sin 2x}}\,dx = \left[-\sqrt{1+\sin 2x}\right]_{\pi/2}^{7\pi/12} = -\sqrt{1+\sin 2\left(\frac{7\pi}{12}\right)} + \sqrt{1+\sin 2\left(\frac{\pi}{2}\right)} = -\sqrt{\frac{1}{2}} + 1 = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

31. $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\, d\theta = \int_0^{\pi/2} \theta\sqrt{2}\,|\sin\theta|\, d\theta = \sqrt{2}\int_0^{\pi/2} \theta \sin\theta\, d\theta = \sqrt{2}\left[-\theta\cos\theta + \sin\theta\right]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$

32. $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\, dt = \int_{-\pi}^{\pi} (\sin^2 t)^{3/2}\, dt = \int_{-\pi}^{\pi} |\sin^3 t|\, dt = -\int_{-\pi}^{0} \sin^3 t\, dt + \int_0^{\pi} \sin^3 t\, dt = -\int_{-\pi}^{0} (1-\cos^2 t)\sin t\, dt$
$+ \int_0^{\pi} (1-\cos^2 t)\sin t\, dt = -\int_{-\pi}^{0} \sin t\, dt + \int_{-\pi}^{0} \cos^2 t \sin t\, dt + \int_0^{\pi} \sin t\, dt - \int_0^{\pi} \cos^2 t\sin t\, dt = \left[\cos t - \frac{\cos^3 t}{3}\right]_{-\pi}^{0}$
$+ \left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{8}{3}$

33. $\int \sec^2 x \tan x\, dx = \int \tan x \sec^2 x\, dx = \frac{1}{2}\tan^2 x + C$

34. $\int \sec x \tan^2 x\, dx = \int \sec x \tan x \tan x dx$; $u = \tan x$, $du = \sec^2 x\, dx$, $dv = \sec x \tan x\, dx$, $v = \sec x$;
$= \sec x \tan x - \int \sec^3 x\, dx = \sec x \tan x - \int \sec^2 x \sec x dx = \sec x \tan x - \int \left(\tan^2 x + 1\right)\sec x dx$
$= \sec x \tan x - \left(\int \tan^2 x \sec x dx + \int \sec x dx\right) = \sec x \tan x - \ln|\sec x + \tan x| - \int \tan^2 x \sec x dx$
$\Rightarrow \int \sec x \tan^2 x\, dx = \sec x \tan x - \ln|\sec x + \tan x| - \int \tan^2 x \sec x dx$
$\Rightarrow 2\int \tan^2 x \sec x dx = \sec x \tan x - \ln|\sec x + \tan x| \Rightarrow \int \tan^2 x \sec x dx = \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$

35. $\int \sec^3 x \tan x\, dx = \int \sec^2 x \sec x \tan x\, dx = \frac{1}{3}\sec^3 x + C$

36. $\int \sec^3 x \tan^3 x\, dx = \int \sec^2 x \tan^2 x \sec x \tan x\, dx = \int \sec^2 x(\sec^2 x - 1)\sec x \tan x\, dx$
$= \int \sec^4 x \sec x \tan x\, dx - \int \sec^2 x \sec x \tan x\, dx = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

37. $\int \sec^2 x \tan^2 x\, dx = \int \tan^2 x \sec^2 x\, dx = \frac{1}{3}\tan^3 x + C$

38. $\int \sec^4 x \tan^2 x\, dx = \int \sec^2 x \tan^2 x \sec^2 x\, dx = \int \left(\tan^2 x + 1\right)\tan^2 x \sec^2 x\, dx = \int \tan^4 x \sec^2 x\, dx + \int \tan^2 x \sec^2 x\, dx$
$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

39. $\int_{-\pi/3}^{0} 2\sec^3 x\, dx$; $u = \sec x$, $du = \sec x \tan x\, dx$, $dv = \sec^2 x\, dx$, $v = \tan x$;
$\int_{-\pi/3}^{0} 2\sec^3 x\, dx = \left[2\sec x \tan x\right]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0} \sec x \tan^2 x\, dx = 2\cdot 1\cdot 0 - 2\cdot 2\cdot\sqrt{3} - 2\int_{-\pi/3}^{0} \sec x\,(\sec^2 x - 1)dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0} \sec^3 x\, dx + 2\int_{-\pi/3}^{0} \sec x\, dx$; $2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + \left[2\ln|\sec x + \tan x|\right]_{-\pi/3}^{0}$
$2\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 4\sqrt{3} + 2\ln|1+0| - 2\ln|2 - \sqrt{3}| = 4\sqrt{3} - 2\ln\left(2 - \sqrt{3}\right)$
$\int_{-\pi/3}^{0} 2\sec^3 x\, dx = 2\sqrt{3} - \ln\left(2 - \sqrt{3}\right)$

40. $\int e^x \sec^3(e^x) dx$; $u = \sec(e^x)$, $du = \sec(e^x)\tan(e^x)e^x dx$, $dv = \sec^2(e^x)e^x dx$, $v = \tan(e^x)$.
$\int e^x \sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) - \int \sec(e^x)\tan^2(e^x)e^x dx$
$= \sec(e^x)\tan(e^x) - \int \sec(e^x)(\sec^2(e^x) - 1)e^x dx$
$= \sec(e^x)\tan(e^x) - \int \sec^3(e^x)e^x dx + \int \sec(e^x)e^x dx$
$2\int e^x \sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right| + C$
$\int e^x \sec^3(e^x)\, dx = \frac{1}{2}\left(\sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right|\right) + C$

41. $\int \sec^4\theta\, d\theta = \int (1+\tan^2\theta)\sec^2\theta\, d\theta = \int \sec^2\theta\, d\theta + \int \tan^2\theta \sec^2\theta\, d\theta = \tan\theta + \frac{1}{3}\tan^3\theta + C$
$= \tan\theta + \frac{1}{3}\tan\theta(\sec^2\theta - 1) + C = \frac{1}{3}\tan\theta\sec^2\theta + \frac{2}{3}\tan\theta + C$

42. $\int 3\sec^4(3x)\, dx = \int (1+\tan^2(3x))\sec^2(3x)3dx = \int \sec^2(3x)3dx + \int \tan^2(3x)\sec^2(3x)3dx = \tan(3x) + \frac{1}{3}\tan^3(3x) + C$

43. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta = \int_{\pi/4}^{\pi/2} (1+\cot^2\theta)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \cot^2\theta\csc^2\theta\, d\theta = \left[-\cot\theta - \frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}$
$= (0) - \left(-1 - \frac{1}{3}\right) = \frac{4}{3}$

44. $\int \sec^6 x\, dx = \int \sec^4 x \sec^2 x\, dx = \int \left(\tan^2 x + 1\right)^2 \sec^2 x\, dx = \int \left(\tan^4 x + 2\tan^2 x + 1\right)\sec^2 x\, dx$
$= \int \tan^4 x \sec^2 x\, dx + 2\int \tan^2 x \sec^2 x\, dx + \int \sec^2 x\, dx = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

45. $\int 4\tan^3 x\, dx = 4\int (\sec^2 x - 1)\tan x\, dx = 4\int \sec^2 x \tan x\, dx - 4\int \tan x\, dx = 4\frac{\tan^2 x}{2} - 4\ln|\sec x| + C$
$= 2\tan^2 x - 4\ln|\sec x| + C = 2\tan^2 x - 2\ln|\sec^2 x| + C = 2\tan^2 x - 2\ln(1+\tan^2 x) + C$

46. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx = 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\tan^2 x\, dx = 6\int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} \tan^2 x\, dx$
$= 6\int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)dx = \left[6\frac{\tan^3 x}{3}\right]_{-\pi/4}^{\pi/4} - 6\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx + 6\int_{-\pi/4}^{\pi/4} dx$
$= 2(1-(-1)) - [6\tan x]_{-\pi/4}^{\pi/4} + [6x]_{-\pi/4}^{\pi/4} = 4 - 6(1-(-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$

47. $\int \tan^5 x\, dx = \int \tan^4 x \tan x\, dx = \int \left(\sec^2 x - 1\right)^2 \tan x\, dx = \int \left(\sec^4 x - 2\sec^2 x + 1\right)\tan x\, dx$
$= \int \sec^4 x \tan x\, dx - 2\int \sec^2 x \tan x\, dx + \int \tan x\, dx = \int \sec^3 x \sec x \tan x\, dx - 2\int \sec x \sec x \tan x\, dx + \int \tan x\, dx$
$= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C = \frac{1}{4}\left(\tan^2 x + 1\right)^2 - \left(\tan^2 x + 1\right) + \ln|\sec x| + C = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$

48. $\int \cot^6 2x\, dx = \int \cot^4 2x \cot^2 2x\, dx = \int \cot^4 2x\,(\csc^2 2x - 1)\, dx = \int \cot^4 2x \csc^2 2x\, dx - \int \cot^4 2x\, dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \cot^2 2x\, dx = \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x(\csc^2 2x - 1)dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \csc^2 2x\, dx + \int \cot^2 2x\, dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \csc^2 2x dx + \int (\csc^2 2x - 1)dx$
$= \int \cot^4 2x \csc^2 2x\, dx - \int \cot^2 2x \csc^2 2x dx + \int \csc^2 2x\, dx - \int dx = -\frac{1}{10}\cot^5 2x + \frac{1}{6}\cot^3 2x - \frac{1}{2}\cot 2x - x + C$

49. $\int_{\pi/6}^{\pi/3} \cot^3 x\, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1)\cot x\, dx = \int_{\pi/6}^{\pi/3} \csc^2 x \cot x\, dx - \int_{\pi/6}^{\pi/3} \cot x\, dx = \left[-\frac{\cot^2 x}{2} + \ln|\csc x|\right]_{\pi/6}^{\pi/3}$
$= -\frac{1}{2}\left(\frac{1}{3} - 3\right) + \left(\ln\frac{2}{\sqrt{3}} - \ln 2\right) = \frac{4}{3} - \ln\sqrt{3}$

50. $\int 8\cot^4 t\, dt = 8\int (\csc^2 t - 1)\cot^2 t\, dt = 8\int \csc^2 t \cot^2 t\, dt - 8\int \cot^2 t\, dt = -\frac{8}{3}\cot^3 t - 8\int (\csc^2 t - 1)dt$
$= -\frac{8}{3}\cot^3 t + 8\cot t + 8t + C$

51. $\int \sin 3x \cos 2x\, dx = \frac{1}{2}\int (\sin x + \sin 5x)\, dx = -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

52. $\int \sin 2x \cos 3x\, dx = \frac{1}{2}\int (\sin(-x) + \sin 5x)\, dx = \frac{1}{2}\int (-\sin x + \sin 5x)\, dx = \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

53. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x\, dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

54. $\int_{0}^{\pi/2} \sin x \cos x\, dx = \frac{1}{2}\int_{0}^{\pi/2} (\sin 0 + \sin 2x)\, dx = \frac{1}{2}\int_{0}^{\pi/2} \sin 2x\, dx = -\frac{1}{4}[\cos 2x]_{0}^{\pi/2} = -\frac{1}{4}(-1-1) = \frac{1}{2}$

55. $\int \cos 3x \cos 4x\, dx = \frac{1}{2}\int (\cos(-x) + \cos 7x)\, dx = \frac{1}{2}\int (\cos x + \cos 7x)\, dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$

56. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x\, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x)\, dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2} = 0$

57. $\int \sin^2 \theta \cos 3\theta\, d\theta = \int \frac{1-\cos 2\theta}{2} \cos 3\theta\, d\theta = \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta\, d\theta$
$= \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{2}\int \frac{1}{2}(\cos(2-3)\theta + \cos(2+3)\theta)\, d\theta = \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{4}\int (\cos(-\theta) + \cos 5\theta)\, d\theta$
$= \frac{1}{2}\int \cos 3\theta\, d\theta - \frac{1}{4}\int \cos \theta\, d\theta - \frac{1}{4}\int \cos 5\theta\, d\theta = \frac{1}{6}\sin 3\theta - \frac{1}{4}\sin \theta - \frac{1}{20}\sin 5\theta + C$

58. $\int \cos^2 2\theta \sin \theta\, d\theta = \int (2\cos^2 \theta - 1)^2 \sin \theta\, d\theta = \int (4\cos^4 \theta - 4\cos^2 \theta + 1) \sin \theta\, d\theta$
$= \int 4\cos^4 \theta \sin \theta\, d\theta - \int 4\cos^2 \theta \sin \theta\, d\theta + \int \sin \theta\, d\theta = -\frac{4}{5}\cos^5 \theta + \frac{4}{3}\cos^3 \theta - \cos \theta + C$

59. $\int \cos^3 \theta \sin 2\theta\, d\theta = \int \cos^3 \theta\, (2\sin \theta \cos \theta)\, d\theta = 2\int \cos^4 \theta \sin \theta\, d\theta = -\frac{2}{5}\cos^5 \theta + C$

60. $\int \sin^3 \theta \cos 2\theta\, d\theta = \int \sin^2 \theta \cos 2\theta \sin \theta\, d\theta = \int \left(1 - \cos^2 \theta\right)(2\cos^2 \theta - 1)\sin \theta\, d\theta$
$= \int (-2\cos^4 \theta + 3\cos^2 \theta - 1)\sin \theta\, d\theta = -2\int \cos^4 \theta \sin \theta\, d\theta + 3\int \cos^2 \theta \sin \theta\, d\theta - \int \sin \theta\, d\theta$
$= \frac{2}{5}\cos^5 \theta - \cos^3 \theta + \cos \theta + C$

61. $\int \sin \theta \cos \theta \cos 3\theta\, d\theta = \frac{1}{2}\int 2\sin \theta \cos \theta \cos 3\theta\, d\theta = \frac{1}{2}\int \sin 2\theta \cos 3\theta\, d\theta = \frac{1}{2}\int \frac{1}{2}(\sin(2-3)\theta + \sin(2+3)\theta)\, d\theta$
$= \frac{1}{4}\int (\sin(-\theta) + \sin 5\theta)\, d\theta = \frac{1}{4}\int (-\sin \theta + \sin 5\theta)\, d\theta = \frac{1}{4}\cos \theta - \frac{1}{20}\cos 5\theta + C$

62. $\int \sin \theta \sin 2\theta \sin 3\theta\, d\theta = \int \frac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right) \sin 3\theta\, d\theta = \frac{1}{2}\int \left(\cos(-\theta) - \cos 3\theta\right) \sin 3\theta\, d\theta$
$= \frac{1}{2}\int \sin 3\theta \cos \theta\, d\theta - \frac{1}{2}\int \sin 3\theta \cos 3\theta\, d\theta = \frac{1}{2}\int \frac{1}{2}(\sin(3-1)\theta + \sin(3+1)\theta) d\theta - \frac{1}{4}\int 2\sin 3\theta \cos 3\theta\, d\theta$
$= \frac{1}{4}\int (\sin 2\theta + \sin 4\theta) d\theta - \frac{1}{4}\int \sin 6\theta\, d\theta = \frac{1}{4}\int (\sin 2\theta + \sin 4\theta) d\theta - \frac{1}{4}\int \sin 6\theta\, d\theta$
$= -\frac{1}{8}\cos 2\theta - \frac{1}{16}\cos 4\theta + \frac{1}{24}\cos 6\theta + C$

63. $\int \frac{\sec^3 x}{\tan x}\, dx = \int \frac{\sec^2 x \sec x}{\tan x}\, dx = \int \frac{(\tan^2 x + 1)\sec x}{\tan x}\, dx = \int \frac{\tan^2 x \sec x}{\tan x}\, dx + \int \frac{\sec x}{\tan x}\, dx = \int \tan x \sec x\, dx + \int \csc x\, dx$
$= \sec x - \ln|\csc x + \cot x| + C$

64. $\int \frac{\sin^3 x}{\cos^4 x}\, dx = \int \frac{\sin^2 x \sin x}{\cos^4 x}\, dx = \int \frac{(1-\cos^2 x) \sin x}{\cos^4 x}\, dx = \int \frac{\sin x}{\cos^4 x}\, dx - \int \frac{\cos^2 x \sin x}{\cos^4 x}\, dx = \int \sec^3 x \tan x\, dx - \int \sec x \tan x\, dx$
$= \int \sec^2 x \sec x \tan x\, dx - \int \sec x \tan x\, dx = \frac{1}{3}\sec^3 x - \sec x + C$

65. $\int \frac{\tan^2 x}{\csc x}\, dx = \int \frac{\sin^2 x}{\cos^2 x} \sin x\, dx = \int \frac{(1-\cos^2 x)}{\cos^2 x} \sin x\, dx = \int \frac{1}{\cos^2 x} \sin x\, dx - \int \frac{\cos^2 x}{\cos^2 x} \sin x\, dx = \int \sec x \tan x\, dx - \int \sin x\, dx$
$= \sec x + \cos x + C$

66. $\int \frac{\cot x}{\cos^2 x}\,dx = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x}\,dx = \int \frac{2}{2\sin x \cos x}\,dx = \int \frac{2}{\sin 2x}\,dx = \int \csc 2x\, 2dx = -\ln|\csc 2x + \cot 2x| + C$

67. $\int x \sin^2 x\,dx = \int x\,\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx \left[u = x,\ du = dx,\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x\right]$
$= \frac{1}{4}x^2 - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right] = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

68. $\int x\cos^3 x\,dx = \int x\cos^2 x\cos x\,dx = \int x(1-\sin^2 x)\cos x\,dx = \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx;$
$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x;$
$\left[u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x\right]$
$\int x\sin^2 x\cos x\,dx = \frac{1}{3}x\sin^3 x - \int \frac{1}{3}\sin^3 x\,dx;$
$\left[u = x,\ du = dx,\ dv = \sin^2 x\cos x\,dx,\ v = \frac{1}{3}\sin^3 x\right]$
$= \frac{1}{3}x\sin^3 x - \frac{1}{3}\int (1-\cos^2 x)\sin x\,dx = \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin x\,dx + \frac{1}{3}\int \cos^2 x\sin x\,dx = \frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x;$
$\Rightarrow \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx = (x\sin x + \cos x) - \left(\frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x\right) + C$
$= x\sin x - \frac{1}{3}x\sin^3 x + \frac{2}{3}\cos x + \frac{1}{9}\cos^3 x + C$

69. $y = \ln(\sec x);\ y' = \frac{\sec x\tan x}{\sec x} = \tan x; (y')^2 = \tan^2 x;\ \int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4}|\sec x|\,dx = [\ln|\sec x + \tan x|]_0^{\pi/4}$
$= \ln\left(\sqrt{2}+1\right) - \ln(0+1) = \ln\left(\sqrt{2}+1\right)$

70. $M = \int_{-\pi/4}^{\pi/4}\sec x\,dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln|\sqrt{2}-1| = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$
$\overline{y} = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{2}\,dx = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}(1-(-1)) = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}$
$\Rightarrow (\overline{x}, \overline{y}) = \left(0, \left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{-1}\right)$

71. $V = \pi\int_0^{\pi}\sin^2 x\,dx = \pi\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx = \frac{\pi}{2}\int_0^{\pi}dx - \frac{\pi}{2}\int_0^{\pi}\cos 2x\,dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{4}[\sin 2x]_0^{\pi} = \frac{\pi}{2}(\pi-0) - \frac{\pi}{4}(0-0) = \frac{\pi^2}{2}$

72. $A = \int_0^{\pi}\sqrt{1+\cos 4x}\,dx = \int_0^{\pi}\sqrt{2}|\cos 2x|dx = \sqrt{2}\int_0^{\pi/4}\cos 2x\,dx - \sqrt{2}\int_{\pi/4}^{3\pi/4}\cos 2x\,dx + \sqrt{2}\int_{3\pi/4}^{\pi}\cos 2x\,dx$
$= \frac{\sqrt{2}}{2}[\sin 2x]_0^{\pi/4} - \frac{\sqrt{2}}{2}[\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2}[\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2}(1-0) - \frac{\sqrt{2}}{2}(-1-1) + \frac{\sqrt{2}}{2}(0+1) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

73. $M = \int_0^{2\pi}(x+\cos x)dx = \left[\frac{1}{2}x^2 + \sin x\right]_0^{2\pi} = \left(\frac{1}{2}(2\pi)^2 + \sin(2\pi)\right) - \left(\frac{1}{2}(0)^2 + \sin(0)\right) = 2\pi^2;$
$\overline{x} = \frac{1}{2\pi^2}\int_0^{2\pi}x(x+\cos x)dx = \frac{1}{2\pi^2}\int_0^{2\pi}(x^2 + x\cos x)dx = \frac{1}{2\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x dx$
$\left[u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x\right]$
$= \frac{1}{6\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left(\left[x\sin x\right]_0^{2\pi} - \int_0^{2\pi}\sin x dx\right) = \frac{1}{6\pi^2}(8\pi^3 - 0) + \frac{1}{2\pi^2}\left(2\pi\sin 2\pi - 0 - \int_0^{2\pi}\sin x dx\right)$
$= \frac{4\pi}{3} + \frac{1}{2\pi^2}\left[\cos x\right]_0^{2\pi} = \frac{4\pi}{3} + \frac{1}{2\pi^2}(\cos 2\pi - \cos 0) = \frac{4\pi}{3} + 0 = \frac{4\pi}{3};\ \overline{y} = \frac{1}{2\pi^2}\int_0^{2\pi}\frac{1}{2}(x+\cos x)^2dx$
$= \frac{1}{4\pi^2}\int_0^{2\pi}(x^2 + 2x\cos x + \cos^2 x)dx = \frac{1}{4\pi^2}\int_0^{2\pi}x^2dx + \frac{1}{2\pi^2}\int_0^{2\pi}x\cos x dx + \frac{1}{4\pi^2}\int_0^{2\pi}\cos^2 x\,dx$

$= \frac{1}{12\pi^2}\left[x^3\right]_0^{2\pi} + \frac{1}{2\pi^2}\left[x\sin x + \cos x\right]_0^{2\pi} + \frac{1}{4\pi^2}\int_0^{2\pi}\frac{\cos 2x + 1}{2}\,dx = \frac{2\pi}{3} + 0 + \frac{1}{8\pi^2}\int_0^{2\pi}\cos 2x\,dx + \frac{1}{8\pi^2}\int_0^{2\pi}dx$

$= \frac{2\pi}{3} + \frac{1}{16\pi^2}\left[\sin 2x\right]_0^{2\pi} + \frac{1}{8\pi^2}\left[x\right]_0^{2\pi} = \frac{2\pi}{3} + 0 + \frac{1}{4\pi} = \frac{8\pi^2+3}{12\pi} \Rightarrow$ The centroid is $\left(\frac{4\pi}{3}, \frac{8\pi^2+3}{12\pi}\right)$.

74. $V = \int_0^{\pi/3}\pi(\sin x + \sec x)^2\,dx = \pi\int_0^{\pi/3}(\sin^2 x + 2\sin x\sec x + \sec^2 x)\,dx$

$= \pi\int_0^{\pi/3}\sin^2 x\,dx + \pi\int_0^{\pi/3}2\tan x\,dx + \pi\int_0^{\pi/3}\sec^2 x\,dx = \pi\int_0^{\pi/3}\frac{1-\cos 2x}{2}\,dx + 2\pi\left[\ln|\sec x|\right]_0^{\pi/3} + \pi\left[\tan x\right]_0^{\pi/3}$

$= \frac{\pi}{2}\int_0^{\pi/3}dx - \frac{\pi}{2}\int_0^{\pi/3}\cos 2x\,dx + 2\pi\left(\ln|\sec\frac{\pi}{3}| - \ln|\sec 0|\right) + \pi\left(\tan\frac{\pi}{3} - \tan 0\right)$

$= \frac{\pi}{2}\left[x\right]_0^{\pi/3} - \frac{\pi}{4}\left[\sin 2x\right]_0^{\pi/3} + 2\pi\ln 2 + \pi\sqrt{3} = \frac{\pi}{2}\left(\frac{\pi}{3} - 0\right) - \frac{\pi}{4}\left(\sin 2\left(\frac{\pi}{3}\right) - \sin 2(0)\right) + 2\pi\ln 2 + \pi\sqrt{3}$

$= \frac{\pi^2}{6} - \frac{\pi\sqrt{3}}{8} + 2\pi\ln 2 + \pi\sqrt{3} = \frac{\pi\left(4\pi + 21\sqrt{3} - 48\ln 2\right)}{24}$

8.3 TRIGONOMETRIC SUBSTITUTIONS

1. $x = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{3\,d\theta}{\cos^2\theta},\ 9 + x^2 = 9\left(1 + \tan^2\theta\right) = 9\sec^2\theta \Rightarrow \frac{1}{\sqrt{9+x^2}} = \frac{1}{3|\sec\theta|} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$;
$\left(\text{because } \cos\theta > 0 \text{ when } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$;
$\int\frac{dx}{\sqrt{9+x^2}} = 3\int\frac{\cos\theta\,d\theta}{3\cos^2\theta} = \int\frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C' = \ln\left|\sqrt{9+x^2} + x\right| + C$

2. $\int\frac{3\,dx}{\sqrt{1+9x^2}}$; $[3x = u] \to \int\frac{du}{\sqrt{1+u^2}}$; $u = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ du = \frac{dt}{\cos^2 t},\ \sqrt{1+u^2} = |\sec t| = \sec t$;
$\int\frac{du}{\sqrt{1+u^2}} = \int\frac{dt}{\cos^2 t\,(\sec t)} = \int\sec t\,dt = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{u^2+1} + u\right| + C = \ln\left|\sqrt{1+9x^2} + 3x\right| + C$

3. $\int_{-2}^{2}\frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}(-1) = \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - \left(\frac{1}{2}\right)\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

4. $\int_0^2\frac{dx}{8+2x^2} = \frac{1}{2}\int_0^2\frac{dx}{4+x^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16}$

5. $\int_0^{3/2}\frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

6. $\int_0^{1/2\sqrt{2}}\frac{2\,dx}{\sqrt{1-4x^2}}$; $[t = 2x] \to \int_0^{1/2\sqrt{2}}\frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1}t\right]_0^{1/\sqrt{2}} = \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

7. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\,d\theta,\ \sqrt{25-t^2} = 5\cos\theta$;
$\int\sqrt{25-t^2}\,dt = \int(5\cos\theta)(5\cos\theta)\,d\theta = 25\int\cos^2\theta\,d\theta = 25\int\frac{1+\cos 2\theta}{2}\,d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$
$= \frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

8. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\,d\theta,\ \sqrt{1-9t^2} = \cos\theta$;
$\int\sqrt{1-9t^2}\,dt = \frac{1}{3}\int(\cos\theta)(\cos\theta)\,d\theta = \frac{1}{3}\int\cos^2\theta\,d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$

9. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\,d\theta,\ \sqrt{4x^2-49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta$;
$\int\frac{dx}{\sqrt{4x^2-49}} = \int\frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta} = \frac{1}{2}\int\sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

10. $x = \frac{3}{5}\sec\theta, 0 < \theta < \frac{\pi}{2}, dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta, \sqrt{25x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$
$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

11. $y = 7\sec\theta, 0 < \theta < \frac{\pi}{2}, dy = 7\sec\theta\tan\theta\, d\theta, \sqrt{y^2 - 49} = 7\tan\theta;$
$\int \frac{\sqrt{y^2-49}}{y}\, dy = \int \frac{(7\tan\theta)(7\sec\theta\tan\theta)\, d\theta}{7\sec\theta} = 7\int \tan^2\theta\, d\theta = 7\int (\sec^2\theta - 1)\, d\theta = 7(\tan\theta - \theta) + C$
$= 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C$

12. $y = 5\sec\theta, 0 < \theta < \frac{\pi}{2}, dy = 5\sec\theta\tan\theta\, d\theta, \sqrt{y^2 - 25} = 5\tan\theta;$
$\int \frac{\sqrt{y^2-25}}{y^3}\, dy = \int \frac{(5\tan\theta)(5\sec\theta\tan\theta)\, d\theta}{125\sec^3\theta} = \frac{1}{5}\int \tan^2\theta\cos^2\theta\, d\theta = \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{10}\int (1 - \cos 2\theta)\, d\theta$
$= \frac{1}{10}(\theta - \sin\theta\cos\theta) + C = \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right] + C = \left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2}\right] + C$

13. $x = \sec\theta, 0 < \theta < \frac{\pi}{2}, dx = \sec\theta\tan\theta\, d\theta, \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$

14. $x = \sec\theta, 0 < \theta < \frac{\pi}{2}, dx = \sec\theta\tan\theta\, d\theta, \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\, d\theta = 2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \theta + \sin\theta\cos\theta + C$
$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1} x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} + C$

15. $u = 9 - x^2 \Rightarrow du = -2x\, dx \Rightarrow -\frac{1}{2}du = x\, dx;$
$\int \frac{x\, dx}{\sqrt{9-x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -\sqrt{u} + C = -\sqrt{9 - x^2} + C$

16. $x = 2\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = 2\sec^2\theta\, d\theta, 4 + x^2 = 4\sec^2\theta$
$\int \frac{x^2\, dx}{4+x^2} = \int \frac{(4\tan^2\theta)(2\sec^2\theta)d\theta}{4\sec^2\theta} = \int 2\tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)d\theta = 2\int \sec^2\theta\, d\theta - 2\int d\theta = 2\tan\theta - 2\theta + C$
$= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$

17. $x = 2\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \frac{2\, d\theta}{\cos^2\theta}, \sqrt{x^2 + 4} = \frac{2}{\cos\theta};$
$\int \frac{x^3\, dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\, d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\, d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta - 1)(-\sin\theta)\, d\theta}{\cos^4\theta};$
$[t = \cos\theta] \rightarrow 8\int \frac{t^2-1}{t^4}\, dt = 8\int \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$
$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C = \frac{1}{3}(x^2 - 8)\sqrt{x^2+4} + C$

18. $x = \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \sec^2\theta\, d\theta, \sqrt{x^2 + 1} = \sec\theta;$
$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$

19. $w = 2\sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dw = 2\cos\theta\, d\theta, \sqrt{4 - w^2} = 2\cos\theta;$
$\int \frac{8\, dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\, d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$

20. $w = 3\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\, d\theta,\ \sqrt{9-w^2} = 3\cos\theta;$

$$\int \frac{\sqrt{9-w^2}}{w^2}\, dw = \int \frac{3\cos\theta \cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \cot^2\theta\, d\theta = \int \left(\frac{1-\sin^2\theta}{\sin^2\theta}\right) d\theta = \int (\csc^2\theta - 1)\, d\theta$$
$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

21. $u = 5x \Rightarrow du = 5dx,\ a = 6$

$$\int \frac{100}{36+25x^2} dx = 20\int \frac{1}{(6)^2+(5x)^2} 5dx = 20\int \frac{1}{a^2+u^2} du = 20 \cdot \frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right) + C = \frac{10}{3}\tan^{-1}\left(\frac{5x}{6}\right) + C$$

22. $u = x^2 - 4 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2}du = x\, dx$

$$\int x\sqrt{x^2-4}\, dx = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2-4)^{3/2} + C$$

23. $x = \sin\theta,\ 0 \le \theta \le \frac{\pi}{3},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$

$$\int_0^{\sqrt{3}/2} \frac{4x^2\, dx}{(1-x^2)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3} \left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta$$
$$= 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

24. $x = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6},\ dx = 2\cos\theta\, d\theta,\ (4-x^2)^{3/2} = 8\cos^3\theta;$

$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$$

25. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{3/2} = \tan^3\theta;$

$$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\frac{x}{\sqrt{x^2-1}} + C$$

26. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{5/2} = \tan^5\theta;$

$$\int \frac{x^2\, dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\, d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C$$

27. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$

$$\int \frac{(1-x^2)^{3/2}\, dx}{x^6} = \int \frac{\cos^3\theta \cdot \cos\theta\, d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\, d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

28. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1-x^2)^{1/2} = \cos\theta;$

$$\int \frac{(1-x^2)^{1/2}\, dx}{x^4} = \int \frac{\cos\theta \cdot \cos\theta\, d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\, d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$$

29. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\, d\theta,\ (4x^2+1)^2 = \sec^4\theta;$

$$\int \frac{8\, dx}{(4x^2+1)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right) d\theta}{\sec^4\theta} = 4\int \cos^2\theta\, d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\tan^{-1}2x + \frac{4x}{(4x^2+1)} + C$$

30. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\, d\theta,\ 9t^2 + 1 = \sec^2\theta;$

$$\int \frac{6\, dt}{(9t^2+1)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right) d\theta}{\sec^4\theta} = 2\int \cos^2\theta\, d\theta = \theta + \sin\theta\cos\theta + C = \tan^{-1}3t + \frac{3t}{(9t^2+1)} + C$$

31. $u = x^2 - 1 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2}du = x\, dx$

$$\int \frac{x^3}{x^2-1} dx = \int \left(x + \frac{x}{x^2-1}\right) dx = \int x\, dx + \int \frac{x}{x^2-1} dx = \frac{1}{2}x^2 + \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}x^2 + \frac{1}{2}\ln|u| + C = \frac{1}{2}x^2 + \frac{1}{2}\ln|x^2-1| + C$$

32. $u = 25 + 4x^2 \Rightarrow du = 8x\,dx \Rightarrow \frac{1}{8}du = x\,dx$

$\int \frac{x}{25+4x^2}dx = \frac{1}{8}\int \frac{1}{u}du = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln(25 + 4x^2) + C$

33. $v = \sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dv = \cos\theta\, d\theta, (1-v^2)^{5/2} = \cos^5\theta;$

$\int \frac{v^2\,dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^5\theta} = \int \tan^2\theta\sec^2\theta\,d\theta = \frac{\tan^3\theta}{3} + C = \frac{1}{3}\left(\frac{v}{\sqrt{1-v^2}}\right)^3 + C$

34. $r = \sin\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2};$

$\int \frac{(1-r^2)^{5/2}\,dr}{r^8} = \int \frac{\cos^5\theta\cdot\cos\theta\,d\theta}{\sin^8\theta} = \int \cot^6\theta\csc^2\theta\,d\theta = -\frac{\cot^7\theta}{7} + C = -\frac{1}{7}\left[\frac{\sqrt{1-r^2}}{r}\right]^7 + C$

35. Let $e^t = 3\tan\theta$, $t = \ln(3\tan\theta)$, $\tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right)$, $dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta$, $\sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta;$

$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta\cdot\sec^2\theta\,d\theta}{\tan\theta\cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\,d\theta = [\ln|\sec\theta + \tan\theta|]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$

$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) = \ln 9 - \ln\left(1 + \sqrt{10}\right)$

36. Let $e^t = \tan\theta$, $t = \ln(\tan\theta)$, $\tan^{-1}\left(\frac{3}{4}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right)$, $dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta$, $1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta;$

$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\,d\theta = [\sin\theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

37. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$; $\left[u = 2\sqrt{t}, du = \frac{1}{\sqrt{t}}\,dt\right] \to \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2}$; $u = \tan\theta$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{4}$, $du = \sec^2\theta\,d\theta$, $1 + u^2 = \sec^2\theta;$

$\int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = [2\theta]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$

38. $y = e^{\tan\theta}, 0 \le \theta \le \frac{\pi}{4}, dy = e^{\tan\theta}\sec^2\theta\,d\theta, \sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta;$

$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\,d\theta = \int_0^{\pi/4} \sec\theta\,d\theta = [\ln|\sec\theta + \tan\theta|]_0^{\pi/4} = \ln\left(1 + \sqrt{2}\right)$

39. $x = \sec\theta, 0 < \theta < \frac{\pi}{2}, dx = \sec\theta\tan\theta\,d\theta, \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta;$

$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1}x + C$

40. $x = \tan\theta, dx = \sec^2\theta\,d\theta, 1 + x^2 = \sec^2\theta;$

$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \theta + C = \tan^{-1}x + C$

41. $x = \sec\theta, dx = \sec\theta\tan\theta\,d\theta, \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta;$

$\int \frac{x\,dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan\theta} = \int \sec^2\theta\,d\theta = \tan\theta + C = \sqrt{x^2-1} + C$

42. $x = \sin\theta, dx = \cos\theta\,d\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2};$

$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\,d\theta}{\cos\theta} = \theta + C = \sin^{-1}x + C$

43. Let $x^2 = \tan\theta, 0 \le \theta < \frac{\pi}{2}, 2x\,dx = \sec^2\theta\,d\theta \Rightarrow x\,dx = \frac{1}{2}\sec^2\theta\,d\theta; \sqrt{1+x^4} = \sqrt{1+\tan^2\theta} = \sec\theta$

$\int \frac{x}{\sqrt{1+x^4}}dx = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\sqrt{1+x^4} + x^2\right| + C$

44. Let $\ln x = \sin\theta,\ -\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$, $\frac{1}{x}dx = \cos\theta\, d\theta$, $\sqrt{1-(\ln x)^2} = \cos\theta$

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx = \int \frac{\cos^2\theta}{\sin\theta}d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}d\theta = \int \csc\theta\, d\theta - \int \sin\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + \cos\theta + C$$

$$= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C = -\ln\left|\frac{1+\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C$$

45. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\, du \Rightarrow \int \sqrt{\frac{4-x}{x}}\, dx = \int \sqrt{\frac{4-u^2}{u^2}}\, 2u\, du = 2\int \sqrt{4-u^2}\, du$;

$u = 2\sin\theta$, $du = 2\cos\theta\, d\theta$, $0 < \theta \le \frac{\pi}{2}$, $\sqrt{4-u^2} = 2\cos\theta$

$$2\int \sqrt{4-u^2}\, du = 2\int (2\cos\theta)(2\cos\theta)\, d\theta = 8\int \cos^2\theta\, d\theta = 8\int \frac{1+\cos 2\theta}{2}\, d\theta = 4\int d\theta + 4\int \cos 2\theta\, d\theta$$

$$= 4\theta + 2\sin 2\theta + C = 4\theta + 4\sin\theta\cos\theta + C = 4\sin^{-1}\left(\frac{u}{2}\right) + 4\left(\frac{u}{2}\right)\left(\frac{\sqrt{4-u^2}}{2}\right) + C = 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{x}\sqrt{4-x} + C$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{4x - x^2} + C$$

46. Let $u = x^{3/2} \Rightarrow x = u^{2/3} \Rightarrow dx = \frac{2}{3}u^{-1/3}du$

$$\int \sqrt{\frac{x}{1-x^3}}dx = \int \sqrt{\frac{u^{2/3}}{1-(u^{2/3})^3}}\left(\frac{2}{3}u^{-1/3}\right)du = \int \frac{u^{1/3}}{\sqrt{1-u^2}}\left(\frac{2}{3u^{1/3}}\right)du = \frac{2}{3}\int \frac{1}{\sqrt{1-u^2}}du = \frac{2}{3}\sin^{-1}u + C = \frac{2}{3}\sin^{-1}\left(x^{3/2}\right) + C$$

47. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\, du \Rightarrow \int \sqrt{x}\sqrt{1-x}\, dx = \int u\sqrt{1-u^2}\, 2u\, du = 2\int u^2\sqrt{1-u^2}\, du$;

$u = \sin\theta$, $du = \cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$, $\sqrt{1-u^2} = \cos\theta$

$$2\int u^2\sqrt{1-u^2}\, du = 2\int \sin^2\theta\cos\theta\cos\theta\, d\theta = 2\int \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{2}\int \sin^2 2\theta\, d\theta = \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \cos 4\theta\, d\theta = \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta + C = \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta\cos 2\theta + C = \frac{1}{4}\theta - \frac{1}{4}\sin\theta\cos\theta\,(2\cos^2\theta - 1) + C$$

$$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C = \frac{1}{4}\sin^{-1}u - \frac{1}{2}u\,(1-u^2)^{3/2} - \frac{1}{4}u\sqrt{1-u^2} + C$$

$$= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\,(1-x)^{3/2} - \frac{1}{4}\sqrt{x}\sqrt{1-x} + C$$

48. Let $w = \sqrt{x-1} \Rightarrow w^2 = x - 1 \Rightarrow 2w\, dw = dx \Rightarrow \int \frac{\sqrt{x-2}}{\sqrt{x-1}}dx = \int \frac{\sqrt{w^2-1}}{w}2w\, dw = 2\int \sqrt{w^2-1}\, dw$

$w = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $0 < \theta < \frac{\pi}{2}$, $\sqrt{w^2-1} = \tan\theta$

$2\int \sqrt{w^2-1}\, dw = 2\int \tan\theta\sec\theta\tan\theta\, d\theta$; $u = \tan\theta$, $du = \sec^2\theta\, d\theta$, $dv = \sec\theta\tan\theta\, d\theta$, $v = \sec\theta$

$$2\int \tan\theta\sec\theta\tan\theta\, d\theta = 2\sec\theta\tan\theta - 2\int \sec^3\theta\, d\theta = 2\sec\theta\tan\theta - 2\int \sec^2\theta\sec\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \left(\tan^2\theta + 1\right)\sec\theta\, d\theta = 2\sec\theta\tan\theta - 2\left(\int \tan^2\theta\sec\theta\, d\theta + \int \sec\theta\, d\theta\right)$$

$$= 2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| - 2\int \tan^2\theta\sec\theta\, d\theta \Rightarrow 2\int \tan^2\theta\sec\theta\, d\theta = \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C$$

$$= w\sqrt{w^2-1} - \ln|w + \sqrt{w^2-1}| + C = \sqrt{x-1}\sqrt{x-2} - \ln|\sqrt{x-1} + \sqrt{x-2}| + C$$

49. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\, dx$; $\begin{bmatrix} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{bmatrix}$

$$\rightarrow\ y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\, d\theta}{2\sec\theta} = 2\int \tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)\, d\theta = 2(\tan\theta - \theta) + C$$

$$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C;\ x = 2 \text{ and } y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$$

50. $\sqrt{x^2-9}\,\frac{dy}{dx}=1,\ dy=\frac{dx}{\sqrt{x^2-9}}\,;\ y=\int\frac{dx}{\sqrt{x^2-9}}\,;\ \begin{bmatrix} x=3\sec\theta,\ 0<\theta<\frac{\pi}{2} \\ dx=3\sec\theta\tan\theta\,d\theta \\ \sqrt{x^2-9}=3\tan\theta \end{bmatrix}\ \to\ y=\int\frac{3\sec\theta\tan\theta\,d\theta}{3\tan\theta}$

$=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|+C;\ x=5$ and $y=\ln 3\ \Rightarrow\ \ln 3=\ln 3+C\ \Rightarrow\ C=0$

$\Rightarrow\ y=\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|$

51. $(x^2+4)\frac{dy}{dx}=3,\ dy=\frac{3\,dx}{x^2+4}\,;\ y=3\int\frac{dx}{x^2+4}=\frac{3}{2}\tan^{-1}\frac{x}{2}+C;\ x=2$ and $y=0\ \Rightarrow\ 0=\frac{3}{2}\tan^{-1}1+C$

$\Rightarrow\ C=-\frac{3\pi}{8}\ \Rightarrow\ y=\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{3\pi}{8}$

52. $(x^2+1)^2\frac{dy}{dx}=\sqrt{x^2+1},\ dy=\frac{dx}{(x^2+1)^{3/2}}\,;\ x=\tan\theta,\ dx=\sec^2\theta\,d\theta,\ (x^2+1)^{3/2}=\sec^3\theta;$

$y=\int\frac{\sec^2\theta\,d\theta}{\sec^3\theta}=\int\cos\theta\,d\theta=\sin\theta+C=\tan\theta\cos\theta+C=\frac{\tan\theta}{\sec\theta}+C=\frac{x}{\sqrt{x^2+1}}+C;\ x=0$ and $y=1$

$\Rightarrow\ 1=0+C\ \Rightarrow\ y=\frac{x}{\sqrt{x^2+1}}+1$

53. $A=\int_0^3\frac{\sqrt{9-x^2}}{3}\,dx;\ x=3\sin\theta,\ 0\le\theta\le\frac{\pi}{2},\ dx=3\cos\theta\,d\theta,\ \sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=3\cos\theta;$

$A=\int_0^{\pi/2}\frac{3\cos\theta\cdot 3\cos\theta\,d\theta}{3}=3\int_0^{\pi/2}\cos^2\theta\,d\theta=\frac{3}{2}\left[\theta+\sin\theta\cos\theta\right]_0^{\pi/2}=\frac{3\pi}{4}$

54. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow y=\pm b\sqrt{1-\frac{x^2}{a^2}};\ A=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=4b\int_0^a\sqrt{1-\frac{x^2}{a^2}}\,dx$

$\left[x=a\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ dx=a\cos\theta\,d\theta,\ \sqrt{1-\frac{x^2}{a^2}}=\cos\theta,\ x=0=a\sin\theta\Rightarrow\theta=0,\ x=a=a\sin\theta\Rightarrow\theta=\frac{\pi}{2}\right]$

$4b\int_0^a\sqrt{1-\frac{x^2}{a^2}}\,dx=4b\int_0^{\pi/2}\cos\theta\,(a\cos\theta)\,d\theta=4ab\int_0^{\pi/2}\cos^2\theta\,d\theta=4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta$

$=2ab\int_0^{\pi/2}d\theta+2ab\int_0^{\pi/2}\cos 2\theta\,d\theta=2ab\Big[\theta\Big]_0^{\pi/2}+ab\Big[\sin 2\theta\Big]_0^{\pi/2}=2ab\left(\frac{\pi}{2}-0\right)+ab(\sin\pi-\sin 0)=\pi ab$

55. (a) $A=\int_0^{1/2}\sin^{-1}x\,dx\ \left[u=\sin^{-1}x,\ du=\frac{1}{\sqrt{1-x^2}}dx,\ dv=dx,\ v=x\right]$

$=\Big[x\sin^{-1}x\Big]_0^{1/2}-\int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx==\left(\frac{1}{2}\sin^{-1}\frac{1}{2}-0\right)+\Big[\sqrt{1-x^2}\Big]_0^{1/2}=\frac{\pi+6\sqrt{3}-12}{12}$

(b) $M=\int_0^{1/2}\sin^{-1}x\,dx=\frac{\pi+6\sqrt{3}-12}{12};\ \overline{x}=\frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2}x\sin^{-1}x\,dx=\frac{12}{\pi+6\sqrt{3}-12}\int_0^{1/2}x\sin^{-1}x\,dx$

$\left[u=\sin^{-1}x,\ du=\frac{1}{\sqrt{1-x^2}}dx,\ dv=x\,dx,\ v=\frac{1}{2}x^2\right]$

$=\frac{12}{\pi+6\sqrt{3}-12}\left(\left[\frac{1}{2}x^2\sin^{-1}x\right]_0^{1/2}-\frac{1}{2}\int_0^{1/2}\frac{x^2}{\sqrt{1-x^2}}\,dx\right)$

$\left[x=\sin\theta,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2},\ dx=\cos\theta\,d\theta,\ \sqrt{1-x^2}=\cos\theta,\ x=0=\sin\theta\Rightarrow\theta=0,\ x=\frac{1}{2}=\sin\theta\Rightarrow\theta=\frac{\pi}{6}\right]$

$=\frac{12}{\pi+6\sqrt{3}-12}\left(\left(\frac{1}{2}\left(\frac{1}{2}\right)^2\sin^{-1}\left(\frac{1}{2}\right)-0\right)-\frac{1}{2}\int_0^{\pi/6}\frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta\right)=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}-\frac{1}{2}\int_0^{\pi/6}\sin^2\theta\,d\theta\right)$

$=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}-\frac{1}{2}\int_0^{\pi/6}\frac{1-\cos 2\theta}{2}\,d\theta\right)=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}-\frac{1}{4}\int_0^{\pi/6}d\theta+\frac{1}{4}\int_0^{\pi/6}\cos 2\theta\,d\theta\right)$

$=\frac{12}{\pi+6\sqrt{3}-12}\left(\frac{\pi}{48}+\left[-\frac{\theta}{4}+\frac{1}{8}\sin 2\theta\right]_0^{\pi/6}\right)=\frac{3\sqrt{3}-\pi}{4\left(\pi+6\sqrt{3}-12\right)};\ \overline{y}=\frac{1}{\frac{\pi+6\sqrt{3}-12}{12}}\int_0^{1/2}\frac{1}{2}(\sin^{-1}x)^2dx$

$\left[u=(\sin^{-1}x)^2,\ du=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}dx,\ dv=dx,\ v=x\right]$

$$= \frac{6}{\pi+6\sqrt{3}-12}\left(\left[x(\sin^{-1}x\,dx)^2\right]_0^{1/2} - \int_0^{1/2} \frac{2x\sin^{-1}x}{\sqrt{1-x^2}}dx\right)$$

$$\left[u = \sin^{-1}x,\ du = \frac{1}{\sqrt{1-x^2}}dx,\ dv = \frac{2x}{\sqrt{1-x^2}}dx,\ v = -2\sqrt{1-x^2}\right]$$

$$= \frac{6}{\pi+6\sqrt{3}-12}\left(\left(\tfrac{1}{2}\left(\sin^{-1}\left(\tfrac{1}{2}\right)\right)^2 - 0\right) + \left[2\sqrt{1-x^2}\sin^{-1}x\right]_0^{1/2} - \int_0^{1/2}\frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}}dx\right)$$

$$= \frac{6}{\pi+6\sqrt{3}-12}\left(\frac{\pi^2}{72} + \left(2\sqrt{1-\left(\tfrac{1}{2}\right)^2}\sin^{-1}\left(\tfrac{1}{2}\right) - 0\right) - \left[2x\right]_0^{1/2}\right) = \frac{6}{\pi+6\sqrt{3}-12}\left(\frac{\pi^2}{72} + \frac{\pi\sqrt{3}}{6} - 1\right) = \frac{\pi^2+12\pi\sqrt{3}-72}{12\left(\pi+6\sqrt{3}-12\right)}$$

56. $V = \int_0^1 \pi\left(\sqrt{x\tan^{-1}x}\right)^2 dx = \pi\int_0^1 x\tan^{-1}x\,dx \qquad \left[u = \tan^{-1}x,\ du = \frac{1}{1+x^2}dx,\ dv = x\,dx,\ v = \tfrac{1}{2}x^2\right]$

$$= \pi\left(\left[\tfrac{1}{2}x^2\tan^{-1}x\right]_0^1 - \tfrac{1}{2}\int_0^1\frac{x^2}{1+x^2}dx\right) = \pi\left(\left(\tfrac{1}{2}\tan^{-1}1 - 0\right) - \tfrac{1}{2}\int_0^1\left(1-\frac{1}{1+x^2}\right)dx\right) = \pi\left(\frac{\pi}{8} - \tfrac{1}{2}\int_0^1\left(1-\frac{1}{1+x^2}\right)dx\right)$$

$$= \pi\left(\frac{\pi}{8} - \tfrac{1}{2}\int_0^1 dx + \tfrac{1}{2}\int_0^1\frac{1}{1+x^2}dx\right) = \pi\left(\frac{\pi}{8} + \left[-\tfrac{1}{2}x + \tfrac{1}{2}\tan^{-1}x\right]_0^1\right) = \pi\left(\frac{\pi}{8} + \left(-\tfrac{1}{2} + \tfrac{1}{2}\tan^{-1}1 + 0 - 0\right)\right) = \frac{\pi(\pi-2)}{4}$$

57. (a) Integration by parts: $u = x^2,\ du = 2x\,dx,\ dv = x\sqrt{1-x^2}\,dx,\ v = -\frac{1}{3}(1-x^2)^{3/2}$

$$\int x^3\sqrt{1-x^2}\,dx = -\tfrac{1}{3}x^2(1-x^2)^{3/2} + \tfrac{1}{3}\int(1-x^2)^{3/2}\,2x\,dx = -\tfrac{1}{3}x^2(1-x^2)^{3/2} - \tfrac{2}{15}(1-x^2)^{5/2} + C$$

(b) Substitution: $u = 1-x^2 \Rightarrow x^2 = 1-u \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

$$\int x^3\sqrt{1-x^2}\,dx = \int x^2\sqrt{1-x^2}\,x\,dx = -\tfrac{1}{2}\int(1-u)\sqrt{u}\,du = -\tfrac{1}{2}\int\left(\sqrt{u}-u^{3/2}\right)du = -\tfrac{1}{3}u^{3/2} + \tfrac{1}{5}u^{5/2} + C$$

$$= -\tfrac{1}{3}(1-x^2)^{3/2} + \tfrac{1}{5}(1-x^2)^{5/2} + C$$

(c) Trig substitution: $x = \sin\theta,\ \frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta$

$$\int x^3\sqrt{1-x^2}\,dx = \int\sin^3\theta\cos\theta\cos\theta\,d\theta = \int\sin^2\theta\cos^2\theta\sin\theta\,d\theta = \int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$$

$$= \int\cos^2\theta\sin\theta\,d\theta - \int\cos^4\theta\sin\theta\,d\theta = -\tfrac{1}{3}\cos^3\theta + \tfrac{1}{5}\cos^5\theta + C = -\tfrac{1}{3}(1-x^2)^{3/2} + \tfrac{1}{5}(1-x^2)^{5/2} + C$$

58. (a) The slope of the line tangent to $y = f(x)$ is given by $f'(x)$. Consider the triangle whose hypotenuse is the 30 ft rope, the length of the base is x and the height $h = \sqrt{900-x^2}$. The slope of the tangent line is also $-\frac{\sqrt{900-x^2}}{x}$, thus $f'(x) = -\frac{\sqrt{900-x^2}}{x}$.

(b) $f(x) = \int -\frac{\sqrt{900-x^2}}{x}dx \qquad \left[x = 30\sin\theta,\ 0 < \theta \le \frac{\pi}{2},\ dx = 30\cos\theta\,d\theta,\ \sqrt{900-x^2} = 30\cos\theta\right]$

$$= -\int\frac{30\cos\theta}{30\sin\theta}30\cos\theta\,d\theta = -30\int\frac{\cos^2\theta}{\sin\theta}d\theta = -30\int\frac{(1-\sin^2\theta)}{\sin\theta}d\theta = -30\int\csc\theta\,d\theta + 30\int\sin\theta\,d\theta$$

$$= 30\ln|\csc\theta+\cot\theta| - 30\cos\theta + C = 30\ln\left|\frac{30}{x} + \frac{\sqrt{900-x^2}}{x}\right| - \sqrt{900-x^2} + C;\ f(30) = 0$$

$$\Rightarrow 0 = 30\ln\left|\frac{30}{30} + \frac{\sqrt{900-30^2}}{30}\right| - \sqrt{900-30^2} + C = C \Rightarrow f(x) = 30\ln\left|\frac{30}{x} + \frac{\sqrt{900-x^2}}{x}\right| - \sqrt{900-x^2}$$

8.4 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. $\frac{5x-13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x-13 = A(x-2) + B(x-3) = (A+B)x - (2A+3B)$

$$\Rightarrow \left.\begin{array}{r} A+B=5 \\ 2A+3B=13 \end{array}\right\} \Rightarrow -B = (10-13) \Rightarrow B = 3 \Rightarrow A = 2;\text{ thus, } \frac{5x-13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1) + B(x-2) = (A+B)x - (A+2B)$

$$\Rightarrow \left.\begin{array}{r} A+B=5 \\ A+2B=7 \end{array}\right\} \Rightarrow B = 2 \Rightarrow A = 3;\text{ thus, } \frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1)+B = Ax+(A+B) \Rightarrow \left.\begin{array}{c} A=1 \\ A+B=4 \end{array}\right\} \Rightarrow A=1$ and $B=3$; thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1)+B = Ax+(-A+B) \Rightarrow \left.\begin{array}{c} A=2 \\ -A+B=2 \end{array}\right\}$
$\Rightarrow A=2$ and $B=4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1)+B(z-1)+Cz^2 \Rightarrow z+1 = (A+C)z^2+(-A+B)z-B$
$\Rightarrow \left.\begin{array}{c} A+C=0 \\ -A+B=1 \\ -B=1 \end{array}\right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2$; thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2)+B(z-3) = (A+B)z+(2A-3B)$
$\Rightarrow \left.\begin{array}{c} A+B=0 \\ 2A-3B=1 \end{array}\right\} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$
$\Rightarrow 5t+2 = A(t-2)+B(t-3) = (A+B)t+(-2A-3B) \Rightarrow \left.\begin{array}{c} A+B=5 \\ -2A-3B=2 \end{array}\right\} \Rightarrow -B=(10+2)=12$
$\Rightarrow B=-12 \Rightarrow A=17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$
$\Rightarrow -9t^2+9 = At(t^2+9)+B(t^2+9)+(Ct+D)t^2 = (A+C)t^3+(B+D)t^2+9At+9B$
$\Rightarrow \left.\begin{array}{c} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{array}\right\} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10$; thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x)+B(1-x); x=1 \Rightarrow A=\frac{1}{2}; x=-1 \Rightarrow B=\frac{1}{2}$;
$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}\left[\ln|1+x| - \ln|1-x|\right] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2)+Bx; x=0 \Rightarrow A=\frac{1}{2}; x=-2 \Rightarrow B=-\frac{1}{2}$;
$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\left[\ln|x| - \ln|x+2|\right] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1)+B(x+6); x=1 \Rightarrow B=\frac{5}{7}; x=-6 \Rightarrow A=\frac{-2}{-7}=\frac{2}{7}$;
$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3)+B(x-4); x=3 \Rightarrow B=\frac{7}{-1}=-7; x=4 \Rightarrow A=\frac{9}{1}=9$;
$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1)+B(y-3); y=-1 \Rightarrow B=\frac{-1}{-4}=\frac{1}{4}; y=3 \Rightarrow A=\frac{3}{4}$;
$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$
$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y=0 \Rightarrow A=4;\ y=-1 \Rightarrow B = \frac{3}{-1} = -3;$
$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^{1} \frac{dy}{y} - 3\int_{1/2}^{1} \frac{dy}{y+1} = [4 \ln|y| - 3\ln|y+1|]_{1/2}^{1} = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$
$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t=0 \Rightarrow A = -\frac{1}{2};\ t=-2$
$\Rightarrow B = \frac{1}{6};\ t=1 \Rightarrow C = \frac{1}{3};\ \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int\frac{dt}{t} + \frac{1}{6}\int\frac{dt}{t+2} + \frac{1}{3}\int\frac{dt}{t-1}$
$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2);\ x=0 \Rightarrow A = \frac{3}{-8};\ x=-2$
$\Rightarrow B = \frac{1}{16};\ x=2 \Rightarrow C = \frac{5}{16};\ \int\frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int\frac{dx}{x} + \frac{1}{16}\int\frac{dx}{x+2} + \frac{5}{16}\int\frac{dx}{x-2}$
$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B$
$= Ax + (A+B) \Rightarrow A=3,\ A+B=2 \Rightarrow A=3,\ B=-1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
$= \int_0^1 (x-2)\,dx + 3\int_0^1\frac{dx}{x+1} - \int_0^1\frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$
$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$
$= Ax + (-A+B) \Rightarrow A=3,\ -A+B=-2 \Rightarrow A=3,\ B=1;\ \int_{-1}^{0}\frac{x^3\,dx}{x^2-2x+1}$
$= \int_{-1}^{0}(x+2)\,dx + 3\int_{-1}^{0}\frac{dx}{x-1} + \int_{-1}^{0}\frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^{0}$
$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$
$x=-1 \Rightarrow C = \frac{1}{4};\ x=1 \Rightarrow D = \frac{1}{4}$; coefficient of $x^3 = A+B \Rightarrow A+B = 0$; constant $= A-B+C+D$
$\Rightarrow A-B+C+D = 1 \Rightarrow A-B = \frac{1}{2}$; thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int\frac{dx}{(x^2-1)^2}$
$= \frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{4}\int\frac{dx}{x-1} + \frac{1}{4}\int\frac{dx}{(x+1)^2} + \frac{1}{4}\int\frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x=-1$
$\Rightarrow C = -\frac{1}{2};\ x=1 \Rightarrow A = \frac{1}{4}$; coefficient of $x^2 = A+B \Rightarrow A+B = 1 \Rightarrow B = \frac{3}{4};\ \int\frac{x^2\,dx}{(x-1)(x^2+2x+1)}$
$= \frac{1}{4}\int\frac{dx}{x-1} + \frac{3}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C = \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x=-1 \Rightarrow A = \frac{1}{2}$; coefficient of x^2
$= A+B \Rightarrow A+B = 0 \Rightarrow B = -\frac{1}{2}$; constant $= A+C \Rightarrow A+C = 1 \Rightarrow C = \frac{1}{2};\ \int_0^1\frac{dx}{(x+1)(x^2+1)}$
$= \frac{1}{2}\int_0^1\frac{dx}{x+1} + \frac{1}{2}\int_0^1\frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$
$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t;\ t=0 \Rightarrow A=4$; coefficient of t^2
$= A+B \Rightarrow A+B = 3 \Rightarrow B = -1$; coefficient of $t = C \Rightarrow C = 1;\ \int_1^{\sqrt{3}}\frac{3t^2+t+4}{t^3+1}\,dt$

$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\, dt = \left[4 \ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1} t\right]_1^{\sqrt{3}}$

$= \left(4 \ln \sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4 \ln 1 - \frac{1}{2}\ln 2 + \tan^{-1} 1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$

$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$

$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A = 0, B = 1; A + C = 2 \Rightarrow C = 2; B + D = 1 \Rightarrow D = 0;$

$\int \frac{y^2+2y+1}{(y^2+1)^2}\, dy = \int \frac{1}{y^2+1}\, dy + 2\int \frac{y}{(y^2+1)^2}\, dy = \tan^{-1} y - \frac{1}{y^2+1} + C$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$

$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D); A = 0, B = 2; A + C = 8 \Rightarrow C = 8; B + D = 2 \Rightarrow D = 0;$

$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\, dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{(4x^2+1)^2} = \tan^{-1} 2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s + 2$

$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$

$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$
$\quad + E(s^2+1)$

$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{aligned} A \quad + C \quad\quad &= 0 \\ -3A + B - 2C + D \quad &= 0 \\ 3A - 3B + 2C - D + E &= 0 \\ -A + 3B - 2C + D \quad &= 2 \\ -B + C - D + E &= 2 \end{aligned}\right\} \text{summing all equations} \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0; C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;

$\int \frac{2s+2}{(s^2+1)(s-1)^3}\, ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1} s + C$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4 + 81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$

$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$

$= (A+B)s^4 + Cs^3 + (18A + 9B + D)s^2 + (9C + E)s + 81A \Rightarrow 81A = 81$ or $A = 1; A + B = 1 \Rightarrow B = 0;$

$C = 0; 9C + E = 0 \Rightarrow E = 0; 18A + 9B + D = 0 \Rightarrow D = -18; \int \frac{s^4+81}{s(s^2+9)^2}\, ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$

$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{x^2-x+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow x^2 - x + 2 = A(x^2+x+1) + (Bx+C)(x-1) = (A+B)x^2 + (A-B+C)x + (A-C)$

$\Rightarrow A + B = 1,\ A - B + C = -1,\ A - C = 2 \Rightarrow$ adding eq(2) and eq(3) $\Rightarrow 2A - B = 1$, add this equation to eq(1)

$\Rightarrow 3A = 2 \Rightarrow A = \frac{2}{3} \Rightarrow B = 1 - A = \frac{1}{3} \Rightarrow C = -1 - A + B = -\frac{4}{3}; \int \frac{x^2-x+2}{x^3-1}dx = \int\left(\frac{2/3}{x-1} + \frac{(1/3)x - 4/3}{x^2+x+1}\right)dx$

$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{x-4}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}dx \quad \left[u = x + \frac{1}{2} \Rightarrow u - \frac{1}{2} = x \Rightarrow du = dx\right]$

$= \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u - \frac{9}{2}}{u^2 + \frac{3}{4}}du = \frac{2}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}du - \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}du$

$= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln\left|\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right| - \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C = \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

28. $\frac{1}{x^4+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} \Rightarrow 1 = A(x+1)(x^2-x+1) + Bx(x^2-x+1) + (Cx+D)x(x+1)$
$= (A+B+C)x^3 + (-B+C+D)x^2 + (B+D)x + A \Rightarrow A = 1, B+D = 0 \Rightarrow D = -B, -B+C+D = 0$
$\Rightarrow -2B + C = 0 \Rightarrow C = 2B, A+B+C = 0 \Rightarrow 1 + B + 2B = 0 \Rightarrow B = -\frac{1}{3} \Rightarrow C = -\frac{2}{3} \Rightarrow D = \frac{1}{3};$
$\int \frac{1}{x^4+x}dx = \int\left(\frac{1}{x} - \frac{1/3}{x+1} + \frac{(-2/3)x+1/3}{x^2-x+1}\right)dx = \int \frac{1}{x}dx - \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{2x-1}{x^2-x+1}dx$
$= \ln|x| - \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x^2-x+1| + C$

29. $\frac{x^2}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x-1)(x+1)$
$= (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x - A + B - D \Rightarrow A+B+C = 0, -A+B+D = 1,$
$A+B-C = 0, -A+B-D = 0 \Rightarrow$ adding eq(1) to eq (3) gives $2A+2B = 0$, adding eq(2) to eq(4) gives
$-2A+2B = 1$, adding these two equations gives $4B = 1 \Rightarrow B = \frac{1}{4}$, using $2A+2B = 0 \Rightarrow A = -\frac{1}{4}$, using
$-A+B-D = 0 \Rightarrow D = \frac{1}{2}$, and using $A+B-C = 0 \Rightarrow C = 0$; $\int \frac{x^2}{x^4-1}dx = \int\left(\frac{-1/4}{x+1} + \frac{1/4}{x-1} + \frac{1/2}{x^2+1}\right)dx$
$= -\frac{1}{4}\int \frac{1}{x+1}dx + \frac{1}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx = -\frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + \frac{1}{2}\tan^{-1}x + C = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x + C$

30. $\frac{x^2+x}{x^4-3x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1} \Rightarrow x^2 + x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x-2)(x+2)$
$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A - 2B - 4D \Rightarrow A+B+C = 0, 2A-2B+D = 1,$
$A+B-4C = 1, 2A-2B-4D = 0 \Rightarrow$ subtractin eq(1) from eq (3) gives $-5C = 1 \Rightarrow C = -\frac{1}{5}$, subtacting eq(2) from eq(4) gives $-5D = -1 \Rightarrow D = \frac{1}{5}$, substituting for C in eq(1) gives $A+B = \frac{1}{5}$, and substituting for D in eq(4) gives
$2A-2B = \frac{4}{5} \Rightarrow A-B = \frac{2}{5}$, adding this equation to the previous equatin gives $2A = \frac{3}{5} \Rightarrow A = \frac{3}{10} \Rightarrow B = -\frac{1}{10};$
$\int \frac{x^2+x}{x^4-3x^2-4}dx = \int\left(\frac{3/10}{x-2} - \frac{1/10}{x+2} + \frac{(-1/5)x+1/5}{x^2+1}\right)dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$
$\frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x^2+1| + \frac{1}{5}\tan^{-1}x + C$

31. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$
$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2; 2A+B = 5 \Rightarrow B = 1; 2A+2B+C = 8 \Rightarrow C = 2;$
$2B+D = 4 \Rightarrow D = 2; \int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$
$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$
$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

32. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1$
$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta + F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta + F$
$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + (B+D+F) \Rightarrow A = 0; B = 1; 2A+C = -4$
$\Rightarrow C = -4; 2B+D = 2 \Rightarrow D = 0; A+C+E = -3 \Rightarrow E = 1; B+D+F = 1 \Rightarrow F = 0;$
$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C$

33. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}; \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + Bx; x = 0 \Rightarrow A = -1;$
$x = 1 \Rightarrow B = 1; \int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

34. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}; \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x+1);$
$x = -1 \Rightarrow A = -\frac{1}{2}; x = 1 \Rightarrow B = \frac{1}{2}; \int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

35. $\frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$
$\Rightarrow 9x^2 - 3x + 1 = Ax(x-1) + B(x-1) + Cx^2$; $x = 1 \Rightarrow C = 7$; $x = 0 \Rightarrow B = -1$; $A + C = 9 \Rightarrow A = 2$;
$\int \frac{9x^3-3x+1}{x^3-x^2}\,dx = \int 9\,dx + 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 7\int \frac{dx}{x-1} = 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$

36. $\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x - 4 = A(2x-1) + B$
$\Rightarrow A = 6$; $-A + B = -4 \Rightarrow B = 2$; $\int \frac{16x^3}{4x^2-4x+1}\,dx = 4\int (x+1)\,dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$
$= 2(x+1)^2 + 3\ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3\ln|2x-1| - (2x-1)^{-1} + C$, where $C = 2 + C_1$

37. $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1) + (By+C)y = (A+B)y^2 + Cy + A$
$7 \Rightarrow A = 1$; $A + B = 0 \Rightarrow B = -1$; $C = 0$; $\int \frac{y^4+y^2-1}{y^3+y}\,dy = \int y\,dy - \int \frac{dy}{y} + \int \frac{y\,dy}{y^2+1}$
$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C$

38. $\frac{2y^4}{y^3-y^2+y-1} = 2y + 2 + \frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1} = \frac{2}{(y^2+1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$
$\Rightarrow 2 = A(y^2+1) + (By+C)(y-1) = (Ay^2+A) + (By^2+Cy-By-C) = (A+B)y^2 + (-B+C)y + (A-C)$
$\Rightarrow A + B = 0$, $-B + C = 0$ or $C = B$, $A - C = A - B = 2 \Rightarrow A = 1, B = -1, C = -1$;
$\int \frac{2y^4}{y^3-y^2+y-1}\,dy = 2\int (y+1)\,dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= (y+1)^2 + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C_1 = y^2 + 2y + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C$,
where $C = C_1 + 1$

39. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y] \int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

40. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt = \int \frac{e^{3t}+2e^t-1}{e^{2t}+1} e^t dt$; $\begin{bmatrix} y = e^t \\ dy = e^t\,dt \end{bmatrix} \to \int \frac{y^3+2y-1}{y^2+1}\,dy = \int \left(y + \frac{y-1}{y^2+1}\right) dy = \frac{y^2}{2} + \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= \frac{y^2}{2} + \frac{1}{2}\ln(y^2+1) - \tan^{-1} y + C = \frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C$

41. $\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t, \cos y\,dy = dt] \to \int \frac{dy}{t^2+t-6} = \frac{1}{5}\int \left(\frac{1}{t-2} - \frac{1}{t+3}\right) dt = \frac{1}{5}\ln\left|\frac{t-2}{t+3}\right| + C$
$= \frac{1}{5}\ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$

42. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \to -\int \frac{dy}{y^2+y-2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$
$= \frac{1}{3}\ln\left|\frac{2+\cos\theta}{1-\cos\theta}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

43. $\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\,dx - 3\int \frac{x}{(x-2)^2}\,dx$
$= \frac{1}{2}\int \tan^{-1}(2x)\,d\left(\tan^{-1}(2x)\right) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2} = \frac{(\tan^{-1} 2x)^2}{4} - 3\ln|x-2| + \frac{6}{x-2} + C$

44. $\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\,dx + \int \frac{x}{(x+1)^2}\,dx$
$= \frac{1}{3}\int \tan^{-1}(3x)\,d\left(\tan^{-1}(3x)\right) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{(\tan^{-1} 3x)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$

45. $\int \frac{1}{x^{3/2} - \sqrt{x}}dx = \int \frac{1}{\sqrt{x}(x-1)}dx$ $\left[\text{Let } u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}dx\right] \to \int \frac{2}{u^2-1}du$;
$\frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0, -A + B = 2$

$\Rightarrow B = 1 \Rightarrow A = -1;\ \int \frac{2}{u^2-1}du = \int \left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = -\int \frac{1}{u+1}du + \int \frac{1}{u-1}du = -\ln|u+1| + \ln|u-1| + C$
$= \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$

46. $\int \frac{1}{(x^{1/3}-1)\sqrt{x}}dx\ \left[\text{Let } x = u^6 \Rightarrow dx = 6u^5\,du\right] \to \int \frac{1}{(u^2-1)u^3}6u^5\,du = \int \frac{6u^2}{u^2-1}du = \int \left(6 + \frac{6}{u^2-1}\right)du$
$= 6\int du + \int \frac{6}{u^2-1}du;\ \frac{6}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 6 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$
$-A + B = 6 \Rightarrow B = 3 \Rightarrow A = -3;\ 6\int du + \int \frac{6}{u^2-1}du = 6u + \int \left(\frac{-3}{u+1} + \frac{3}{u-1}\right)du = 6u - 3\int \frac{1}{u+1}du + 3\int \frac{1}{u-1}du$
$= 6u - 3\ln|u+1| + 3\ln|u-1| + C = 6x^{1/6} + 3\ln\left|\frac{x^{1/6}-1}{x^{1/6}+1}\right| + C$

47. $\int \frac{\sqrt{x+1}}{x}dx\ \left[\text{Let } x + 1 = u^2 \Rightarrow dx = 2u\,du\right] \to \int \frac{u}{u^2-1}\,2u\,du = \int \frac{2u^2}{u^2-1}du = \int \left(2 + \frac{2}{u^2-1}\right)du$
$= 2\int du + \int \frac{2}{u^2-1}du;\ \frac{2}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1) = (A+B)u - A + B \Rightarrow A + B = 0,$
$-A + B = 2 \Rightarrow B = 1 \Rightarrow A = -1;\ 2\int du + \int \frac{2}{u^2-1}du = 2u + \int \left(\frac{-1}{u+1} + \frac{1}{u-1}\right)du = 2u - \int \frac{1}{u+1}du + \int \frac{1}{u-1}du$
$= 2u - \ln|u+1| + \ln|u-1| + C = 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

48. $\int \frac{1}{x\sqrt{x+9}}dx\ \left[\text{Let } x + 9 = u^2 \Rightarrow dx = 2u\,du\right] \to \int \frac{1}{(u^2-9)u}\,2u\,du = \int \frac{2}{u^2-9}\,du;\ \frac{2}{u^2-9} = \frac{A}{u-3} + \frac{B}{u+3}$
$\Rightarrow 2 = A(u+3) + B(u-3) = (A+B)u + 3A - 3B \Rightarrow A + B = 0,\ 3A - 3B = 2 \Rightarrow A = \frac{1}{3} \Rightarrow B = -\frac{1}{3};$
$\int \frac{2}{u^2-9}\,du = \int \left(\frac{1/3}{u-3} - \frac{1/3}{u+3}\right)du = \frac{1}{3}\int \frac{1}{u-3}\,du - \frac{1}{3}\int \frac{1}{u+3}\,du = \frac{1}{3}\ln|u-3| - \frac{1}{3}\ln|u+3| + C = \frac{1}{3}\ln\left|\frac{\sqrt{x+9}-3}{\sqrt{x+9}+3}\right| + C$

49. $\int \frac{1}{x(x^4+1)}dx = \int \frac{x^3}{x^4(x^4+1)}dx\ \left[\text{Let } u = x^4 \Rightarrow du = 4x^3\,dx\right] \to \frac{1}{4}\int \frac{1}{u(u+1)}du;\ \frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$
$\Rightarrow 1 = A(u+1) + Bu = (A+B)u + A \Rightarrow A = 1 \Rightarrow B = -1;\ \frac{1}{4}\int \frac{1}{u(u+1)}du = \frac{1}{4}\int \left(\frac{1}{u} - \frac{1}{u+1}\right)du$
$= \frac{1}{4}\int \frac{1}{u}du - \frac{1}{4}\int \frac{1}{u+1}du = \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u+1| + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$

50. $\int \frac{1}{x^6(x^5+4)}dx = \int \frac{x^4}{x^{10}(x^5+4)}dx = \left[\text{Let } u = x^5 \Rightarrow du = 5x^4\,dx\right] \to \frac{1}{5}\int \frac{1}{u^2(u+4)}du;\ \frac{1}{u^2(u+4)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+4}$
$\Rightarrow 1 = Au(u+4) + B(u+4) + Cu^2 = (A+C)u^2 + (4A+B)u + 4B \Rightarrow A + C = 0,\ 4A + B = 0,\ 4B = 1 \Rightarrow B = \frac{1}{4}$
$\Rightarrow A = -\frac{1}{16} \Rightarrow C = \frac{1}{16};\ \frac{1}{5}\int \frac{1}{u^2(u+4)}du = \frac{1}{5}\int \left(-\frac{1/16}{u} + \frac{1/4}{u^2} + \frac{1/16}{u+4}\right)du = -\frac{1}{80}\int \frac{1}{u}du + \frac{1}{20}\int \frac{1}{u^2}du + \frac{1}{80}\int \frac{1}{u+4}du$
$= -\frac{1}{80}\ln|u| - \frac{1}{20u} + \frac{1}{80}\ln|u+4| + C = -\frac{1}{80}\ln|x^5| - \frac{1}{20x^5} + \frac{1}{80}\ln|x^5+4| + C = \frac{1}{80}\ln\left|\frac{x^5+4}{x^5}\right| - \frac{1}{20x^5} + C$

51. $(t^2 - 3t + 2)\,\frac{dx}{dt} = 1;\ x = \int \frac{dt}{t^2-3t+2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C;\ \frac{t-2}{t-1} = Ce^x;\ t = 3 \text{ and } x = 0$
$\Rightarrow\ \frac{1}{2} = C\ \Rightarrow\ \frac{t-2}{t-1} = \frac{1}{2}e^x\ \Rightarrow\ x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

52. $(3t^4 + 4t^2 + 1)\,\frac{dx}{dt} = 2\sqrt{3};\ x = 2\sqrt{3}\int \frac{dt}{3t^4+4t^2+1} = \sqrt{3}\int \frac{dt}{t^2+\frac{1}{3}} - \sqrt{3}\int \frac{dt}{t^2+1}$
$= 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t + C;\ t = 1 \text{ and } x = \frac{-\pi\sqrt{3}}{4}\ \Rightarrow\ -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C\ \Rightarrow\ C = -\pi$
$\Rightarrow\ x = 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1}t - \pi$

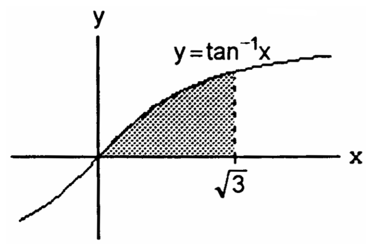
53. $(t^2+2t)\frac{dx}{dt} = 2x+2$; $\frac{1}{2}\int \frac{dx}{x+1} = \int \frac{dt}{t^2+2t} \Rightarrow \frac{1}{2}\ln|x+1| = \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+2} \Rightarrow \ln|x+1| = \ln\left|\frac{t}{t+2}\right| + C$;
$t = 1$ and $x = 1 \Rightarrow \ln 2 = \ln\frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln|x+1| = \ln 6\left|\frac{t}{t+2}\right| \Rightarrow x+1 = \frac{6t}{t+2}$
$\Rightarrow x = \frac{6t}{t+2} - 1,\ t > 0$

54. $(t+1)\frac{dx}{dt} = x^2+1 \Rightarrow \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln|t+1| + C$; $t = 0$ and $x = 0 \Rightarrow \tan^{-1} 0 = \ln|1| + C$
$\Rightarrow C = \tan^{-1} 0 = 0 \Rightarrow \tan^{-1} x = \ln|t+1| \Rightarrow x = \tan(\ln(t+1)),\ t > -1$

55. $V = \pi\int_{0.5}^{2.5} y^2\,dx = \pi\int_{0.5}^{2.5} \frac{9}{3x-x^2}\,dx = 3\pi\left(\int_{0.5}^{2.5}\left(-\frac{1}{x-3} + \frac{1}{x}\right)\right)dx = \left[3\pi\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5} = 3\pi\ln 25$

56. $V = 2\pi\int_0^1 xy\,dx = 2\pi\int_0^1 \frac{2x}{(x+1)(2-x)}\,dx = 4\pi\int_0^1\left(-\frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{2}{3}\left(\frac{1}{2-x}\right)\right)dx$
$= \left[-\frac{4\pi}{3}(\ln|x+1| + 2\ln|2-x|)\right]_0^1 = \frac{4\pi}{3}(\ln 2)$

57. $A = \int_0^{\sqrt{3}} \tan^{-1} x\,dx = [x\tan^{-1} x]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx$
$= \frac{\pi\sqrt{3}}{3} - \left[\frac{1}{2}\ln(x^2+1)\right]_0^{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} - \ln 2$;
$\bar{x} = \frac{1}{A}\int_0^{\sqrt{3}} x\tan^{-1} x\,dx$
$= \frac{1}{A}\left(\left[\frac{1}{2}x^2\tan^{-1} x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\,dx\right)$
$= \frac{1}{A}\left[\frac{\pi}{2} - \left[\frac{1}{2}(x - \tan^{-1} x)\right]_0^{\sqrt{3}}\right]$
$= \frac{1}{A}\left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) = \frac{1}{A}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \cong 1.10$



58. $A = \int_3^5 \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx = 3\int_3^5 \frac{dx}{x} - \int_3^5 \frac{dx}{x+3} + 2\int_3^5 \frac{dx}{x-1} = [3\ln|x| - \ln|x+3| + 2\ln|x-1|]_3^5 = \ln\frac{125}{9}$;
$\bar{x} = \frac{1}{A}\int_3^5 \frac{x(4x^2+13x-9)}{x^3+2x^2-3x}\,dx = \frac{1}{A}\left([4x]_3^5 + 3\int_3^5 \frac{dx}{x+3} + 2\int_3^5 \frac{dx}{x-1}\right) = \frac{1}{A}(8 + 11\ln 2 - 3\ln 6) \cong 3.90$

59. (a) $\frac{dx}{dt} = kx(N-x) \Rightarrow \int \frac{dx}{x(N-x)} = \int k\,dt \Rightarrow \frac{1}{N}\int \frac{dx}{x} + \frac{1}{N}\int \frac{dx}{N-x} = \int k\,dt \Rightarrow \frac{1}{N}\ln\left|\frac{x}{N-x}\right| = kt + C$;
$k = \frac{1}{250}$, $N = 1000$, $t = 0$ and $x = 2 \Rightarrow \frac{1}{1000}\ln\left|\frac{2}{998}\right| = C \Rightarrow \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| = \frac{t}{250} + \frac{1}{1000}\ln\left(\frac{1}{499}\right)$
$\Rightarrow \ln\left|\frac{499x}{1000-x}\right| = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow 499x = e^{4t}(1000-x) \Rightarrow (499 + e^{4t})x = 1000e^{4t} \Rightarrow x = \frac{1000e^{4t}}{499+e^{4t}}$
(b) $x = \frac{1}{2}N = 500 \Rightarrow 500 = \frac{1000e^{4t}}{499+e^{4t}} \Rightarrow 500\cdot 499 + 500e^{4t} = 1000e^{4t} \Rightarrow e^{4t} = 499 \Rightarrow t = \frac{1}{4}\ln 499 \approx 1.55$ days

60. $\frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k\,dt$
(a) $a = b$: $\int \frac{dx}{(a-x)^2} = \int k\,dt \Rightarrow \frac{1}{a-x} = kt + C$; $t = 0$ and $x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a}$
$\Rightarrow \frac{1}{a-x} = \frac{akt+1}{a} \Rightarrow a - x = \frac{a}{akt+1} \Rightarrow x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}$
(b) $a \neq b$: $\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \Rightarrow \frac{1}{b-a}\int \frac{dx}{a-x} - \frac{1}{b-a}\int \frac{dx}{b-x} = \int k\,dt \Rightarrow \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| = kt + C$;
$t = 0$ and $x = 0 \Rightarrow \frac{1}{b-a}\ln\frac{b}{a} = C \Rightarrow \ln\left|\frac{b-x}{a-x}\right| = (b-a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt}$
$\Rightarrow x = \frac{ab\left[1 - e^{(b-a)kt}\right]}{a - be^{(b-a)kt}}$

8.5 INTEGRAL TABLES AND COMPUTER ALGEBRA SYSTEMS

1. $\int \frac{dx}{x\sqrt{x-3}} = \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{x-3}{3}} + C$
(We used FORMULA 13(a) with $a = 1$, $b = 3$)

2. $\int \frac{dx}{x\sqrt{x+4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{x+4}-\sqrt{4}}{\sqrt{x+4}+\sqrt{4}} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x+4}-2}{\sqrt{x+4}+2} \right| + C$
(We used FORMULA 13(b) with a = 1, b = 4)

3. $\int \frac{x\,dx}{\sqrt{x-2}} = \int \frac{(x-2)\,dx}{\sqrt{x-2}} + 2\int \frac{dx}{\sqrt{x-2}} = \int \left(\sqrt{x-2}\right)^1 dx + 2\int \left(\sqrt{x-2}\right)^{-1} dx$
$= \left(\frac{2}{1}\right) \frac{\left(\sqrt{x-2}\right)^3}{3} + 2\left(\frac{2}{1}\right) \frac{\left(\sqrt{x-2}\right)^1}{1} = \sqrt{x-2}\left[\frac{2(x-2)}{3} + 4\right] + C$
(We used FORMULA 11 with a = 1, b = −2, n = 1 and a = 1, b = −2, n = −1)

4. $\int \frac{x\,dx}{(2x+3)^{3/2}} = \frac{1}{2}\int \frac{(2x+3)\,dx}{(2x+3)^{3/2}} - \frac{3}{2}\int \frac{dx}{(2x+3)^{3/2}} = \frac{1}{2}\int \frac{dx}{\sqrt{2x+3}} - \frac{3}{2}\int \frac{dx}{\left(\sqrt{2x+3}\right)^3}$
$= \frac{1}{2}\int \left(\sqrt{2x+3}\right)^{-1} dx - \frac{3}{2}\int \left(\sqrt{2x+3}\right)^{-3} dx = \left(\frac{1}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x+3}\right)^1}{1} - \left(\frac{3}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x+3}\right)^{-1}}{(-1)} + C$
$= \frac{1}{2\sqrt{2x+3}}(2x+3+3) + C = \frac{(x+3)}{\sqrt{2x+3}} + C$
(We used FORMULA 11 with a = 2, b = 3, n = −1 and a = 2, b = 3, n = −3)

5. $\int x\sqrt{2x-3}\,dx = \frac{1}{2}\int (2x-3)\sqrt{2x-3}\,dx + \frac{3}{2}\int \sqrt{2x-3}\,dx = \frac{1}{2}\int \left(\sqrt{2x-3}\right)^3 dx + \frac{3}{2}\int \left(\sqrt{2x-3}\right)^1 dx$
$= \left(\frac{1}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x-3}\right)^5}{5} + \left(\frac{3}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x-3}\right)^3}{3} + C = \frac{(2x-3)^{3/2}}{2}\left[\frac{2x-3}{5} + 1\right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C$
(We used FORMULA 11 with a = 2, b = −3, n = 3 and a = 2, b = −3, n = 1)

6. $\int x(7x+5)^{3/2}\,dx = \frac{1}{7}\int (7x+5)(7x+5)^{3/2}\,dx - \frac{5}{7}\int (7x+5)^{3/2}\,dx = \frac{1}{7}\int \left(\sqrt{7x+5}\right)^5 dx - \frac{5}{7}\int \left(\sqrt{7x+5}\right)^3 dx$
$= \left(\frac{1}{7}\right)\left(\frac{2}{7}\right) \frac{\left(\sqrt{7x+5}\right)^7}{7} - \left(\frac{5}{7}\right)\left(\frac{2}{7}\right) \frac{\left(\sqrt{7x+5}\right)^5}{5} + C = \left[\frac{(7x+5)^{5/2}}{49}\right]\left[\frac{2(7x+5)}{7} - 2\right] + C$
$= \left[\frac{(7x+5)^{5/2}}{49}\right]\left(\frac{14x-4}{7}\right) + C$
(We used FORMULA 11 with a = 7, b = 5, n = 5 and a = 7, b = 5, n = 3)

7. $\int \frac{\sqrt{9-4x}}{x^2}\,dx = -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2}\int \frac{dx}{x\sqrt{9-4x}} + C$
(We used FORMULA 14 with a = −4, b = 9)
$= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right) \ln \left| \frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}} \right| + C$
(We used FORMULA 13(b) with a = −4, b = 9)
$= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3} \ln \left| \frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3} \right| + C$

8. $\int \frac{dx}{x^2\sqrt{4x-9}} = -\frac{\sqrt{4x-9}}{(-9)x} + \frac{4}{18}\int \frac{dx}{x\sqrt{4x-9}} + C$
(We used FORMULA 15 with a = 4, b = −9)
$= \frac{\sqrt{4x-9}}{9x} + \left(\frac{2}{9}\right)\left(\frac{2}{\sqrt{9}}\right) \tan^{-1} \sqrt{\frac{4x-9}{9}} + C$
(We used FORMULA 13(a) with a = 4, b = 9)
$= \frac{\sqrt{4x-9}}{9x} + \frac{4}{27} \tan^{-1} \sqrt{\frac{4x-9}{9}} + C$

9. $\int x\sqrt{4x-x^2}\,dx = \int x\sqrt{2\cdot 2x - x^2}\,dx = \frac{(x+2)(2x-3\cdot 2)\sqrt{2\cdot 2\cdot x - x^2}}{6} + \frac{2^3}{2}\sin^{-1}\left(\frac{x-2}{2}\right) + C$
$= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C = \frac{(x+2)(x-3)\sqrt{4x-x^2}}{3} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$
(We used FORMULA 51 with a = 2)

10. $\int \frac{\sqrt{x-x^2}}{x}\,dx = \int \frac{\sqrt{2\cdot\frac{1}{2}x - x^2}}{x}\,dx = \sqrt{2\cdot\frac{1}{2}x - x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C = \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C$

(We used FORMULA 52 with $a = \frac{1}{2}$)

11. $\int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 + x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2+x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right| + C$

(We used FORMULA 26 with $a = \sqrt{7}$)

12. $\int \frac{dx}{x\sqrt{7-x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 - x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2-x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7-x^2}}{x}\right| + C$

(We used FORMULA 34 with $a = \sqrt{7}$)

13. $\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{2^2-x^2}}{x}\,dx = \sqrt{2^2-x^2} - 2\ln\left|\frac{2+\sqrt{2^2-x^2}}{x}\right| + C = \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$

(We used FORMULA 31 with $a = 2$)

14. $\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{x^2-2^2}}{x}\,dx = \sqrt{x^2-2^2} - 2\sec^{-1}\left|\frac{x}{2}\right| + C = \sqrt{x^2-4} - 2\sec^{-1}\left|\frac{x}{2}\right| + C$

(We used FORMULA 42 with $a = 2$)

15. $\int e^{2t}\cos 3t\,dt = \frac{e^{2t}}{2^2+3^2}(2\cos 3t + 3\sin 3t) + C = \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$

(We used FORMULA 108 with $a = 2$, $b = 3$)

16. $\int e^{-3t}\sin 4t\,dt = \frac{e^{-3t}}{(-3)^2+4^2}(-3\sin 4t - 4\cos 4t) + C = \frac{e^{-3t}}{25}(-3\sin 4t - 4\cos 4t) + C$

(We used FORMULA 107 with $a = -3$, $b = 4$)

17. $\int x\cos^{-1}x\,dx = \int x^1\cos^{-1}x\,dx = \frac{x^{1+1}}{1+1}\cos^{-1}x + \frac{1}{1+1}\int \frac{x^{1+1}\,dx}{\sqrt{1-x^2}} = \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

(We used FORMULA 100 with $a = 1$, $n = 1$)

$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x\right) - \frac{1}{2}\left(\frac{1}{2}x\sqrt{1-x^2}\right) + C = \frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + C$

(We used FORMULA 33 with $a = 1$)

18. $\int x\tan^{-1}x\,dx = \int x^1\tan^{-1}(1x)\,dx = \frac{x^{1+1}}{1+1}\tan^{-1}(1x) - \frac{1}{1+1}\int \frac{x^{1+1}\,dx}{1+(1)^2x^2} = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2\,dx}{1+x^2}$

(We used FORMULA 101 with $a = 1$, $n = 1$)

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$ (after long division)

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C = \frac{1}{2}\left((x^2+1)\tan^{-1}x - x\right) + C$

19. $\int x^2\tan^{-1}x\,dx = \frac{x^{2+1}}{2+1}\tan^{-1}x - \frac{1}{2+1}\int \frac{x^{2+1}}{1+x^2}\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$

(We used FORMULA 101 with $a = 1$, $n = 2$);

$\int \frac{x^3}{1+x^2}\,dx = \int x\,dx - \int \frac{x\,dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C \Rightarrow \int x^2\tan^{-1}x\,dx$

$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$

20. $\int \frac{\tan^{-1} x}{x^2}\, dx = \int x^{-2} \tan^{-1} x\, dx = \frac{x^{(-2+1)}}{(-2+1)} \tan^{-1} x - \frac{1}{(-2+1)} \int \frac{x^{(-2+1)}}{1+x^2}\, dx = \frac{x^{-1}}{(-1)} \tan^{-1} x + \int \frac{x^{-1}}{(1+x^2)}\, dx$

(We used FORMULA 101 with $a = 1$, $n = -2$);

$\int \frac{x^{-1}\, dx}{1+x^2} = \int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} - \int \frac{x\, dx}{1+x^2} = \ln|x| - \frac{1}{2} \ln(1+x^2) + C$

$\Rightarrow \int \frac{\tan^{-1} x}{x^2}\, dx = -\frac{1}{x} \tan^{-1} x + \ln|x| - \frac{1}{2} \ln(1+x^2) + C$

21. $\int \sin 3x \cos 2x\, dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a = 3$, $b = 2$)

22. $\int \sin 2x \cos 3x\, dx = -\frac{\cos 5x}{10} + \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with $a = 2$, $b = 3$)

23. $\int 8 \sin 4t \sin \frac{t}{2}\, dx = \frac{8}{7} \sin\left(\frac{7t}{2}\right) - \frac{8}{9} \sin\left(\frac{9t}{2}\right) + C = 8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9}\right] + C$

(We used FORMULA 62(b) with $a = 4$, $b = \frac{1}{2}$)

24. $\int \sin \frac{t}{3} \sin \frac{t}{6}\, dt = 3 \sin\left(\frac{t}{6}\right) - \sin\left(\frac{t}{2}\right) + C$

(We used FORMULA 62(b) with $a = \frac{1}{3}$, $b = \frac{1}{6}$)

25. $\int \cos \frac{\theta}{3} \cos \frac{\theta}{4}\, d\theta = 6 \sin\left(\frac{\theta}{12}\right) + \frac{6}{7} \sin\left(\frac{7\theta}{12}\right) + C$

(We used FORMULA 62(c) with $a = \frac{1}{3}$, $b = \frac{1}{4}$)

26. $\int \cos \frac{\theta}{2} \cos 7\theta\, d\theta = \frac{1}{13} \sin\left(\frac{13\theta}{2}\right) + \frac{1}{15} \sin\left(\frac{15\theta}{2}\right) + C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13} + \frac{\sin\left(\frac{15\theta}{2}\right)}{15} + C$

(We used FORMULA 62(c) with $a = \frac{1}{2}$, $b = 7$)

27. $\int \frac{x^3+x+1}{(x^2+1)^2}\, dx = \int \frac{x\, dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2} = \frac{1}{2} \int \frac{d(x^2+1)}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$

$= \frac{1}{2} \ln(x^2+1) + \frac{x}{2(1+x^2)} + \frac{1}{2} \tan^{-1} x + C$

(For the second integral we used FORMULA 17 with $a = 1$)

28. $\int \frac{x^2+6x}{(x^2+3)^2}\, dx = \int \frac{dx}{x^2+3} + \int \frac{6x\, dx}{(x^2+3)^2} - \int \frac{3\, dx}{(x^2+3)^2} = \int \frac{dx}{x^2+\left(\sqrt{3}\right)^2} + 3 \int \frac{d(x^2+3)}{(x^2+3)^2} - 3 \int \frac{dx}{\left[x^2+\left(\sqrt{3}\right)^2\right]^2}$

$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{(x^2+3)} - 3\left(\frac{x}{2\left(\sqrt{3}\right)^2\left(\left(\sqrt{3}\right)^2+x^2\right)} + \frac{1}{2\left(\sqrt{3}\right)^3} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$

(For the first integral we used FORMULA 16 with $a = \sqrt{3}$; for the third integral we used FORMULA 17 with $a = \sqrt{3}$)

$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - \frac{x}{2(x^2+3)} + C$

29. $\int \sin^{-1} \sqrt{x}\, dx;\ \left[\begin{array}{c} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\, du \end{array}\right] \rightarrow 2 \int u^1 \sin^{-1} u\, du = 2\left(\frac{u^{1+1}}{1+1} \sin^{-1} u - \frac{1}{1+1} \int \frac{u^{1+1}}{\sqrt{1-u^2}}\, du\right)$

$= u^2 \sin^{-1} u - \int \frac{u^2\,du}{\sqrt{1-u^2}}$

(We used FORMULA 99 with a = 1, n = 1)

$= u^2 \sin^{-1} u - \left(\frac{1}{2}\sin^{-1} u - \frac{1}{2}u\sqrt{1-u^2}\right) + C = \left(u^2 - \frac{1}{2}\right)\sin^{-1} u + \frac{1}{2}u\sqrt{1-u^2} + C$

(We used FORMULA 33 with a = 1)

$= \left(x - \frac{1}{2}\right)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x - x^2} + C$

30. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{\cos^{-1} u}{u}\cdot 2u\,du = 2\int \cos^{-1} u\,du = 2\left(u\cos^{-1} u - \frac{1}{1}\sqrt{1-u^2}\right) + C$

(We used FORMULA 97 with a = 1)

$= 2\left(\sqrt{x}\cos^{-1}\sqrt{x} - \sqrt{1-x}\right) + C$

31. $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{u\cdot 2u}{\sqrt{1-u^2}}\,du = 2\int \frac{u^2}{\sqrt{1-u^2}}\,du = 2\left(\frac{1}{2}\sin^{-1} u - \frac{1}{2}u\sqrt{1-u^2}\right) + C$

$= \sin^{-1} u - u\sqrt{1-u^2} + C$

(We used FORMULA 33 with a = 1)

$= \sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + C = \sin^{-1}\sqrt{x} - \sqrt{x - x^2} + C$

32. $\int \frac{\sqrt{2-x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{\sqrt{2-u^2}}{u}\cdot 2u\,du = 2\int \sqrt{\left(\sqrt{2}\right)^2 - u^2}\,du$

$= 2\left[\frac{u}{2}\sqrt{\left(\sqrt{2}\right)^2 - u^2} + \frac{\left(\sqrt{2}\right)^2}{2}\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)\right] + C = u\sqrt{2-u^2} + 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$

(We used FORMULA 29 with $a = \sqrt{2}$)

$= \sqrt{2x - x^2} + 2\sin^{-1}\sqrt{\frac{x}{2}} + C$

33. $\int (\cot t)\sqrt{1-\sin^2 t}\,dt = \int \frac{\sqrt{1-\sin^2 t}\,(\cos t)\,dt}{\sin t};\ \begin{bmatrix} u = \sin t \\ du = \cos t\,dt \end{bmatrix} \to \int \frac{\sqrt{1-u^2}\,du}{u}$

$= \sqrt{1-u^2} - \ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C$

(We used FORMULA 31 with a = 1)

$= \sqrt{1-\sin^2 t} - \ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right| + C$

34. $\int \frac{dt}{(\tan t)\sqrt{4-\sin^2 t}} = \int \frac{\cos t\,dt}{(\sin t)\sqrt{4-\sin^2 t}};\ \begin{bmatrix} u = \sin t \\ du = \cos t\,dt \end{bmatrix} \to \int \frac{du}{u\sqrt{4-u^2}} = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C$

(We used FORMULA 34 with a = 2)

$= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right| + C$

35. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}};\ \begin{bmatrix} u = \ln y \\ y = e^u \\ dy = e^u\,du \end{bmatrix} \to \int \frac{e^u\,du}{e^u\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln\left|u + \sqrt{3+u^2}\right| + C$

$= \ln\left|\ln y + \sqrt{3+(\ln y)^2}\right| + C$

(We used FORMULA 20 with $a = \sqrt{3}$)

36. $\int \tan^{-1}\sqrt{y}\,dy;\ \begin{bmatrix} t=\sqrt{y} \\ y=t^2 \\ dy=2t\,dt \end{bmatrix} \rightarrow 2\int t\tan^{-1}t\,dt = 2\left[\frac{t^2}{2}\tan^{-1}t - \frac{1}{2}\int \frac{t^2}{1+t^2}\,dt\right] = t^2\tan^{-1}t - \int \frac{t^2}{1+t^2}\,dt$

(We used FORMULA 101 with n = 1, a = 1)

$= t^2\tan^{-1}t - \int \frac{t^2+1}{t^2+1}\,dt + \int \frac{dt}{1+t^2} = t^2\tan^{-1}t - t + \tan^{-1}t + C = y\tan^{-1}\sqrt{y} + \tan^{-1}\sqrt{y} - \sqrt{y} + C$

37. $\int \frac{1}{\sqrt{x^2+2x+5}}dx = \int \frac{1}{\sqrt{(x+1)^2+4}}dx;\ \begin{bmatrix} t=x+1 \\ dt=dx \end{bmatrix} \rightarrow \int \frac{1}{\sqrt{t^2+4}}dt$

(We used FORMULA 20 with a = 2)

$= \ln\left|t+\sqrt{t^2+4}\right| + C = \ln\left|(x+1)+\sqrt{(x+1)^2+4}+C = \ln\right|(x+1)+\sqrt{x^2+2x+5}\,\Big| + C$

38. $\int \frac{x^2}{\sqrt{x^2-4x+5}}dx = \int \frac{x^2}{\sqrt{(x-2)^2+1}}dx;\ \begin{bmatrix} t=x-2 \\ dt=dx \end{bmatrix} \rightarrow \int \frac{(t+2)^2}{\sqrt{t^2+1}}dt = \int \frac{t^2+4t+2}{\sqrt{t^2+1}}dt = \int \frac{t^2}{\sqrt{t^2+1}}dt + \int \frac{4t}{\sqrt{t^2+1}}dt + \int \frac{4}{\sqrt{t^2+1}}dt$

(We used FORMULA 25 with a = 1) (We used FORMULA 20 with a = 1)

$= \left[-\frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right| + \frac{t\sqrt{t^2+1}}{2}\right] + 4\sqrt{t^2+1} + \left[4\ln\left|t+\sqrt{t^2+1}\right|\right] + C$

$= -\frac{1}{2}\ln\left|(x-2)+\sqrt{(x-2)^2+1}\right| + \frac{(x-2)\sqrt{(x-2)^2+1}}{2} + 4\sqrt{(x-2)^2+1} + 4\ln\left|(x-2)+\sqrt{(x-2)^2+1}\right| + C$

$= \frac{7}{2}\ln\left|(x-2)+\sqrt{x^2-4x+5}\right| + \frac{(x+6)\sqrt{x^2-4x+5}}{2} + C$

39. $\int \sqrt{5-4x-x^2}dx = \int \sqrt{9-(x+2)^2}dx;\ \begin{bmatrix} t=x+2 \\ dt=dx \end{bmatrix} \rightarrow \int \sqrt{9-t^2}dt;$

(We used FORMULA 29 with a = 3)

$= \frac{t}{2}\sqrt{9-t^2} + \frac{3^2}{2}\sin^{-1}\left(\frac{t}{3}\right) + C = \frac{x+2}{2}\sqrt{9-(x+2)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C = \frac{x+2}{2}\sqrt{5-4x-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + C$

40. $\int x^2\sqrt{2x-x^2}\,dx = \int x^2\sqrt{1-(x-1)^2}\,dx;\ \begin{bmatrix} t=x-1 \\ dt=dx \end{bmatrix} \rightarrow \int (t+1)^2\sqrt{1-t^2}\,dt = \int \left(t^2+2t+1\right)\sqrt{1-t^2}\,dt$

$= \int t^2\sqrt{1-t^2}\,dt + \int 2t\sqrt{1-t^2}\,dt + \int \sqrt{1-t^2}\,dt$

(We used FORMULA 30 with a = 1) (We used FORMULA 29 with a = 1)

$= \left[\frac{1^4}{8}\sin^{-1}\left(\frac{t}{1}\right) - \frac{1}{8}t\sqrt{1-t^2}(1^2-2t^2)\right] - \frac{2}{3}(1-t^2)^{3/2} + \left[\frac{t}{2}\sqrt{1-t^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{t}{1}\right)\right] + C$

$= \frac{1}{8}\sin^{-1}(x-1) - \frac{1}{8}(x-1)\sqrt{1-(x-1)^2}\left(1^2-2(x-1)^2\right) - \frac{2}{3}\left(1-(x-1)^2\right)^{3/2} + \frac{x-1}{2}\sqrt{1-(x-1)^2}$

$+ \frac{1}{2}\sin^{-1}(x-1) + C = \frac{5}{8}\sin^{-1}(x-1) - \frac{2}{3}(2x-x^2)^{3/2} + \frac{x-1}{8}\sqrt{2x-x^2}(2x^2-4x+5) + C$

41. $\int \sin^5 2x\,dx = -\frac{\sin^4 2x\cos 2x}{5\cdot 2} + \frac{5-1}{5}\int \sin^3 2x\,dx = -\frac{\sin^4 2x\cos 2x}{10} + \frac{4}{5}\left[-\frac{\sin^2 2x\cos 2x}{3\cdot 2} + \frac{3-1}{3}\int \sin 2x\,dx\right]$

(We used FORMULA 60 with a = 2, n = 5 and a = 2, n = 3)

$= -\frac{\sin^4 2x\cos 2x}{10} - \frac{2}{15}\sin^2 2x\cos 2x + \frac{8}{15}\left(-\frac{1}{2}\right)\cos 2x + C = -\frac{\sin^4 2x\cos 2x}{10} - \frac{2\sin^2 2x\cos 2x}{15} - \frac{4\cos 2x}{15} + C$

42. $\int 8\cos^4 2\pi t\,dt = 8\left(\frac{\cos^3 2\pi t\sin 2\pi t}{4\cdot 2\pi} + \frac{4-1}{4}\int \cos^2 2\pi t\,dt\right)$

(We used FORMULA 61 with a = 2π, n = 4)

$= \frac{\cos^3 2\pi t\sin 2\pi t}{\pi} + 6\left[\frac{t}{2} + \frac{\sin(2\cdot 2\pi\cdot t)}{4\cdot 2\pi}\right] + C$

(We used FORMULA 59 with a = 2π)

$= \frac{\cos^3 2\pi t\sin 2\pi t}{\pi} + 3t + \frac{3\sin 4\pi t}{4\pi} + C = \frac{\cos^3 2\pi t\sin 2\pi t}{\pi} + \frac{3\cos 2\pi t\sin 2\pi t}{2\pi} + 3t + C$

43. $\int \sin^2 2\theta \cos^3 2\theta \, d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{2(2+3)} + \frac{3-1}{3+2} \int \sin^2 2\theta \cos 2\theta \, d\theta$

(We used FORMULA 69 with a = 2, m = 3, n = 2)

$= \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5} \int \sin^2 2\theta \cos 2\theta \, d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5}\left[\frac{1}{2}\int \sin^2 2\theta \, d(\sin 2\theta)\right] = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{\sin^3 2\theta}{15} + C$

44. $\int 2 \sin^2 t \sec^4 t \, dt = \int 2 \sin^2 t \cos^{-4} t \, dt = 2\left(-\frac{\sin t \cos^{-3} t}{2-4} + \frac{2-1}{2-4}\int \cos^{-4} t \, dt\right)$

(We used FORMULA 68 with a = 1, n = 2, m = −4)

$= \sin t \cos^{-3} t - \int \cos^{-4} t \, dt = \sin t \cos^{-3} t - \int \sec^4 t \, dt = \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{4-1} + \frac{4-2}{4-1}\int \sec^2 t \, dt\right)$

(We used FORMULA 92 with a = 1, n = 4)

$= \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{3}\right) - \frac{2}{3}\tan t + C = \frac{2}{3}\sec^2 t \tan t - \frac{2}{3}\tan t + C = \frac{2}{3}\tan t\,(\sec^2 t - 1) + C$

$= \frac{2}{3}\tan^3 t + C$

An easy way to find the integral using substitution:

$\int 2 \sin^2 t \cos^{-4} t \, dt = \int 2 \tan^2 t \sec^2 t \, dt = \int 2 \tan^2 t \, d(\tan t) = \frac{2}{3}\tan^3 t + C$

45. $\int 4 \tan^3 2x \, dx = 4\left(\frac{\tan^2 2x}{2\cdot 2} - \int \tan 2x \, dx\right) = \tan^2 2x - 4\int \tan 2x \, dx$

(We used FORMULA 86 with n = 3, a = 2)

$= \tan^2 2x - \frac{4}{2}\ln|\sec 2x| + C = \tan^2 2x - 2\ln|\sec 2x| + C$

46. $\int 8 \cot^4 t \, dt = 8\left(-\frac{\cot^3 t}{3} - \int \cot^2 t \, dt\right)$

(We used FORMULA 87 with a = 1, n = 4)

$= 8\left(-\frac{1}{3}\cot^3 t + \cot t + t\right) + C$

(We used FORMULA 85 with a = 1)

47. $\int 2 \sec^3 \pi x \, dx = 2\left[\frac{\sec \pi x \tan \pi x}{\pi(3-1)} + \frac{3-2}{3-1}\int \sec \pi x \, dx\right]$

(We used FORMULA 92 with n = 3, a = π)

$= \frac{1}{\pi}\sec \pi x \tan \pi x + \frac{1}{\pi}\ln|\sec \pi x + \tan \pi x| + C$

(We used FORMULA 88 with a = π)

48. $\int 3 \sec^4 3x \, dx = 3\left[\frac{\sec^2 3x \tan 3x}{3(4-1)} + \frac{4-2}{4-1}\int \sec^2 3x \, dx\right]$

(We used FORMULA 92 with n = 4, a = 3)

$= \frac{\sec^2 3x \tan 3x}{3} + \frac{2}{3}\tan 3x + C$

(We used FORMULA 90 with a = 3)

49. $\int \csc^5 x \, dx = -\frac{\csc^3 x \cot x}{5-1} + \frac{5-2}{5-1}\int \csc^3 x \, dx = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4}\left(-\frac{\csc x \cot x}{3-1} + \frac{3-2}{3-1}\int \csc x \, dx\right)$

(We used FORMULA 93 with n = 5, a = 1 and n = 3, a = 1)

$= -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x - \frac{3}{8}\ln|\csc x + \cot x| + C$

(We used FORMULA 89 with a = 1)

50. $\int 16x^3(\ln x)^2 \, dx = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{2}{4}\int x^3 \ln x \, dx\right] = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{1}{2}\left[\frac{x^4(\ln x)}{4} - \frac{1}{4}\int x^3 \, dx\right]\right]$

(We used FORMULA 110 with a = 1, n = 3, m = 2 and a = 1, n = 3, m = 1)

$= 16\left(\frac{x^4(\ln x)^2}{4} - \frac{x^4(\ln x)}{8} + \frac{x^4}{32}\right) + C = 4x^4(\ln x)^2 - 2x^4 \ln x + \frac{x^4}{2} + C$

51. $\int e^t \sec^3(e^t-1)\,dt$; $\begin{bmatrix} x = e^t - 1 \\ dx = e^t\,dt \end{bmatrix} \to \int \sec^3 x\,dx = \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1}\int \sec x\,dx$

(We used FORMULA 92 with a = 1, n = 3)

$= \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C = \frac{1}{2}\left[\sec(e^t-1)\tan(e^t-1) + \ln|\sec(e^t-1) + \tan(e^t-1)|\right] + C$

52. $\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$; $\begin{bmatrix} t = \sqrt{\theta} \\ \theta = t^2 \\ d\theta = 2t\,dt \end{bmatrix} \to 2\int \csc^3 t\,dt = 2\left[-\frac{\csc t \cot t}{3-1} + \frac{3-2}{3-1}\int \csc t\,dt\right]$

(We used FORMULA 93 with a = 1, n = 3)

$= 2\left[-\frac{\csc t \cot t}{2} - \frac{1}{2}\ln|\csc t + \cot t|\right] + C = -\csc\sqrt{\theta}\cot\sqrt{\theta} - \ln\left|\csc\sqrt{\theta} + \cot\sqrt{\theta}\right| + C$

53. $\int_0^1 2\sqrt{x^2+1}\,dx$; $[x = \tan t] \to 2\int_0^{\pi/4} \sec t \cdot \sec^2 t\,dt = 2\int_0^{\pi/4}\sec^3 t\,dt = 2\left[\left[\frac{\sec t \cdot \tan t}{3-1}\right]_0^{\pi/4} + \frac{3-2}{3-1}\int_0^{\pi/4}\sec t\,dt\right]$

(We used FORMULA 92 with n = 3, a = 1)

$= [\sec t \cdot \tan t + \ln|\sec t + \tan t|]_0^{\pi/4} = \sqrt{2} + \ln\left(\sqrt{2}+1\right)$

54. $\int_0^{\sqrt{3}/2} \frac{dy}{(1-y^2)^{5/2}}$; $[y = \sin x] \to \int_0^{\pi/3}\frac{\cos x\,dx}{\cos^5 x} = \int_0^{\pi/3}\sec^4 x\,dx = \left[\frac{\sec^2 x \tan x}{4-1}\right]_0^{\pi/3} + \frac{4-2}{4-1}\int_0^{\pi/3}\sec^2 x\,dx$

(We used FORMULA 92 with a = 1, n = 4)

$= \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x\right]_0^{\pi/3} = \left(\frac{4}{3}\right)\sqrt{3} + \left(\frac{2}{3}\right)\sqrt{3} = 2\sqrt{3}$

55. $\int_1^2 \frac{(r^2-1)^{3/2}}{r}\,dr$; $[r = \sec\theta] \to \int_0^{\pi/3}\frac{\tan^3\theta}{\sec\theta}(\sec\theta\tan\theta)\,d\theta = \int_0^{\pi/3}\tan^4\theta\,d\theta = \left[\frac{\tan^3\theta}{4-1}\right]_0^{\pi/3} - \int_0^{\pi/3}\tan^2\theta\,d\theta$

$= \left[\frac{\tan^3\theta}{3} - \tan\theta + \theta\right]_0^{\pi/3} = \frac{3\sqrt{3}}{3} - \sqrt{3} + \frac{\pi}{3} = \frac{\pi}{3}$

(We used FORMULA 86 with a = 1, n = 4 and FORMULA 84 with a = 1)

56. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2+1)^{7/2}}$; $[t = \tan\theta] \to \int_0^{\pi/6}\frac{\sec^2\theta\,d\theta}{\sec^7\theta} = \int_0^{\pi/6}\cos^5\theta\,d\theta = \left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6} + \left(\frac{5-1}{5}\right)\int_0^{\pi/6}\cos^3\theta\,d\theta$

$= \left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6} + \frac{4}{5}\left[\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/6} + \left(\frac{3-1}{3}\right)\int_0^{\pi/6}\cos\theta\,d\theta\right] = \left[\frac{\cos^4\theta\sin\theta}{5} + \frac{4}{15}\cos^2\theta\sin\theta + \frac{8}{15}\sin\theta\right]_0^{\pi/6}$

(We used FORMULA 61 with a = 1, n = 5 and a = 1, n = 3)

$= \frac{\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}{5} + \left(\frac{4}{15}\right)\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right) + \left(\frac{8}{15}\right)\left(\frac{1}{2}\right) = \frac{9}{160} + \frac{1}{10} + \frac{4}{15} = \frac{3\cdot 9 + 48 + 32\cdot 4}{480} = \frac{203}{480}$

57. $S = \int_0^{\sqrt{2}} 2\pi y\sqrt{1+(y')^2}\,dx$

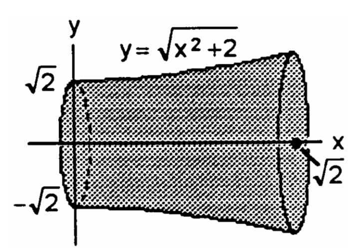
$= 2\pi\int_0^{\sqrt{2}}\sqrt{x^2+2}\sqrt{1+\frac{x^2}{x^2+2}}\,dx$

$= 2\sqrt{2}\pi\int_0^{\sqrt{2}}\sqrt{x^2+1}\,dx$

$= 2\sqrt{2}\pi\left[\frac{x\sqrt{x^2+1}}{2} + \frac{1}{2}\ln\left|x + \sqrt{x^2+1}\right|\right]_0^{\sqrt{2}}$

(We used FORMULA 21 with a = 1)

$= \sqrt{2}\pi\left[\sqrt{6} + \ln\left(\sqrt{2}+\sqrt{3}\right)\right] = 2\pi\sqrt{3} + \pi\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$



58. $L = \int_0^{\sqrt{3}/2}\sqrt{1+(2x)^2}\,dx = 2\int_0^{\sqrt{3}/2}\sqrt{\frac{1}{4}+x^2}\,dx = 2\left[\frac{x}{2}\sqrt{\frac{1}{4}+x^2} + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\ln\left(x + \sqrt{\frac{1}{4}+x^2}\right)\right]_0^{\sqrt{3}/2}$

(We used FORMULA 2 with $a = \frac{1}{2}$)

$= \left[\frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\ln\left(x + \frac{1}{2}\sqrt{1+4x^2}\right)\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}}{4}\sqrt{1+4\left(\frac{3}{4}\right)} + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)}\right) - \frac{1}{4}\ln\frac{1}{2}$

$= \frac{\sqrt{3}}{4}(2) + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2}+1\right) + \frac{1}{4}\ln 2 = \frac{\sqrt{3}}{2} + \frac{1}{4}\ln\left(\sqrt{3}+2\right)$

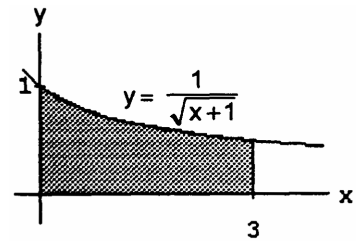
59. $A = \int_0^3 \frac{dx}{\sqrt{x+1}} = \left[2\sqrt{x+1}\right]_0^3 = 2;\ \overline{x} = \frac{1}{A}\int_0^3 \frac{x\,dx}{\sqrt{x+1}}$

$= \frac{1}{A}\int_0^3 \sqrt{x+1}\,dx - \frac{1}{A}\int_0^3 \frac{dx}{\sqrt{x+1}}$

$= \frac{1}{2}\cdot\frac{2}{3}\left[(x+1)^{3/2}\right]_0^3 - 1 = \frac{4}{3};$

(We used FORMULA 11 with $a = 1, b = 1, n = 1$ and $a = 1, b = 1, n = -1$)

$\overline{y} = \frac{1}{2A}\int_0^3 \frac{dx}{x+1} = \frac{1}{4}\left[\ln(x+1)\right]_0^3 = \frac{1}{4}\ln 4 = \frac{1}{2}\ln 2 = \ln\sqrt{2}$



60. $M_y = \int_0^3 x\left(\frac{36}{2x+3}\right)dx = 18\int_0^3 \frac{2x+3}{2x+3}\,dx - 54\int_0^3 \frac{dx}{2x+3} = \left[18x - 27\ln|2x+3|\right]_0^3$

$= 18\cdot 3 - 27\ln 9 - (-27\ln 3) = 54 - 27\cdot 2\ln 3 + 27\ln 3 = 54 - 27\ln 3$

61. $S = 2\pi\int_{-1}^{1} x^2\sqrt{1+4x^2}\,dx;$

$\begin{bmatrix} u = 2x \\ du = 2\,dx \end{bmatrix} \rightarrow \frac{\pi}{4}\int_{-2}^{2} u^2\sqrt{1+u^2}\,du$

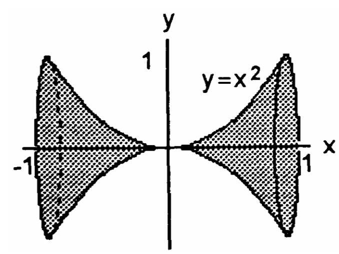
$= \frac{\pi}{4}\left[\frac{u}{8}(1+2u^2)\sqrt{1+u^2} - \frac{1}{8}\ln\left(u+\sqrt{1+u^2}\right)\right]_{-2}^{2}$

(We used FORMULA 22 with $a = 1$)

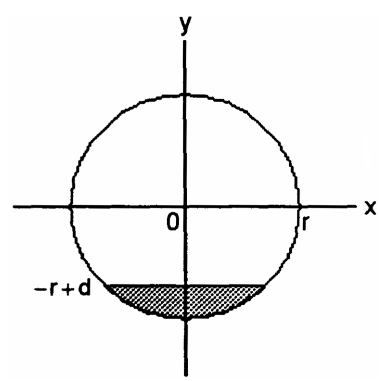
$= \frac{\pi}{4}\left[\frac{2}{8}(1+2\cdot 4)\sqrt{1+4} - \frac{1}{8}\ln\left(2+\sqrt{1+4}\right)\right.$

$\left. + \frac{2}{8}(1+2\cdot 4)\sqrt{1+4} + \frac{1}{8}\ln\left(-2+\sqrt{1+4}\right)\right]$

$= \frac{\pi}{4}\left[\frac{9}{2}\sqrt{5} - \frac{1}{8}\ln\left(\frac{2+\sqrt{5}}{-2+\sqrt{5}}\right)\right] \approx 7.62$



62. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r + d$. The width of this layer is $2\sqrt{r^2 - y^2}$. Therefore, $A = 2\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy$ and $V = L\cdot A = 2L\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy$



(b) $2L\int_{-r}^{-r+d}\sqrt{r^2-y^2}\,dy = 2L\left[\frac{y\sqrt{r^2-y^2}}{2} + \frac{r^2}{2}\sin^{-1}\frac{y}{r}\right]_{-r}^{-r+d}$

(We used FORMULA 29 with $a = r$)

$= 2L\left[\frac{(d-r)}{2}\sqrt{2rd-d^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{r^2}{2}\left(\frac{\pi}{2}\right)\right] = 2L\left[\left(\frac{d-r}{2}\right)\sqrt{2rd-d^2} + \left(\frac{r^2}{2}\right)\left(\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{\pi}{2}\right)\right]$

63. The integrand $f(x) = \sqrt{x - x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$\Rightarrow \int_0^1 \sqrt{x - x^2}\, dx = \int_0^1 \sqrt{2 \cdot \frac{1}{2}\, x - x^2}\, dx = \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{2 \cdot \frac{1}{2}\, x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$

(We used FORMULA 48 with $a = \frac{1}{2}$)

$= \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{x - x^2} + \frac{1}{8} \sin^{-1}(2x - 1)\right]_0^1 = \frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8}$

64. The integrand is maximized by integrating $g(x) = x\sqrt{2x - x^2}$ over the largest domain on which g is nonnegative, namely $[0, 2]$

$\Rightarrow \int_0^2 x\sqrt{2x - x^2}\, dx = \left[\frac{(x+1)(2x-3)\sqrt{2x - x^2}}{6} + \frac{1}{2} \sin^{-1}(x - 1)\right]_0^2$

(We used FORMULA 51 with $a = 1$)

$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

CAS EXPLORATIONS

65. Example CAS commands:

Maple:

```
q1 := Int( x*ln(x), x );                    # (a)
q1 = value( q1 );
q2 := Int( x^2*ln(x), x );                  # (b)
q2 = value( q2 );
q3 := Int( x^3*ln(x), x );                  # (c)
q3 = value( q3 );
q4 := Int( x^4*ln(x), x );                  # (d)
q4 = value( q4 );
q5 := Int( x^n*ln(x), x );                  # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

66. Example CAS commands:

Maple:

```
q1 := Int( ln(x)/x, x );                    # (a)
q1 = value( q1 );
q2 := Int( ln(x)/x^2, x );                  # (b)
q2 = value( q2 );
q3 := Int( ln(x)/x^3, x );                  # (c)
q3 = value( q3 );
q4 := Int( ln(x)/x^4, x );                  # (d)
q4 = value( q4 );
q5 := Int( ln(x)/x^n, x );                  # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

67. Example CAS commands:

Maple:

```
q := Int( sin(x)^n/(sin(x)^n+cos(x)^n), x=0..Pi/2 );      # (a)
q = value( q );
q1 := eval( q, n=1 ):                                       # (b)
q1 = value( q1 );
for N in [1,2,3,5,7] do
  q1 := eval( q, n=N );
  print( q1 = evalf(q1) );
end do:
qq1 := PDEtools[dchange]( x=Pi/2-u, q, [u] );              # (c)
qq2 := subs( u=x, qq1 );
qq3 := q + q = q + qq2;
qq4 := combine( qq3 );
qq5 := value( qq4 );
simplify( qq5/2 );
```

65-67. Example CAS commands:

Mathematica: (functions may vary)

In Mathematica, the natural log is denoted by Log rather than Ln, Log base 10 is Log[x,10]

Mathematica does not include an arbitrary constant when computing an indefinite integral,

```
Clear[x, f, n]
f[x_]:=Log[x] / x^n
Integrate[f[x], x]
```

For exercise 67, Mathematica cannot evaluate the integral with arbitrary n. It does evaluate the integral (value is $\pi/4$ in each case) for small values of n, but for large values of n, it identifies this integral as Indeterminate

65. (e) $\int x^n \ln x \, dx = \frac{x^{n+1} \ln x}{n+1} - \frac{1}{n+1} \int x^n \, dx, \; n \neq -1$

(We used FORMULA 110 with $a = 1$, $m = 1$)

$= \frac{x^{n+1} \ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C$

66. (e) $\int x^{-n} \ln x \, dx = \frac{x^{-n+1} \ln x}{-n+1} - \frac{1}{(-n)+1} \int x^{-n} \, dx, \; n \neq 1$

(We used FORMULA 110 with $a = 1$, $m = 1$, $n = -n$)

$= \frac{x^{1-n} \ln x}{1-n} - \frac{1}{1-n} \left(\frac{x^{1-n}}{1-n} \right) + C = \frac{x^{1-n}}{1-n} \left(\ln x - \frac{1}{1-n} \right) + C$

67. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$I = \int_0^{\pi/2} \frac{\sin^n x \, dx}{\sin^n x + \cos^n x} = \int_{\pi/2}^0 \frac{-\sin^n \left(\frac{\pi}{2} - u\right) du}{\sin^n \left(\frac{\pi}{2} - u\right) + \cos^n \left(\frac{\pi}{2} - u\right)} = \int_0^{\pi/2} \frac{\cos^n u \, du}{\cos^n u + \sin^n u} = \int_0^{\pi/2} \frac{\cos^n x \, dx}{\cos^n x + \sin^n x}$

$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} \right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

8.6 NUMERICAL INTEGRATION

1. $\int_1^2 x\,dx$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
$\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2}$;
$f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0 \Rightarrow M = 0$
$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	2	5/2
x_2	3/2	3/2	2	3
x_3	7/4	7/4	2	7/2
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_T| = \int_1^2 x\,dx - T = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;
$\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2}$;
$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	5/4	5/4	4	5
x_2	3/2	3/2	2	3
x_3	7/4	7/4	4	7
x_4	2	2	1	2

(b) $\int_1^2 x\,dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x\,dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

2. $\int_1^3 (2x-1)\,dx$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
$\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6$;
$f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0 \Rightarrow M = 0$
$\Rightarrow |E_T| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	2	4
x_2	2	3	2	6
x_3	5/2	4	2	8
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx = [x^2 - x]_1^3 = (9-3) - (1-1) = 6 \Rightarrow |E_T| = \int_1^3 (2x-1)\,dx - T = 6 - 6 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(x_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;
$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	1	1	1	1
x_1	3/2	2	4	8
x_2	2	3	2	6
x_3	5/2	4	4	16
x_4	3	5	1	5

(b) $\int_1^3 (2x-1)\,dx = 6 \Rightarrow |E_S| = \int_1^3 (2x-1)\,dx - S$
$= 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

3. $\int_{-1}^1 (x^2+1)\,dx$

I. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
$\sum mf(x_i) = 11 \Rightarrow T = \frac{1}{4}(11) = 2.75$;
$f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow M = 2$
$\Rightarrow |E_T| \le \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12}$ or 0.08333

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−1	2	1	2
x_1	−1/2	5/4	2	5/2
x_2	0	1	2	2
x_3	1/2	5/4	2	5/2
x_4	1	2	1	2

(b) $\int_{-1}^1 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_{-1}^1 = \left(\frac{1}{3}+1\right) - \left(-\frac{1}{3}-1\right) = \frac{8}{3} \Rightarrow E_T = \int_{-1}^1 (x^2+1)\,dx - T = \frac{8}{3} - \frac{11}{4} = -\frac{1}{12}$
$\Rightarrow |E_T| = \left|-\frac{1}{12}\right| \approx 0.08333$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{8}{3}}\right) \times 100 \approx 3\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6}(16) = \frac{8}{3} = 2.66667$;
$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−1	2	1	2
x_1	−1/2	5/4	4	5
x_2	0	1	2	2
x_3	1/2	5/4	4	5
x_4	1	2	1	2

(b) $\int_{-1}^{1}(x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_{-1}^{1} = \frac{8}{3}$
$\Rightarrow |E_S| = \int_{-1}^{1}(x^2+1)\,dx - S = \frac{8}{3} - \frac{8}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

4. $\int_{-2}^{0}(x^2-1)\,dx$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$
$\sum mf(x_i) = 3 \Rightarrow T = \frac{1}{4}(3) = \frac{3}{4}$;
$f(x) = x^2 - 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$
$\Rightarrow M = 2 \Rightarrow |E_T| \le \frac{0-(-2)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12} = 0.08333$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−2	3	1	3
x_1	−3/2	5/4	2	5/2
x_2	−1	0	2	0
x_3	− 1/2	−3/4	2	−3/2
x_4	0	−1	1	−1

(b) $\int_{-2}^{0}(x^2-1)\,dx = \left[\frac{x^3}{3} - x\right]_{-2}^{0} = 0 - \left(-\frac{8}{3} + 2\right) = \frac{2}{3} \Rightarrow E_T = \int_{-2}^{0}(x^2-1)\,dx - T = \frac{2}{3} - \frac{3}{4} = -\frac{1}{12}$
$\Rightarrow |E_T| = \frac{1}{12}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{2}{3}}\right) \times 100 \approx 13\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$; $\sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6}(4) = \frac{2}{3}$;
$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	−2	3	1	3
x_1	−3/2	5/4	4	5
x_2	−1	0	2	0
x_3	− 1/2	−3/4	4	−3
x_4	0	−1	1	−1

(b) $\int_{-2}^{0}(x^2-1)\,dx = \frac{2}{3} \Rightarrow |E_S| = \int_{-2}^{0}(x^2-1)\,dx - S$
$= \frac{2}{3} - \frac{2}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

5. $\int_{0}^{2}(t^3+t)\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$; $\sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4}(25) = \frac{25}{4}$;
$f(t) = t^3 + t \Rightarrow f'(t) = 3t^2 + 1 \Rightarrow f''(t) = 6t$
$\Rightarrow M = 12 = f''(2) \Rightarrow |E_T| \le \frac{2-0}{12}\left(\frac{1}{2}\right)^2(12) = \frac{1}{2}$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	2	5/4
t_2	1	2	2	4
t_3	3/2	39/8	2	39/4
t_4	2	10	1	10

(b) $\int_{0}^{2}(t^3+t)\,dt = \left[\frac{t^4}{4} + \frac{t^2}{2}\right]_{0}^{2} = \left(\frac{2^4}{4} + \frac{2^2}{2}\right) - 0 = 6 \Rightarrow |E_T| = \int_{0}^{2}(t^3+t)\,dt - T = 6 - \frac{25}{4} = -\frac{1}{4} \Rightarrow |E_T| = \frac{1}{4}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{\left|-\frac{1}{4}\right|}{6} \times 100 \approx 4\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;
$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/2	5/8	4	5/2
t_2	1	2	2	4
t_3	3/2	39/8	4	39/2
t_4	2	10	1	10

(b) $\int_{0}^{2}(t^3+t)\,dt = 6 \Rightarrow |E_S| = \int_{0}^{2}(t^3+t)\,dt - S$
$= 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

6. $\int_{-1}^{1}(t^3+1)\,dt$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	-1	0	1	0
t_1	$-1/2$	7/8	2	7/4
t_2	0	1	2	2
t_3	1/2	9/8	2	9/4
t_4	1	2	1	2

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{1-(-1)}{4}=\frac{2}{4}=\frac{1}{2}$
$\Rightarrow \frac{\Delta x}{2}=\frac{1}{4}$; $\sum mf(t_i)=8 \Rightarrow T=\frac{1}{4}(8)=2$;
$f(t)=t^3+1 \Rightarrow f'(t)=3t^2 \Rightarrow f''(t)=6t$
$\Rightarrow M=6=f''(1) \Rightarrow |E_T|\le\frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(6)=\frac{1}{4}$

(b) $\int_{-1}^{1}(t^3+1)\,dt=\left[\frac{t^4}{4}+t\right]_{-1}^{1}=\left(\frac{1^4}{4}+1\right)-\left(\frac{(-1)^4}{4}+(-1)\right)=2 \Rightarrow |E_T|=\int_{-1}^{1}(t^3+1)\,dt-T=2-2=0$

(c) $\frac{|E_T|}{\text{True Value}}\times 100=0\%$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	-1	0	1	0
t_1	$-1/2$	7/8	4	7/2
t_2	0	1	2	2
t_3	1/2	9/8	4	9/2
t_4	1	2	1	2

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{1-(-1)}{4}=\frac{2}{4}=\frac{1}{2}$
$\Rightarrow \frac{\Delta x}{3}=\frac{1}{6}$; $\sum mf(t_i)=12 \Rightarrow S=\frac{1}{6}(12)=2$;
$f^{(3)}(t)=6 \Rightarrow f^{(4)}(t)=0 \Rightarrow M=0 \Rightarrow |E_S|=0$

(b) $\int_{-1}^{1}(t^3+1)\,dt=2 \Rightarrow |E_S|=\int_{-1}^{1}(t^3+1)\,dt-S$
$=2-2=0$

(c) $\frac{|E_S|}{\text{True Value}}\times 100=0\%$

7. $\int_{1}^{2}\frac{1}{s^2}\,ds$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	2	32/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	2	32/49
s_4	2	1/4	1	1/4

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4} \Rightarrow \frac{\Delta x}{2}=\frac{1}{8}$;
$\sum mf(s_i)=\frac{179{,}573}{44{,}100} \Rightarrow T=\frac{1}{8}\left(\frac{179{,}573}{44{,}100}\right)=\frac{179{,}573}{352{,}800}$
≈ 0.50899; $f(s)=\frac{1}{s^2} \Rightarrow f'(s)=-\frac{2}{s^3}$
$\Rightarrow f''(s)=\frac{6}{s^4} \Rightarrow M=6=f''(1)$
$\Rightarrow |E_T|\le\frac{2-1}{12}\left(\frac{1}{4}\right)^2(6)=\frac{1}{32}=0.03125$

(b) $\int_{1}^{2}\frac{1}{s^2}\,ds=\int_{1}^{2}s^{-2}\,ds=\left[-\frac{1}{s}\right]_{1}^{2}=-\frac{1}{2}-\left(-\frac{1}{1}\right)=\frac{1}{2} \Rightarrow E_T=\int_{1}^{2}\frac{1}{s^2}\,ds-T=\frac{1}{2}-0.50899=-0.00899$
$\Rightarrow |E_T|=0.00899$

(c) $\frac{|E_T|}{\text{True Value}}\times 100=\frac{0.00899}{0.5}\times 100\approx 2\%$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	1	1	1	1
s_1	5/4	16/25	4	64/25
s_2	3/2	4/9	2	8/9
s_3	7/4	16/49	4	64/49
s_4	2	1/4	1	1/4

II. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4} \Rightarrow \frac{\Delta x}{3}=\frac{1}{12}$;
$\sum mf(s_i)=\frac{264{,}821}{44{,}100} \Rightarrow S=\frac{1}{12}\left(\frac{264{,}821}{44{,}100}\right)=\frac{264{,}821}{529{,}200}$
≈ 0.50042; $f^{(3)}(s)=-\frac{24}{s^5} \Rightarrow f^{(4)}(s)=\frac{120}{s^6}$
$\Rightarrow M=120 \Rightarrow |E_S|\le\left|\frac{2-1}{180}\right|\left(\frac{1}{4}\right)^4(120)$
$=\frac{1}{384}\approx 0.00260$

(b) $\int_{1}^{2}\frac{1}{s^2}\,ds=\frac{1}{2} \Rightarrow E_S=\int_{1}^{2}\frac{1}{s^2}\,ds-S=\frac{1}{2}-0.50042=-0.00042 \Rightarrow |E_S|=0.00042$

(c) $\frac{|E_S|}{\text{True Value}}\times 100=\frac{0.0004}{0.5}\times 100\approx 0.08\%$

8. $\int_{2}^{4}\frac{1}{(s-1)^2}\,ds$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	2	8/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	2	8/25
s_4	4	1/9	1	1/9

I. (a) For $n=4$, $\Delta x=\frac{b-a}{n}=\frac{4-2}{4}=\frac{1}{2} \Rightarrow \frac{\Delta x}{2}=\frac{1}{4}$;
$\sum mf(s_i)=\frac{1269}{450}$
$\Rightarrow T=\frac{1}{4}\left(\frac{1269}{450}\right)=\frac{1269}{1800}=0.70500$;
$f(s)=(s-1)^{-2} \Rightarrow f'(s)=-\frac{2}{(s-1)^3}$
$\Rightarrow f''(s)=\frac{6}{(s-1)^4} \Rightarrow M=6$
$\Rightarrow |E_T|\le\frac{4-2}{12}\left(\frac{1}{2}\right)^2(6)=\frac{1}{4}=0.25$

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \left[\frac{-1}{(s-1)}\right]_2^4 = \left(\frac{-1}{4-1}\right) - \left(\frac{-1}{2-1}\right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2}\,ds - T = \frac{2}{3} - 0.705 \approx -0.03833$
$\Rightarrow |E_T| \approx 0.03833$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(s_i) = \frac{1813}{450}$
$\Rightarrow S = \frac{1}{6}\left(\frac{1813}{450}\right) = \frac{1813}{2700} \approx 0.67148$;
$f^{(3)}(s) = \frac{-24}{(s-1)^5} \Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^6} \Rightarrow M = 120$
$\Rightarrow |E_S| \le \frac{4-2}{180}\left(\frac{1}{2}\right)^4(120) = \frac{1}{12} \approx 0.08333$

	s_i	$f(s_i)$	m	$mf(s_i)$
s_0	2	1	1	1
s_1	5/2	4/9	4	16/9
s_2	3	1/4	2	1/2
s_3	7/2	4/25	4	16/25
s_4	4	1/9	1	1/9

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3} \Rightarrow E_S = \int_2^4 \frac{1}{(s-1)^2}\,ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow |E_S| \approx 0.00481$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$

9. $\int_0^\pi \sin t\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{8}$;
$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284$
$\Rightarrow T = \frac{\pi}{8}\left(2 + 2\sqrt{2}\right) \approx 1.89612$;
$f(t) = \sin t \Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t$
$\Rightarrow M = 1 \Rightarrow |E_T| \le \frac{\pi-0}{12}\left(\frac{\pi}{4}\right)^2(1) = \frac{\pi^3}{192}$
≈ 0.16149

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^\pi \sin t\,dt = [-\cos t]_0^\pi = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow |E_T| = \int_0^\pi \sin t\,dt - T \approx 2 - 1.89612 = 0.10388$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{12}$;
$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569$
$\Rightarrow S = \frac{\pi}{12}\left(2 + 4\sqrt{2}\right) \approx 2.00456$;
$f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$
$\Rightarrow M = 1 \Rightarrow |E_S| \le \frac{\pi-0}{180}\left(\frac{\pi}{4}\right)^4(1) \approx 0.00664$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	$\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	$\pi/2$	1	2	2
t_3	$3\pi/4$	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	π	0	1	0

(b) $\int_0^\pi \sin t\,dt = 2 \Rightarrow E_S = \int_0^\pi \sin t\,dt - S \approx 2 - 2.00456 = -0.00456 \Rightarrow |E_S| \approx 0.00456$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0\%$

10. $\int_0^1 \sin \pi t\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828$
$\Rightarrow T = \frac{1}{8}\left(2 + 2\sqrt{2}\right) \approx 0.60355$; $f(t) = \sin \pi t$
$\Rightarrow f'(t) = \pi \cos \pi t$
$\Rightarrow f''(t) = -\pi^2 \sin \pi t \Rightarrow M = \pi^2$
$\Rightarrow |E_T| \le \frac{1-0}{12}\left(\frac{1}{4}\right)^2(\pi^2) \approx 0.05140$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	2	$\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t\,dt = \left[-\frac{1}{\pi}\cos \pi t\right]_0^1 = \left(-\frac{1}{\pi}\cos \pi\right) - \left(-\frac{1}{\pi}\cos 0\right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \int_0^1 \sin \pi t\,dt - T$
$\approx \frac{2}{\pi} - 0.60355 = 0.03307$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$

II. (a) For n = 4, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685$

$\Rightarrow S = \frac{1}{12}\left(2 + 4\sqrt{2}\right) \approx 0.63807$;

$f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$

$\Rightarrow M = \pi^4 \Rightarrow |E_S| \le \frac{1-0}{180}\left(\frac{1}{4}\right)^4(\pi^4) \approx 0.00211$

	t_i	$f(t_i)$	m	$mf(t_i)$
t_0	0	0	1	0
t_1	1/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_2	1/2	1	2	2
t_3	3/4	$\sqrt{2}/2$	4	$2\sqrt{2}$
t_4	1	0	1	0

(b) $\int_0^1 \sin \pi t\, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_S = \int_0^1 \sin \pi t\, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_S| \approx 0.00145$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00145}{\left(\frac{2}{\pi}\right)} \times 100 \approx 0\%$

11. (a) $M = 0$ (see Exercise 1): Then $n = 1 \Rightarrow \Delta x = 1 \Rightarrow |E_T| = \frac{1}{12}(1)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 1): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = \frac{1}{2} \Rightarrow |E_S| = \frac{1}{180}\left(\frac{1}{2}\right)^4(0) = 0 < 10^{-4}$

12. (a) $M = 0$ (see Exercise 2): Then $n = 1 \Rightarrow \Delta x = 2 \Rightarrow |E_T| = \frac{2}{12}(2)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 2): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

13. (a) $M = 2$ (see Exercise 3): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$

$\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 3): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

14. (a) $M = 2$ (see Exercise 4): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$

$\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 4): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

15. (a) $M = 12$ (see Exercise 5): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(12) = \frac{8}{n^2} < 10^{-4} \Rightarrow n^2 > 8(10^4) \Rightarrow n > \sqrt{8(10^4)}$

$\Rightarrow n > 282.8$, so let $n = 283$

(b) $M = 0$ (see Exercise 5): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

16. (a) $M = 6$ (see Exercise 6): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$

$= 200$, so let $n = 201$

(b) $M = 0$ (see Exercise 6): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

17. (a) $M = 6$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_T| \le \frac{1}{12}\left(\frac{1}{n}\right)^2(6) = \frac{1}{2n^2} < 10^{-4} \Rightarrow n^2 > \frac{1}{2}(10^4) \Rightarrow n > \sqrt{\frac{1}{2}(10^4)}$

$\Rightarrow n > 70.7$, so let $n = 71$

(b) $M = 120$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_S| = \frac{1}{180}\left(\frac{1}{n}\right)^4(120) = \frac{2}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{2}{3}(10^4)$

$\Rightarrow n > \sqrt[4]{\frac{2}{3}(10^4)} \Rightarrow n > 9.04$, so let $n = 10$ (n must be even)

18. (a) $M = 6$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$

$\Rightarrow n > 200$, so let $n = 201$

(b) $M = 120$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(120) = \frac{64}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{64}{3}(10^4)$

$\Rightarrow n > \sqrt[4]{\frac{64}{3}(10^4)} \Rightarrow n > 21.5$, so let $n = 22$ (n must be even)

19. (a) $f(x) = \sqrt{x+1} \Rightarrow f'(x) = \frac{1}{2}(x+1)^{-1/2} \Rightarrow f''(x) = -\frac{1}{4}(x+1)^{-3/2} = -\frac{1}{4\left(\sqrt{x+1}\right)^3} \Rightarrow M = \frac{1}{4\left(\sqrt{1}\right)^3} = \frac{1}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{1}{4}\right) = \frac{9}{16n^2} < 10^{-4} \Rightarrow n^2 > \frac{9}{16}\left(10^4\right) \Rightarrow n > \sqrt{\frac{9}{16}\left(10^4\right)} \Rightarrow n > 75$,

so let $n = 76$

(b) $f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2} = -\frac{15}{16\left(\sqrt{x+1}\right)^7} \Rightarrow M = \frac{15}{16\left(\sqrt{1}\right)^7} = \frac{15}{16}$. Then $\Delta x = \frac{3}{n}$

$\Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{15}{16}\right) = \frac{3^5(15)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(15)\left(10^4\right)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(15)\left(10^4\right)}{16(180)}} \Rightarrow n > 10.6$, so let

$n = 12$ (n must be even)

20. (a) $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow f'(x) = -\frac{1}{2}(x+1)^{-3/2} \Rightarrow f''(x) = \frac{3}{4}(x+1)^{-5/2} = \frac{3}{4\left(\sqrt{x+1}\right)^5} \Rightarrow M = \frac{3}{4\left(\sqrt{1}\right)^5} = \frac{3}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{3}{4}\right) = \frac{3^4}{48n^2} < 10^{-4} \Rightarrow n^2 > \frac{3^4\left(10^4\right)}{48} \Rightarrow n > \sqrt{\frac{3^4\left(10^4\right)}{48}} \Rightarrow n > 129.9$, so let $n = 130$

(b) $f^{(3)}(x) = -\frac{15}{8}(x+1)^{-7/2} \Rightarrow f^{(4)}(x) = \frac{105}{16}(x+1)^{-9/2} = \frac{105}{16\left(\sqrt{x+1}\right)^9} \Rightarrow M = \frac{105}{16\left(\sqrt{1}\right)^9} = \frac{105}{16}$. Then $\Delta x = \frac{3}{n}$

$\Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{105}{16}\right) = \frac{3^5(105)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(105)\left(10^4\right)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(105)\left(10^4\right)}{16(180)}} \Rightarrow n > 17.25$, so

let $n = 18$ (n must be even)

21. (a) $f(x) = \sin(x+1) \Rightarrow f'(x) = \cos(x+1) \Rightarrow f''(x) = -\sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(1)$

$= \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8\left(10^4\right)}{12} \Rightarrow n > \sqrt{\frac{8\left(10^4\right)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = -\cos(x+1) \Rightarrow f^{(4)}(x) = \sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(1) = \frac{32}{180n^4} < 10^{-4}$

$\Rightarrow n^4 > \frac{32\left(10^4\right)}{180} \Rightarrow n > \sqrt[4]{\frac{32\left(10^4\right)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

22. (a) $f(x) = \cos(x+\pi) \Rightarrow f'(x) = -\sin(x+\pi) \Rightarrow f''(x) = -\cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n}$

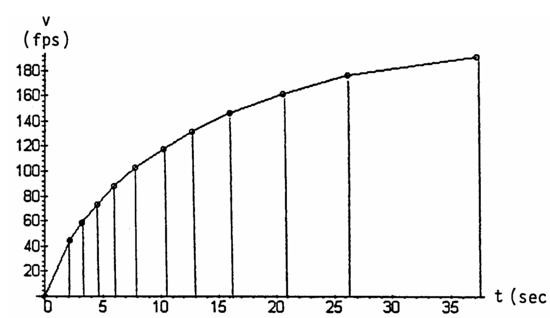
$\Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8\left(10^4\right)}{12} \Rightarrow n > \sqrt{\frac{8\left(10^4\right)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = \sin(x+\pi) \Rightarrow f^{(4)}(x) = \cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(1) = \frac{32}{180n^4} < 10^{-4}$

$\Rightarrow n^4 > \frac{32\left(10^4\right)}{180} \Rightarrow n > \sqrt[4]{\frac{32\left(10^4\right)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

23. $\frac{5}{2}(6.0 + 2(8.2) + 2(9.1) \ldots + 2(12.7) + 13.0)(30) = 15{,}990 \text{ ft}^3$.

24. Use the conversion 30 mph = 44 fps (ft per sec) since time is measured in seconds. The distance traveled as the car accelerates from, say, 40 mph = 58.67 fps to 50 mph = 73.33 fps in $(4.5 - 3.2) = 1.3$ sec is the area of the trapezoid (see figure) associated with that time interval: $\frac{1}{2}(58.67 + 73.33)(1.3) = 85.8$ ft. The total distance traveled by the Ford Mustang Cobra is the sum of all these eleven trapezoids (using $\frac{\Delta t}{2}$ and the table below):



v (mph)	0	30	40	50	60	70	80	90	100	110	120	130
v (fps)	0	44	58.67	73.33	88	102.67	117.33	132	146.67	161.33	176	190.67
t (sec)	0	2.2	3.2	4.5	5.9	7.8	10.2	12.7	16	20.6	26.2	37.1
Δt/2	0	1.1	0.5	0.65	0.7	0.95	1.2	1.25	1.65	2.3	2.8	5.45

$s = (44)(1.1) + (102.67)(0.5) + (132)(0.65) + (161.33)(0.7) + (190.67)(0.95) + (220)(1.2) + (249.33)(1.25)$
$+ (278.67)(1.65) + (308)(2.3) + (337.33)(2.8) + (366.67)(5.45) = 5166.346 \text{ ft} \approx 0.9785 \text{ mi}$

25. Using Simpson's Rule, $\Delta x = 1 \Rightarrow \frac{\Delta x}{3} = \frac{1}{3}$; $\sum my_i = 33.6 \Rightarrow$ Cross Section Area $\approx \frac{1}{3}(33.6)$ $= 11.2 \text{ ft}^2$. Let x be the length of the tank. Then the Volume V = (Cross Sectional Area) x = 11.2x. Now 5000 lb of gasoline at 42 lb/ft^3 $\Rightarrow V = \frac{5000}{42} = 119.05 \text{ ft}^3$ $\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63$ ft

	x_i	y_i	m	my_i
x_0	0	1.5	1	1.5
x_1	1	1.6	4	6.4
x_2	2	1.8	2	3.6
x_3	3	1.9	4	7.6
x_4	4	2.0	2	4.0
x_5	5	2.1	4	8.4
x_6	6	2.1	1	2.1

26. $\frac{24}{2}[0.019 + 2(0.020) + 2(0.021) + \ldots + 2(0.031) + 0.035] = 4.2$ L

27. (a) $|E_S| \le \frac{b-a}{180}(\Delta x^4)M$; $n = 4 \Rightarrow \Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$; $\left|f^{(4)}\right| \le 1 \Rightarrow M = 1 \Rightarrow |E_S| \le \frac{(\frac{\pi}{2}-0)}{180}\left(\frac{\pi}{8}\right)^4(1) \approx 0.00021$

(b) $\Delta x = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$;
$\sum mf(x_i) = 10.47208705$
$\Rightarrow S = \frac{\pi}{24}(10.47208705) \approx 1.37079$

	x_i	$f(x_i)$	m	$mf(x_{1i})$
x_0	0	1	1	1
x_1	$\pi/8$	0.974495358	4	3.897981432
x_2	$\pi/4$	0.900316316	2	1.800632632
x_3	$3\pi/8$	0.784213303	4	3.136853212
x_4	$\pi/2$	0.636619772	1	0.636619772

(c) $\approx \left(\frac{0.00021}{1.37079}\right) \times 100 \approx 0.015\%$

28. (a) $\Delta x = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \text{erf}(1) = \frac{2}{\sqrt{3}}\left(\frac{0.1}{3}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_9 + y_{10})$
$\frac{2}{30\sqrt{\pi}}(e^0 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \ldots + 4e^{-0.81} + e^{-1}) \approx 0.843$

(b) $|E_S| \le \frac{1-0}{180}(0.1)^4(12) \approx 6.7 \times 10^{-6}$

29. $T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n)$ where $\Delta x = \frac{b-a}{n}$ and f is continuous on [a, b]. So
$T = \frac{b-a}{n}\frac{(y_0 + y_1 + y_1 + y_2 + y_2 + \ldots + y_{n-1} + y_{n-1} + y_n)}{2} = \frac{b-a}{n}\left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \ldots + \frac{f(x_{n-1})+f(x_n)}{2}\right)$.
Since f is continuous on each interval $[x_{k-1}, x_k]$, and $\frac{f(x_{k-1})+f(x_k)}{2}$ is always between $f(x_{k-1})$ and $f(x_k)$, there is a point c_k in $[x_{k-1}, x_k]$ with $f(c_k) = \frac{f(x_{k-1})+f(x_k)}{2}$; this is a consequence of the Intermediate Value Theorem. Thus our sum is $\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f(c_k)$ which has the form $\sum_{k=1}^{n}\Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{n}$ for all k. This is a Riemann Sum for f on [a, b].

30. $S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$ where n is even, $\Delta x = \frac{b-a}{n}$ and f is continuous on [a, b]. So
$S = \frac{b-a}{n}\left(\frac{y_0 + 4y_1 + y_2}{3} + \frac{y_2 + 4y_3 + y_4}{3} + \frac{y_4 + 4y_5 + y_6}{3} + \ldots + \frac{y_{n-2} + 4y_{n-1} + y_n}{3}\right)$
$= \frac{b-a}{\frac{n}{2}}\left(\frac{f(x_0)+4f(x_1)+f(x_2)}{6} + \frac{f(x_2)+4f(x_3)+f(x_4)}{6} + \frac{f(x_4)+4f(x_5)+f(x_6)}{6} + \ldots + \frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right)$
$\frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6}$ is the average of the six values of the continuous function on the interval $[x_{2k}, x_{2k+2}]$, so it is between the minimum and maximum of f on this interval. By the Extreme Value Theorem for continuous functions, f takes on its maximum and minimum in this interval, so there are x_a and x_b with $x_{2k} \le x_a, x_b \le x_{2k+2}$ and $f(x_a) \le \frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6} \le f(x_b)$. By the Intermediate Value Theorem, there is c_k in $[x_{2k}, x_{2k+2}]$ with $f(c_k) = \frac{f(x_{2k})+4f(x_{2k+1})+f(x_{2k+2})}{6}$. So our sum has the form $\sum_{k=1}^{n/2}\Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{(n/2)}$, a Riemann sum for f on [a, b].

31. (a) $a = 1, e = \frac{1}{2} \Rightarrow \text{Length} = 4\int_0^{\pi/2}\sqrt{1 - \frac{1}{4}\cos^2 t}\,dt$
$= 2\int_0^{\pi/2}\sqrt{4 - \cos^2 t}\,dt = \int_0^{\pi/2} f(t)\,dt$; use the Trapezoid Rule with $n = 10 \Rightarrow \Delta t = \frac{b-a}{n} = \frac{\left(\frac{\pi}{2}\right) - 0}{10}$
$= \frac{\pi}{20}$. $\int_0^{\pi/2}\sqrt{4 - \cos^2 t}\,dt \approx \sum_{n=0}^{10} mf(x_n) = 37.3686183$
$\Rightarrow T = \frac{\Delta t}{2}(37.3686183) = \frac{\pi}{40}(37.3686183)$
$= 2.934924419 \Rightarrow \text{Length} = 2(2.934924419)$
≈ 5.870

(b) $|f''(t)| < 1 \Rightarrow M = 1$
$\Rightarrow |E_T| \le \frac{b-a}{12}(\Delta t^2 M) \le \frac{\left(\frac{\pi}{2}\right) - 0}{12}\left(\frac{\pi}{20}\right)^2 1 \le 0.0032$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1.732050808	1	1.732050808
x_1	$\pi/20$	1.739100843	2	3.478201686
x_2	$\pi/10$	1.759400893	2	3.518801786
x_3	$3\pi/20$	1.790560631	2	3.581121262
x_4	$\pi/5$	1.82906848	1	3.658136959
x_5	$\pi/4$	1.870828693	1	3.741657387
x_6	$3\pi/10$	1.911676881	2	3.823353762
x_7	$7\pi/20$	1.947791731	2	3.895583461
x_8	$2\pi/5$	1.975982919	2	3.951965839
x_9	$9\pi/20$	1.993872679	2	3.987745357
x_{10}	$\pi/2$	2	1	2

32. $\Delta x = \frac{\pi - 0}{8} = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$; $\sum mf(x_i) = 29.184807792$
$\Rightarrow S = \frac{\pi}{24}(29.18480779) \approx 3.82028$

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	1.414213562	1	1.414213562
x_1	$\pi/8$	1.361452677	4	5.445810706
x_2	$\pi/4$	1.224744871	2	2.449489743
x_3	$3\pi/8$	1.070722471	4	4.282889883
x_4	$\pi/2$	1	2	2
x_5	$5\pi/8$	1.070722471	4	4.282889883
x_6	$3\pi/4$	1.224744871	2	2.449489743
x_7	$7\pi/8$	1.361452677	4	5.445810706
x_8	π	1.414213562	1	1.414213562

33. The length of the curve $y = \sin\left(\frac{3\pi}{20}x\right)$ from 0 to 20 is: $L = \int_0^{20}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx} = \frac{3\pi}{20}\cos\left(\frac{3\pi}{20}x\right) \Rightarrow \left(\frac{dy}{dx}\right)^2$ $= \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right) \Rightarrow L = \int_0^{20}\sqrt{1 + \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right)}\,dx$. Using numerical integration we find $L \approx 21.07$ in

34. First, we'll find the length of the cosine curve: $L = \int_{-25}^{25}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx} = -\frac{25\pi}{50}\sin\left(\frac{\pi x}{50}\right)$ $\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right) \Rightarrow L = \int_{-25}^{25}\sqrt{1 + \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right)}\,dx$. Using a numerical integrator we find $L \approx 73.1848$ ft. Surface area is: $A = \text{length} \cdot \text{width} \approx (73.1848)(300) = 21{,}955.44$ ft.
Cost $= 1.75A = (1.75)(21{,}955.44) = \$38{,}422.02$. Answers may vary slightly, depending on the numerical integration used.

35. $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \cos^2 x \Rightarrow S = \int_0^{\pi} 2\pi(\sin x)\sqrt{1 + \cos^2 x}\,dx$; a numerical integration gives $S \approx 14.4$

36. $y = \frac{x^2}{4} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4} \Rightarrow S = \int_0^2 2\pi\left(\frac{x^2}{4}\right)\sqrt{1 + \frac{x^2}{4}}\,dx$; a numerical integration gives $S \approx 5.28$

37. A calculator or computer numerical integrator yields $\sin^{-1} 0.6 \approx 0.643501109$.

38. A calculator or computer numerical integrator yields $\pi \approx 3.1415929$.

8.7 IMPROPER INTEGRALS

1. $\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

2. $\int_1^{\infty} \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^{1.001}} = \lim_{b\to\infty} \left[-1000x^{-0.001}\right]_1^b = \lim_{b\to\infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

3. $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 x^{-1/2}\, dx = \lim_{b\to 0^+} \left[2x^{1/2}\right]_b^1 = \lim_{b\to 0^+} \left(2 - 2\sqrt{b}\right) = 2 - 0 = 2$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^b (4-x)^{-1/2} dx = \lim_{b\to 4^-} \left[-2\sqrt{4-b} - \left(-2\sqrt{4}\right)\right] = 0 + 4 = 4$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}} = \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim_{b\to 0^-} \left[3x^{1/3}\right]_{-1}^b + \lim_{c\to 0^+} \left[3x^{1/3}\right]_c^1$
$= \lim_{b\to 0^-} \left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim_{c\to 0^+} \left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = \lim_{b\to 0^-} \left[\frac{3}{2} x^{2/3}\right]_{-8}^b + \lim_{c\to 0^+} \left[\frac{3}{2} x^{2/3}\right]_c^1$
$= \lim_{b\to 0^-} \left[\frac{3}{2} b^{2/3} - \frac{3}{2}(-8)^{2/3}\right] + \lim_{c\to 0^+} \left[\frac{3}{2}(1)^{2/3} - \frac{3}{2} c^{2/3}\right] = \left[0 - \frac{3}{2}(4)\right] + \left(\frac{3}{2} - 0\right) = -\frac{9}{2}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \left[\sin^{-1} x\right]_0^b = \lim_{b\to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

8. $\int_0^1 \frac{dr}{r^{0.999}} = \lim_{b\to 0^+} \left[1000r^{0.001}\right]_b^1 = \lim_{b\to 0^+} \left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

9. $\int_{-\infty}^{-2} \frac{2\, dx}{x^2-1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim_{b\to -\infty} \left[\ln|x-1|\right]_b^{-2} - \lim_{b\to -\infty} \left[\ln|x+1|\right]_b^{-2} = \lim_{b\to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$
$= \lim_{b\to -\infty} \left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim_{b\to -\infty} \frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

10. $\int_{-\infty}^2 \frac{2\, dx}{x^2+4} = \lim_{b\to -\infty} \left[\tan^{-1} \frac{x}{2}\right]_b^2 = \lim_{b\to -\infty} \left(\tan^{-1} 1 - \tan^{-1} \frac{b}{2}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

11. $\int_2^{\infty} \frac{2\, dv}{v^2-v} = \lim_{b\to\infty} \left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim_{b\to\infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

12. $\int_2^{\infty} \frac{2\, dt}{t^2-1} = \lim_{b\to\infty} \left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim_{b\to\infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

13. $\int_{-\infty}^{\infty} \frac{2x\, dx}{(x^2+1)^2} = \int_{-\infty}^0 \frac{2x\, dx}{(x^2+1)^2} + \int_0^{\infty} \frac{2x\, dx}{(x^2+1)^2};\ \begin{bmatrix} u = x^2+1 \\ du = 2x\, dx \end{bmatrix} \to \int_{\infty}^1 \frac{du}{u^2} + \int_1^{\infty} \frac{du}{u^2} = \lim_{b\to\infty} \left[-\frac{1}{u}\right]_b^1 + \lim_{c\to\infty} \left[-\frac{1}{u}\right]_1^c$
$= \lim_{b\to\infty} \left(-1 + \frac{1}{b}\right) + \lim_{c\to\infty} \left[-\frac{1}{c} - (-1)\right] = (-1+0) + (0+1) = 0$

14. $\int_{-\infty}^{\infty} \frac{x\, dx}{(x^2+4)^{3/2}} = \int_{-\infty}^0 \frac{x\, dx}{(x^2+4)^{3/2}} + \int_0^{\infty} \frac{x\, dx}{(x^2+4)^{3/2}};\ \begin{bmatrix} u = x^2+4 \\ du = 2x\, dx \end{bmatrix} \to \int_{\infty}^4 \frac{du}{2u^{3/2}} + \int_4^{\infty} \frac{du}{2u^{3/2}}$
$= \lim_{b\to\infty} \left[-\frac{1}{\sqrt{u}}\right]_b^4 + \lim_{c\to\infty} \left[-\frac{1}{\sqrt{u}}\right]_4^c = \lim_{b\to\infty} \left(-\frac{1}{2} + \frac{1}{\sqrt{b}}\right) + \lim_{c\to\infty} \left(-\frac{1}{\sqrt{c}} + \frac{1}{2}\right) = \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$; $\begin{bmatrix} u=\theta^2+2\theta \\ du=2(\theta+1)\,d\theta \end{bmatrix} \to \int_0^3 \frac{du}{2\sqrt{u}} = \lim\limits_{b\to 0^+} \int_b^3 \frac{du}{2\sqrt{u}} = \lim\limits_{b\to 0^+} \left[\sqrt{u}\right]_b^3 = \lim\limits_{b\to 0^+} \left(\sqrt{3}-\sqrt{b}\right) = \sqrt{3}-0$
$= \sqrt{3}$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_0^2 \frac{2s\,ds}{\sqrt{4-s^2}} + \int_0^2 \frac{ds}{\sqrt{4-s^2}}$; $\begin{bmatrix} u=4-s^2 \\ du=-2s\,ds \end{bmatrix} \to -\frac{1}{2}\int_4^0 \frac{du}{\sqrt{u}} + \lim\limits_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}}$
$= \lim\limits_{b\to 0^+} \int_b^4 \frac{du}{2\sqrt{u}} + \lim\limits_{c\to 2^-} \int_0^c \frac{ds}{\sqrt{4-s^2}} = \lim\limits_{b\to 0^+} \left[\sqrt{u}\right]_b^4 + \lim\limits_{c\to 2^-} \left[\sin^{-1}\frac{s}{2}\right]_0^c$
$= \lim\limits_{b\to 0^+} \left(2-\sqrt{b}\right) + \lim\limits_{c\to 2^-} \left(\sin^{-1}\frac{c}{2} - \sin^{-1}0\right) = (2-0) + \left(\frac{\pi}{2}-0\right) = \frac{4+\pi}{2}$

17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$; $\begin{bmatrix} u=\sqrt{x} \\ du=\frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int_0^\infty \frac{2\,du}{u^2+1} = \lim\limits_{b\to\infty} \int_0^b \frac{2\,du}{u^2+1} = \lim\limits_{b\to\infty} \left[2\tan^{-1}u\right]_0^b$
$= \lim\limits_{b\to\infty} \left(2\tan^{-1}b - 2\tan^{-1}0\right) = 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$

18. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim\limits_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim\limits_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$
$= \lim\limits_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim\limits_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c = \lim\limits_{b\to 1^+} \left(\sec^{-1}2 - \sec^{-1}b\right) + \lim\limits_{c\to\infty} \left(\sec^{-1}c - \sec^{-1}2\right)$
$= \left(\frac{\pi}{3}-0\right) + \left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1}v)} = \lim\limits_{b\to\infty} \left[\ln|1+\tan^{-1}v|\right]_0^b = \lim\limits_{b\to\infty} \left[\ln|1+\tan^{-1}b|\right] - \ln|1+\tan^{-1}0|$
$= \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0) = \ln\left(1+\frac{\pi}{2}\right)$

20. $\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim\limits_{b\to\infty} \left[8\left(\tan^{-1}x\right)^2\right]_0^b = \lim\limits_{b\to\infty} \left[8\left(\tan^{-1}b\right)^2\right] - 8\left(\tan^{-1}0\right)^2 = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

21. $\int_{-\infty}^0 \theta e^\theta\,d\theta = \lim\limits_{b\to -\infty} \left[\theta e^\theta - e^\theta\right]_b^0 = \left(0\cdot e^0 - e^0\right) - \lim\limits_{b\to -\infty} \left[be^b - e^b\right] = -1 - \lim\limits_{b\to -\infty} \left(\frac{b-1}{e^{-b}}\right)$
$= -1 - \lim\limits_{b\to -\infty} \left(\frac{1}{-e^{-b}}\right)$ (l'Hôpital's rule for $\frac{\infty}{\infty}$ form)
$= -1 - 0 = -1$

22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta = \lim\limits_{b\to\infty} \int_0^b 2e^{-\theta}\sin\theta\,d\theta$
$= \lim\limits_{b\to\infty} 2\left[\frac{e^{-\theta}}{1+1}(-\sin\theta - \cos\theta)\right]_0^b$ (FORMULA 107 with $a=-1$, $b=1$)
$= \lim\limits_{b\to\infty} \frac{-2(\sin b + \cos b)}{2e^b} + \frac{2(\sin 0 + \cos 0)}{2e^0} = 0 + \frac{2(0+1)}{2} = 1$

23. $\int_{-\infty}^0 e^{-|x|}\,dx = \int_{-\infty}^0 e^x\,dx = \lim\limits_{b\to -\infty} \left[e^x\right]_b^0 = \lim\limits_{b\to -\infty} \left(1-e^b\right) = (1-0) = 1$

24. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx = \int_{-\infty}^0 2xe^{-x^2}\,dx + \int_0^\infty 2xe^{-x^2}\,dx = \lim\limits_{b\to -\infty} \left[-e^{-x^2}\right]_b^0 + \lim\limits_{c\to\infty} \left[-e^{-x^2}\right]_0^c$
$= \lim\limits_{b\to -\infty} \left[-1 - \left(-e^{-b^2}\right)\right] + \lim\limits_{c\to\infty} \left[-e^{-c^2} - (-1)\right] = (-1-0) + (0+1) = 0$

25. $\int_0^1 x\ln x\,dx = \lim\limits_{b\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 = \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) - \lim\limits_{b\to 0^+} \left(\frac{b^2}{2}\ln b - \frac{b^2}{4}\right) = -\frac{1}{4} - \lim\limits_{b\to 0^+} \frac{\ln b}{\left(\frac{2}{b^2}\right)} + 0$
$= -\frac{1}{4} - \lim\limits_{b\to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^3}\right)} = -\frac{1}{4} + \lim\limits_{b\to 0^+} \left(\frac{b^2}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$

26. $\int_0^1 (-\ln x)\,dx = \lim_{b \to 0^+} [x - x \ln x]_b^1 = [1 - 1 \ln 1] - \lim_{b \to 0^+} [b - b \ln b] = 1 - 0 + \lim_{b \to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = 1 + \lim_{b \to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$
$= 1 - \lim_{b \to 0^+} b = 1 - 0 = 1$

27. $\int_0^2 \frac{ds}{\sqrt{4 - s^2}} = \lim_{b \to 2^-} \left[\sin^{-1} \frac{s}{2}\right]_0^b = \lim_{b \to 2^-} \left(\sin^{-1} \frac{b}{2}\right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

28. $\int_0^1 \frac{4r\,dr}{\sqrt{1 - r^4}} = \lim_{b \to 1^-} \left[2 \sin^{-1} (r^2)\right]_0^b = \lim_{b \to 1^-} \left[2 \sin^{-1} (b^2)\right] - 2 \sin^{-1} 0 = 2 \cdot \frac{\pi}{2} - 0 = \pi$

29. $\int_1^2 \frac{ds}{s\sqrt{s^2 - 1}} = \lim_{b \to 1^+} \left[\sec^{-1} s\right]_b^2 = \sec^{-1} 2 - \lim_{b \to 1^+} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

30. $\int_2^4 \frac{dt}{t\sqrt{t^2 - 4}} = \lim_{b \to 2^+} \left[\frac{1}{2} \sec^{-1} \frac{t}{2}\right]_b^4 = \lim_{b \to 2^+} \left[\left(\frac{1}{2} \sec^{-1} \frac{4}{2}\right) - \frac{1}{2} \sec^{-1} \left(\frac{b}{2}\right)\right] = \frac{1}{2} \left(\frac{\pi}{3}\right) - \frac{1}{2} \cdot 0 = \frac{\pi}{6}$

31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}} = \lim_{b \to 0^-} \int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{c \to 0^+} \int_c^4 \frac{dx}{\sqrt{x}} = \lim_{b \to 0^-} \left[-2\sqrt{-x}\right]_{-1}^b + \lim_{c \to 0^+} \left[2\sqrt{x}\right]_c^4$
$= \lim_{b \to 0^-} \left(-2\sqrt{-b}\right) - \left(-2\sqrt{-(-1)}\right) + 2\sqrt{4} - \lim_{c \to 0^+} 2\sqrt{c} = 0 + 2 + 2 \cdot 2 - 0 = 6$

32. $\int_0^2 \frac{dx}{\sqrt{|x - 1|}} = \int_0^1 \frac{dx}{\sqrt{1 - x}} + \int_1^2 \frac{dx}{\sqrt{x - 1}} = \lim_{b \to 1^-} \left[-2\sqrt{1-x}\right]_0^b + \lim_{c \to 1^+} \left[2\sqrt{x - 1}\right]_c^2$
$= \lim_{b \to 1^-} \left(-2\sqrt{1-b}\right) - \left(-2\sqrt{1 - 0}\right) + 2\sqrt{2 - 1} - \lim_{c \to 1^+} \left(2\sqrt{c - 1}\right) = 0 + 2 + 2 - 0 = 4$

33. $\int_{-1}^\infty \frac{d\theta}{\theta^2 + 5\theta + 6} = \lim_{b \to \infty} \left[\ln \left|\frac{\theta + 2}{\theta + 3}\right|\right]_{-1}^b = \lim_{b \to \infty} \left[\ln \left|\frac{b + 2}{b + 3}\right|\right] - \ln \left|\frac{-1 + 2}{-1 + 3}\right| = 0 - \ln \left(\frac{1}{2}\right) = \ln 2$

34. $\int_0^\infty \frac{dx}{(x + 1)(x^2 + 1)} = \lim_{b \to \infty} \left[\frac{1}{2} \ln |x + 1| - \frac{1}{4} \ln (x^2 + 1) + \frac{1}{2} \tan^{-1} x\right]_0^b = \lim_{b \to \infty} \left[\frac{1}{2} \ln \left(\frac{x + 1}{\sqrt{x^2 + 1}}\right) + \frac{1}{2} \tan^{-1} x\right]_0^b$
$= \lim_{b \to \infty} \left[\frac{1}{2} \ln \left(\frac{b + 1}{\sqrt{b^2 + 1}}\right) + \frac{1}{2} \tan^{-1} b\right] - \left[\frac{1}{2} \ln \frac{1}{\sqrt{1}} + \frac{1}{2} \tan^{-1} 0\right] = \frac{1}{2} \ln 1 + \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \ln 1 - \frac{1}{2} \cdot 0 = \frac{\pi}{4}$

35. $\int_0^{\pi/2} \tan \theta\,d\theta = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln |\cos \theta|\right]_0^b = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln |\cos b|\right] + \ln 1 = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln |\cos b|\right] = +\infty$, the integral diverges

36. $\int_0^{\pi/2} \cot \theta\,d\theta = \lim_{b \to 0^+} \left[\ln |\sin \theta|\right]_b^{\pi/2} = \ln 1 - \lim_{b \to 0^+} \left[\ln |\sin b|\right] = -\lim_{b \to 0^+} \left[\ln |\sin b|\right] = +\infty$, the integral diverges

37. $\int_0^\pi \frac{\sin \theta\,d\theta}{\sqrt{\pi - \theta}}$; $[\pi - \theta = x] \rightarrow -\int_\pi^0 \frac{\sin x\,dx}{\sqrt{x}} = \int_0^\pi \frac{\sin x\,dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_0^\pi \frac{dx}{\sqrt{x}}$ converges, then $\int_0^\pi \frac{\sin x}{\sqrt{x}}\,dx$ converges by the Direct Comparison Test.

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos \theta\,d\theta}{(\pi - 2\theta)^{1/3}}$; $\begin{bmatrix} x = \pi - 2\theta \\ \theta = \frac{\pi}{2} - \frac{x}{2} \\ d\theta = -\frac{dx}{2} \end{bmatrix} \rightarrow \int_{2\pi}^0 \frac{-\cos\left(\frac{\pi}{2} - \frac{x}{2}\right) dx}{2x^{1/3}} = \int_0^{2\pi} \frac{\sin\left(\frac{x}{2}\right) dx}{2x^{1/3}}$. Since $0 \le \frac{\sin \frac{x}{2}}{2x^{1/3}} \le \frac{1}{2x^{1/3}}$ for all $0 \le x \le 2\pi$ and $\int_0^{2\pi} \frac{dx}{2x^{1/3}}$ converges, then $\int_0^{2\pi} \frac{\sin \frac{x}{2}\,dx}{2x^{1/3}}$ converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2} e^{-1/x}\,dx$; $\left[\frac{1}{x} = y\right] \rightarrow \int_\infty^{1/\ln 2} \frac{y^2 e^{-y}\,dy}{-y^2} = \int_{1/\ln 2}^\infty e^{-y}\,dy = \lim_{b \to \infty} \left[-e^{-y}\right]_{1/\ln 2}^b = \lim_{b \to \infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$
$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$; $\left[y=\sqrt{x}\right] \to 2\int_0^1 e^{-y}\,dy = 2-\frac{2}{e}$, so the integral converges.

41. $\int_0^\pi \frac{dt}{\sqrt{t+\sin t}}$. Since for $0 \le t \le \pi$, $0 \le \frac{1}{\sqrt{t+\sin t}} \le \frac{1}{\sqrt{t}}$ and $\int_0^\pi \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t-\sin t}$; let $f(t)=\frac{1}{t-\sin t}$ and $g(t)=\frac{1}{t^3}$, then $\lim\limits_{t\to 0} \frac{f(t)}{g(t)} = \lim\limits_{t\to 0} \frac{t^3}{t-\sin t} = \lim\limits_{t\to 0} \frac{3t^2}{1-\cos t} = \lim\limits_{t\to 0} \frac{6t}{\sin t}$ $= \lim\limits_{t\to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim\limits_{b\to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim\limits_{b\to 0^+}\left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t-\sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$ and $\int_0^1 \frac{dx}{1-x^2} = \lim\limits_{b\to 1^-}\left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_0^b = \lim\limits_{b\to 1^-}\left[\frac{1}{2}\ln\left|\frac{1+b}{1-b}\right|\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1-x} = \int_0^1 \frac{dx}{1-x} + \int_1^2 \frac{dx}{1-x}$ and $\int_0^1 \frac{dx}{1-x} = \lim\limits_{b\to 1^-}\left[-\ln(1-x)\right]_0^b = \lim\limits_{b\to 1^-}\left[-\ln(1-b)\right] - 0 = \infty$, which diverges $\Rightarrow \int_0^2 \frac{dx}{1-x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\,dx = \int_{-1}^0 \ln(-x)\,dx + \int_0^1 \ln x\,dx$; $\int_0^1 \ln x\,dx = \lim\limits_{b\to 0^+}\left[x\ln x - x\right]_b^1 = [1\cdot 0 - 1] - \lim\limits_{b\to 0^+}[b\ln b - b]$ $= -1-0=-1$; $\int_{-1}^0 \ln(-x)\,dx = -1 \Rightarrow \int_{-1}^1 \ln|x|\,dx = -2$ converges.

46. $\int_{-1}^1 (-x\ln|x|)\,dx = \int_{-1}^0 [-x\ln(-x)]\,dx + \int_0^1 (-x\ln x)\,dx = \lim\limits_{b\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 - \lim\limits_{c\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_c^1$ $= \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] - \lim\limits_{b\to 0^+}\left[\frac{b^2}{2}\ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2}\ln 1 - \frac{1}{4}\text{-}\right] + \lim\limits_{c\to 0^+}\left[\frac{c^2}{2}\ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 25 for the limit calculations).

47. $\int_1^\infty \frac{dx}{1+x^3}$; $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_1^\infty \frac{dx}{x^3}$ converges $\Rightarrow \int_1^\infty \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

48. $\int_4^\infty \frac{dx}{\sqrt{x}-1}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim\limits_{x\to\infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_4^\infty \frac{dx}{\sqrt{x}} = \lim\limits_{b\to\infty}\left[2\sqrt{x}\right]_4^b = \infty$, which diverges $\Rightarrow \int_4^\infty \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

49. $\int_2^\infty \frac{dv}{\sqrt{v-1}}$; $\lim\limits_{v\to\infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim\limits_{v\to\infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim\limits_{v\to\infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_2^\infty \frac{dv}{\sqrt{v}} = \lim\limits_{b\to\infty}\left[2\sqrt{v}\right]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.

50. $\int_0^\infty \frac{d\theta}{1+e^\theta}$; $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ for $0 \le \theta < \infty$ and $\int_0^\infty \frac{d\theta}{e^\theta} = \lim\limits_{b\to\infty}\left[-e^{-\theta}\right]_0^b = \lim\limits_{b\to\infty}\left(-e^{-b}+1\right) = 1 \Rightarrow \int_0^\infty \frac{d\theta}{e^\theta}$ converges $\Rightarrow \int_0^\infty \frac{d\theta}{1+e^\theta}$ converges by the Direct Comparison Test.

51. $\int_0^\infty \frac{dx}{\sqrt{x^6+1}} = \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6+1}} < \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{x^3}$ and $\int_1^\infty \frac{dx}{x^3} = \lim\limits_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b$ $= \lim\limits_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2} \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.

52. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_2^\infty \frac{1}{x}\,dx = \lim\limits_{b\to\infty} [\ln b]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$; $\lim\limits_{x\to\infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty \frac{dx}{x^{3/2}}$ $= \lim\limits_{b\to\infty} \left[-2x^{-1/2}\right]_1^b = \lim\limits_{b\to\infty} \left(\frac{-2}{\sqrt{b}}+2\right) = 2 \Rightarrow \int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges by the Limit Comparison Test.

54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x\,dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim\limits_{b\to\infty} [\ln x]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{dx}{x} = \lim\limits_{b\to\infty} [\ln x]_\pi^b = \infty$, which diverges $\Rightarrow \int_\pi^\infty \frac{2+\cos x}{x}\,dx$ diverges by the Direct Comparison Test.

56. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{2}{x^2}\,dx = \lim\limits_{b\to\infty} \left[-\frac{2}{x}\right]_\pi^b = \lim\limits_{b\to\infty} \left(-\frac{2}{b}+\frac{2}{\pi}\right) = \frac{2}{\pi}$ $\Rightarrow \int_\pi^\infty \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ converges by the Direct Comparison Test.

57. $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$; $\lim\limits_{t\to\infty} \frac{t^{3/2}}{t^{3/2}-1} = 1$ and $\int_4^\infty \frac{2\,dt}{t^{3/2}} = \lim\limits_{b\to\infty} \left[-4t^{-1/2}\right]_4^b = \lim\limits_{b\to\infty} \left(\frac{-4}{\sqrt{b}}+2\right) = 2 \Rightarrow \int_4^\infty \frac{2\,dt}{t^{3/2}}$ converges $\Rightarrow \int_4^\infty \frac{2\,dt}{t^{3/2}+1}$ converges by the Limit Comparison Test.

58. $\int_2^\infty \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_2^\infty \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^\infty \frac{e^x}{x}\,dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_1^\infty \frac{e^x\,dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^\infty \ln(\ln x)\,dx$; $[x = e^y] \rightarrow \int_e^\infty (\ln y)\,e^y\,dy$; $0 < \ln y < (\ln y)\,e^y$ for $y \ge e$ and $\int_e^\infty \ln y\,dy = \lim\limits_{b\to\infty} [y\ln y - y]_e^b = \infty$, which diverges $\Rightarrow \int_e^\infty \ln e^y\,dy$ diverges $\Rightarrow \int_{e^e}^\infty \ln(\ln x)\,dx$ diverges by the Direct Comparison Test.

61. $\int_1^\infty \frac{dx}{\sqrt{e^x-x}}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim\limits_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x-x}} = \lim\limits_{x\to\infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^\infty \frac{dx}{\sqrt{e^x}} = \int_1^\infty e^{-x/2}\,dx$ $= \lim\limits_{b\to\infty} \left[-2e^{-x/2}\right]_1^b = \lim\limits_{b\to\infty} \left(-2e^{-b/2}+2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^\infty e^{-x/2}\,dx$ converges $\Rightarrow \int_1^\infty \frac{dx}{\sqrt{e^x-x}}$ converges by the Limit Comparison Test.

62. $\int_1^\infty \frac{dx}{e^x-2^x}$; $\lim\limits_{x\to\infty} \frac{\left(\frac{1}{e^x-2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim\limits_{x\to\infty} \frac{e^x}{e^x-2^x} = \lim\limits_{x\to\infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^\infty \frac{dx}{e^x} = \lim\limits_{b\to\infty} \left[-e^{-x}\right]_1^b$ $= \lim\limits_{b\to\infty} \left(-e^{-b}+e^{-1}\right) = \frac{1}{e} \Rightarrow \int_1^\infty \frac{dx}{e^x}$ converges $\Rightarrow \int_1^\infty \frac{dx}{e^x-2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}} = 2\int_0^\infty \frac{dx}{\sqrt{x^4+1}}$; $\int_0^\infty \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^\infty \frac{dx}{\sqrt{x^4+1}} < \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^\infty \frac{dx}{x^2}$ and $\int_1^\infty \frac{dx}{x^2} = \lim\limits_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim\limits_{b\to\infty} \left(-\frac{1}{b}+1\right) = 1 \Rightarrow \int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

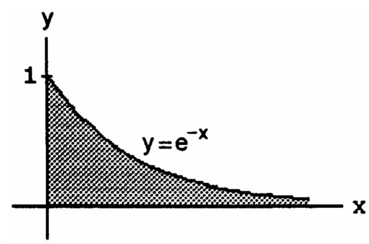
64. $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$; $0 < \frac{1}{e^x + e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^{\infty} \frac{dx}{e^x}$ converges $\Rightarrow 2\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$ converges by the Direct Comparison Test.

65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \rightarrow \int_0^{\ln 2} \frac{dt}{t^p} = \lim\limits_{b \to 0^+} \left[\frac{1}{-p+1} t^{1-p}\right]_b^{\ln 2} = \lim\limits_{b \to 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p}(\ln 2)^{1-p}$
$\Rightarrow$ the integral converges for $p < 1$ and diverges for $p \geq 1$

(b) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \rightarrow \int_{\ln 2}^{\infty} \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 65(a): it converges for $p > 1$ and diverges for $p \leq 1$

66. $\int_0^{\infty} \frac{2x\,dx}{x^2+1} = \lim\limits_{b \to \infty} \left[\ln(x^2+1)\right]_0^b = \lim\limits_{b \to \infty} \left[\ln(b^2+1)\right] - 0 = \lim\limits_{b \to \infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$ diverges. But $\lim\limits_{b \to \infty} \int_{-\infty}^{b} \frac{2x\,dx}{x^2+1} = \lim\limits_{b \to \infty} \left[\ln(x^2+1)\right]_{-b}^{b} = \lim\limits_{b \to \infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim\limits_{b \to \infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$
$= \lim\limits_{b \to \infty} (\ln 1) = 0$

67. $A = \int_0^{\infty} e^{-x}\,dx = \lim\limits_{b \to \infty} \left[-e^{-x}\right]_0^b = \lim\limits_{b \to \infty} (-e^{-b}) - (-e^{-0})$
$= 0 + 1 = 1$



68. $\bar{x} = \frac{1}{A}\int_0^{\infty} xe^{-x}\,dx = \lim\limits_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^b = \lim\limits_{b \to \infty} (-be^{-b} - e^{-b}) - (-0 \cdot e^{-0} - e^{-0}) = 0 + 1 = 1$;
$\bar{y} = \frac{1}{2A}\int_0^{\infty} (e^{-x})^2\,dx = \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx = \lim\limits_{b \to \infty} \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^b = \lim\limits_{b \to \infty} \frac{1}{2}\left(-\frac{1}{2}e^{-2b}\right) - \frac{1}{2}\left(-\frac{1}{2}e^{-2 \cdot 0}\right) = 0 + \frac{1}{4} = \frac{1}{4}$

69. $V = \int_0^{\infty} 2\pi xe^{-x}\,dx = 2\pi\int_0^{\infty} xe^{-x}\,dx = 2\pi \lim\limits_{b \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^b = 2\pi\left[\lim\limits_{b \to \infty} (-be^{-b} - e^{-b}) - 1\right] = 2\pi$

70. $V = \int_0^{\infty} \pi(e^{-x})^2\,dx = \pi\int_0^{\infty} e^{-2x}\,dx = \pi \lim\limits_{b \to \infty} \left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi \lim\limits_{b \to \infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = \frac{\pi}{2}$

71. $A = \int_0^{\pi/2} (\sec x - \tan x)\,dx = \lim\limits_{b \to \frac{\pi}{2}^-} \left[\ln|\sec x + \tan x| - \ln|\sec x|\right]_0^b = \lim\limits_{b \to \frac{\pi}{2}^-} \left(\ln\left|1 + \frac{\tan b}{\sec b}\right| - \ln|1+0|\right)$
$= \lim\limits_{b \to \frac{\pi}{2}^-} \ln|1 + \sin b| = \ln 2$

72. (a) $V = \int_0^{\pi/2} \pi\sec^2 x\,dx - \int_0^{\pi/2} \pi\tan^2 x\,dx = \pi\int_0^{\pi/2} (\sec^2 x - \tan^2 x)\,dx = \int_0^{\pi/2} \pi\left[\sec^2 x - (\sec^2 x - 1)\right]dx$
$= \pi\int_0^{\pi/2} dx = \frac{\pi^2}{2}$

(b) $S_{outer} = \int_0^{\pi/2} 2\pi\sec x\sqrt{1 + \sec^2 x\tan^2 x}\,dx \geq \int_0^{\pi/2} 2\pi\sec x(\sec x\tan x)\,dx = \pi \lim\limits_{b \to \frac{\pi}{2}^-} \left[\tan^2 x\right]_0^b$
$= \pi\left[\lim\limits_{b \to \frac{\pi}{2}^-} \left[\tan^2 b\right] - 0\right] = \pi \lim\limits_{b \to \frac{\pi}{2}^-} (\tan^2 b) = \infty \Rightarrow S_{outer}$ diverges; $S_{inner} = \int_0^{\pi/2} 2\pi\tan x\sqrt{1 + \sec^4 x}\,dx$
$\geq \int_0^{\pi/2} 2\pi\tan x\sec^2 x\,dx = \pi \lim\limits_{b \to \frac{\pi}{2}^-} \left[\tan^2 x\right]_0^b = \pi\left[\lim\limits_{b \to \frac{\pi}{2}^-} \left[\tan^2 b\right] - 0\right] = \pi \lim\limits_{b \to \frac{\pi}{2}^-} (\tan^2 b) = \infty$
$\Rightarrow S_{inner}$ diverges

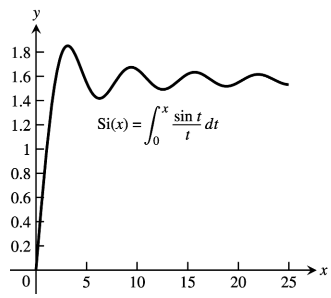
73. (a) $\int_3^\infty e^{-3x}\,dx = \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b = \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b}\right) - \left(-\frac{1}{3}e^{-3\cdot 3}\right) = 0 + \frac{1}{3}\cdot e^{-9} = \frac{1}{3}e^{-9}$
$\approx 0.0000411 < 0.000042$. Since $e^{-x^2} \le e^{-3x}$ for $x > 3$, then $\int_3^\infty e^{-x^2}\,dx < 0.000042$ and therefore $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error greater than 0.000042.

(b) $\int_0^3 e^{-x^2}\,dx \cong 0.88621$

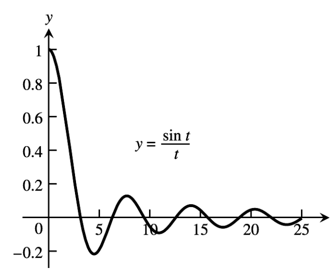
74. (a) $V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \pi\left[\lim_{b\to\infty}\left(-\frac{1}{b}\right) - \left(-\frac{1}{1}\right)\right] = \pi(0+1) = \pi$

(b) When you take the limit to ∞, you are no longer modeling the real world which is finite. The comparison step in the modeling process discussed in Section 4.2 relating the mathematical world to the real world fails to hold.

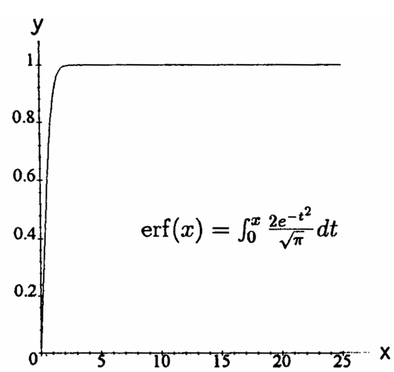
75. (a)



(b) > int((sin(t))/t, t=0..infinity); (answer is $\frac{\pi}{2}$)

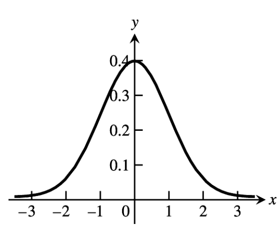


76. (a)



(b) > f:= 2*exp(−t^2)/sqrt(Pi);
> int(f, t=0..infinity); (answer is 1)

77. (a) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$
f is increasing on $(-\infty, 0]$. f is decreasing on $[0, \infty)$.
f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$



(b) Maple commands:

>f: = exp(−x^2/2)(sqrt(2*pi);

>int(f, x = −1..1); ≈ 0.683

>int(f, x = −2..2); ≈ 0.954

>int(f, x = −3..3); ≈ 0.997

(c) Part (b) suggests that as n increases, the integral approaches 1. We can take $\int_{-n}^{n} f(x)\,dx$ as close to 1 as we want by choosing $n > 1$ large enough. Also, we can make $\int_{n}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-n} f(x)\,dx$ as small as we want by choosing n large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.)

Thus, $\int_{n}^{\infty} f(x)\,dx < \int_{n}^{\infty} e^{-x/2}dx$.

$\int_{n}^{\infty} e^{-x/2}dx = \lim_{c\to\infty}\int_{n}^{c} e^{-x/2}dx = \lim_{c\to\infty}\left[-2e^{-x/2}\right]_{n}^{c} = \lim_{c\to\infty}\left[-2e^{-c/2} + 2e^{-n/2}\right] = 2e^{-n/2}$

As $n \to \infty$, $2e^{-n/2} \to 0$, for large enough n, $\int_{n}^{\infty} f(x)\,dx$ is as small as we want. Likewise for large enough n, $\int_{-\infty}^{-n} f(x)\,dx$ is as small as we want.

78. (a) The statement is true since $\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx$, $\int_{b}^{\infty} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx - \int_{a}^{b} f(x)\,dx$ and $\int_{a}^{b} f(x)\,dx$ exists since f(x) is integrable on every interval [a, b].

(b) $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx - \int_{a}^{b} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$

$= \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$

79. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..exp(1);
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=domain, y=-50..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p= -2","p = -1","p = 0","p = 1","p = 2"], title="#79 (Section 8.7)" );
q1 := Int( f(x,p), x=domain );
q2 := value( q1 );
q3 := simplify( q2 ) assuming p>-1;
q4 := simplify( q2 ) assuming p<-1;
q5 := value( eval( q1, p=-1 ) );
i1 := q1 = piecewise( p<-1, q4, p=-1, q5, p>-1, q3 );
```

80. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := exp(1)..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=exp(1)..10, y=0..100, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#80 (Section 8.7)" );
q6 := Int( f(x,p), x=domain );
q7 := value( q6 );
q8 := simplify( q7 ) assuming p>-1;
q9 := simplify( q7 ) assuming p<-1;
```

```
q10 := value( eval( q6, p=-1 ) );
i2 := q6 = piecewise( p<-1, q9, p=-1, q10, p>-1, q8 );
```

81. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=0..10, y=-50..50, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#81 (Section 8.7)" );
q11 := Int( f(x,p), x=domain ):
q11 = lhs(i1+i2);
`` = rhs(i1+i2);
`` = piecewise( p<-1, q4+q9, p=-1, q5+q10, p>-1, q3+q8 );
`` = piecewise( p<-1, -infinity, p=-1, undefined, p>-1, infinity );
```

82. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(abs(x));
domain := -infinity..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=-4..4, y=-20..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#82 (Section 8.7)" );
q12 := Int( f(x,p), x=domain );
q12p := Int( f(x,p), x=0..infinity );
q12n := Int( f(x,p), x=-infinity..0 );
q12 = q12p + q12n;
`` = simplify( q12p+q12n );
```

79-82. Example CAS commands:

Mathematica: (functions and domains may vary)

```
Clear[x, f, p]
f[x_]:= x^p Log[Abs[x]]
int = Integrate[f[x], {x, e, 100)]
int /. p → 2.5
```

In order to plot the function, a value for p must be selected.

```
p = 3;
Plot[f[x], {x, 2.72, 10}]
```

CHAPTER 8 PRACTICE EXERCISES

1. $u = \ln(x+1)$, $du = \frac{dx}{x+1}$; $dv = dx$, $v = x$;

$\int \ln(x+1)\,dx = x\ln(x+1) - \int \frac{x}{x+1}\,dx = x\ln(x+1) - \int dx + \int \frac{dx}{x+1} = x\ln(x+1) - x + \ln(x+1) + C_1$
$= (x+1)\ln(x+1) - x + C_1 = (x+1)\ln(x+1) - (x+1) + C$, where $C = C_1 + 1$

2. $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^2\,dx$, $v = \frac{1}{3}x^3$;

$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3\left(\frac{1}{x}\right)dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

3. $u = \tan^{-1} 3x,\ du = \frac{3\,dx}{1+9x^2}$; $dv = dx,\ v = x$;
$\int \tan^{-1} 3x\,dx = x\tan^{-1} 3x - \int \frac{3x\,dx}{1+9x^2}$; $\begin{bmatrix} y = 1+9x^2 \\ dy = 18x\,dx \end{bmatrix} \to x\tan^{-1} 3x - \frac{1}{6}\int \frac{dy}{y}$
$= x\tan^{-1}(3x) - \frac{1}{6}\ln(1+9x^2) + C$

4. $u = \cos^{-1}\left(\frac{x}{2}\right),\ du = \frac{-\,dx}{\sqrt{4-x^2}}$; $dv = dx,\ v = x$;
$\int \cos^{-1}\left(\frac{x}{2}\right) dx = x\cos^{-1}\left(\frac{x}{2}\right) + \int \frac{x\,dx}{\sqrt{4-x^2}}$; $\begin{bmatrix} y = 4-x^2 \\ dy = -2x\,dx \end{bmatrix} \to x\cos^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{dy}{\sqrt{y}}$
$= x\cos^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C = x\cos^{-1}\left(\frac{x}{2}\right) - 2\sqrt{1-\left(\frac{x}{2}\right)^2} + C$

5.
$$\begin{array}{lcl} & & e^x \\ (x+1)^2 & \xrightarrow{(+)} & e^x \\ 2(x+1) & \xrightarrow{(-)} & e^x \\ 2 & \xrightarrow{(+)} & e^x \\ 0 & & \end{array}$$
$\Rightarrow \int (x+1)^2 e^x\,dx = \left[(x+1)^2 - 2(x+1) + 2\right] e^x + C$

6.
$$\begin{array}{lcl} & & \sin(1-x) \\ x^2 & \xrightarrow{(+)} & \cos(1-x) \\ 2x & \xrightarrow{(-)} & -\sin(1-x) \\ 2 & \xrightarrow{(+)} & -\cos(1-x) \\ 0 & & \end{array}$$
$\Rightarrow \int x^2 \sin(1-x)\,dx = x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$

7. $u = \cos 2x,\ du = -2\sin 2x\,dx$; $dv = e^x\,dx,\ v = e^x$;
$I = \int e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx$;
$u = \sin 2x,\ du = 2\cos 2x\,dx$; $dv = e^x\,dx,\ v = e^x$;
$I = e^x\cos 2x + 2\left[e^x \sin 2x - 2\int e^x\cos 2x\,dx\right] = e^x\cos 2x + 2e^x\sin 2x - 4I \Rightarrow I = \frac{e^x\cos 2x}{5} + \frac{2e^x\sin 2x}{5} + C$

8. $u = \sin 3x,\ du = 3\cos 3x\,dx$; $dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x}$;
$I = \int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\int e^{-2x}\cos 3x\,dx$;
$u = \cos 3x,\ du = -3\sin 3x\,dx$; $dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x}$;
$I = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\left[-\frac{1}{2}e^{-2x}\cos 3x - \frac{3}{2}\int e^{-2x}\sin 3x\,dx\right] = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x - \frac{9}{4}I$
$\Rightarrow I = \frac{4}{13}\left(-\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x\right) + C = -\frac{2}{13}e^{-2x}\sin 3x - \frac{3}{13}e^{-2x}\cos 3x + C$

9. $\int \frac{x\,dx}{x^2-3x+2} = \int \frac{2\,dx}{x-2} - \int \frac{dx}{x-1} = 2\ln|x-2| - \ln|x-1| + C$

10. $\int \frac{x\,dx}{x^2+4x+3} = \frac{3}{2}\int \frac{dx}{x+3} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{3}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C$

11. $\int \frac{dx}{x(x+1)^2} = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{-1}{(x+1)^2}\right) dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$

12. $\int \frac{x+1}{x^2(x-1)}\,dx = \int \left(\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2}\right) dx = 2\ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + C = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$

13. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y]$ $\rightarrow$ $-\int \frac{dy}{y^2+y-2} = -\frac{1}{3}\int \frac{dy}{y-1} + \frac{1}{3}\int \frac{dy}{y+2} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C$
$= \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

14. $\int \frac{\cos\theta\, d\theta}{\sin^2\theta + \sin\theta - 6}$; $[\sin\theta = x]$ $\rightarrow$ $\int \frac{dx}{x^2+x-6} = \frac{1}{5}\int \frac{dx}{x-2} - \frac{1}{5}\int \frac{dx}{x+3} = \frac{1}{5}\ln\left|\frac{\sin\theta-2}{\sin\theta+3}\right| + C$

15. $\int \frac{3x^2+4x+4}{x^3+x}\,dx = \int \frac{4}{x}\,dx - \int \frac{x-4}{x^2+1}\,dx = 4\ln|x| - \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}x + C$

16. $\int \frac{4x\,dx}{x^3+4x} = \int \frac{4\,dx}{x^2+4} = 2\tan^{-1}\left(\frac{x}{2}\right) + C$

17. $\int \frac{(v+3)\,dv}{2v^3-8v} = \frac{1}{2}\int \left(-\frac{3}{4v} + \frac{5}{8(v-2)} + \frac{1}{8(v+2)}\right) dv = -\frac{3}{8}\ln|v| + \frac{5}{16}\ln|v-2| + \frac{1}{16}\ln|v+2| + C$
$= \frac{1}{16}\ln\left|\frac{(v-2)^5(v+2)}{v^6}\right| + C$

18. $\int \frac{(3v-7)\,dv}{(v-1)(v-2)(v-3)} = \int \frac{(-2)\,dv}{v-1} + \int \frac{dv}{v-2} + \int \frac{dv}{v-3} = \ln\left|\frac{(v-2)(v-3)}{(v-1)^2}\right| + C$

19. $\int \frac{dt}{t^4+4t^2+3} = \frac{1}{2}\int \frac{dt}{t^2+1} - \frac{1}{2}\int \frac{dt}{t^2+3} = \frac{1}{2}\tan^{-1}t - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{2}\tan^{-1}t - \frac{\sqrt{3}}{6}\tan^{-1}\frac{t}{\sqrt{3}} + C$

20. $\int \frac{t\,dt}{t^4-t^2-2} = \frac{1}{3}\int \frac{t\,dt}{t^2-2} - \frac{1}{3}\int \frac{t\,dt}{t^2+1} = \frac{1}{6}\ln|t^2-2| - \frac{1}{6}\ln(t^2+1) + C$

21. $\int \frac{x^3+x^2}{x^2+x-2}\,dx = \int \left(x + \frac{2x}{x^2+x-2}\right) dx = \int x\,dx + \frac{2}{3}\int \frac{dx}{x-1} + \frac{4}{3}\int \frac{dx}{x+2} = \frac{x^2}{2} + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

22. $\int \frac{x^3+1}{x^3-x}\,dx = \int \left(1 + \frac{x+1}{x^3-x}\right) dx = \int \left[1 + \frac{1}{x(x-1)}\right] dx = \int dx + \int \frac{dx}{x-1} - \int \frac{dx}{x} = x + \ln|x-1| - \ln|x| + C$

23. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int \left(x - \frac{3x}{x^2+4x+3}\right) dx = \int x\,dx + \frac{3}{2}\int \frac{dx}{x+1} - \frac{9}{2}\int \frac{dx}{x+3} = \frac{x^2}{2} - \frac{9}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$

24. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int \left[(2x-3) + \frac{x}{x^2+2x-8}\right] dx = \int (2x-3)\,dx + \frac{1}{3}\int \frac{dx}{x-2} + \frac{2}{3}\int \frac{dx}{x+4}$
$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$

25. $\int \frac{dx}{x\left(3\sqrt{x+1}\right)}$; $\begin{bmatrix} u = \sqrt{x+1} \\ du = \frac{dx}{2\sqrt{x+1}} \\ dx = 2u\,du \end{bmatrix}$ $\rightarrow$ $\frac{2}{3}\int \frac{u\,du}{(u^2-1)\,u} = \frac{1}{3}\int \frac{du}{u-1} - \frac{1}{3}\int \frac{du}{u+1} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+1| + C$
$= \frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

26. $\int \frac{dx}{x\left(1+\sqrt[3]{x}\right)}$; $\begin{bmatrix} u = \sqrt[3]{x} \\ du = \frac{dx}{3x^{2/3}} \\ dx = 3u^2\,du \end{bmatrix}$ $\rightarrow \int \frac{3u^2\,du}{u^3(1+u)} = 3\int \frac{du}{u(1+u)} = 3\ln\left|\frac{u}{u+1}\right| + C = 3\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right| + C$

27. $\int \frac{ds}{e^s-1}$; $\begin{bmatrix} u = e^s - 1 \\ du = e^s\,ds \\ ds = \frac{du}{u+1} \end{bmatrix}$ $\rightarrow$ $\int \frac{du}{u(u+1)} = -\int \frac{du}{u+1} + \int \frac{du}{u} = \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{e^s-1}{e^s}\right| + C = \ln|1-e^{-s}| + C$

28. $\int \frac{ds}{\sqrt{e^s+1}}$; $\begin{bmatrix} u = \sqrt{e^s+1} \\ du = \frac{e^s\,ds}{2\sqrt{e^s+1}} \\ ds = \frac{2u\,du}{u^2-1} \end{bmatrix}$ $\rightarrow \int \frac{2u\,du}{u(u^2-1)} = 2\int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln\left|\frac{u-1}{u+1}\right| + C$
$= \ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right| + C$

29. (a) $\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \frac{d(16-y^2)}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$
(b) $\int \frac{y\,dy}{\sqrt{16-y^2}}$; $[y = 4\sin x] \rightarrow 4\int \frac{\sin x\cos x\,dx}{\cos x} = -4\cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$

30. (a) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{d(4+x^2)}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$
(b) $\int \frac{x\,dx}{\sqrt{4+x^2}}$; $[x = 2\tan y] \rightarrow \int \frac{2\tan y \cdot 2\sec^2 y\,dy}{2\sec y} = 2\int \sec y\tan y\,dy = 2\sec y + C = \sqrt{4+x^2} + C$

31. (a) $\int \frac{x\,dx}{4-x^2} = -\frac{1}{2}\int \frac{d(4-x^2)}{4-x^2} = -\frac{1}{2}\ln|4-x^2| + C$
(b) $\int \frac{x\,dx}{4-x^2}$; $[x = 2\sin\theta] \rightarrow \int \frac{2\sin\theta \cdot 2\cos\theta\,d\theta}{4\cos^2\theta} = \int \tan\theta\,d\theta = -\ln|\cos\theta| + C = -\ln\left(\frac{\sqrt{4-x^2}}{2}\right) + C$
$= -\frac{1}{2}\ln|4-x^2| + C$

32. (a) $\int \frac{t\,dt}{\sqrt{4t^2-1}} = \frac{1}{8}\int \frac{d(4t^2-1)}{\sqrt{4t^2-1}} = \frac{1}{4}\sqrt{4t^2-1} + C$
(b) $\int \frac{t\,dt}{\sqrt{4t^2-1}}$; $\left[t = \frac{1}{2}\sec\theta\right] \rightarrow \int \frac{\frac{1}{2}\sec\theta\tan\theta \cdot \frac{1}{2}\sec\theta\,d\theta}{\tan\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{\tan\theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$

33. $\int \frac{x\,dx}{9-x^2}$; $\begin{bmatrix} u = 9-x^2 \\ du = -2x\,dx \end{bmatrix}$ $\rightarrow -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = \ln\frac{1}{\sqrt{u}} + C = \ln\frac{1}{\sqrt{9-x^2}} + C$

34. $\int \frac{dx}{x(9-x^2)} = \frac{1}{9}\int \frac{dx}{x} + \frac{1}{18}\int \frac{dx}{3-x} - \frac{1}{18}\int \frac{dx}{3+x} = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| - \frac{1}{18}\ln|3+x| + C$
$= \frac{1}{9}\ln|x| - \frac{1}{18}\ln|9-x^2| + C$

35. $\int \frac{dx}{9-x^2} = \frac{1}{6}\int \frac{dx}{3-x} + \frac{1}{6}\int \frac{dx}{3+x} = -\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x| + C = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

36. $\int \frac{dx}{\sqrt{9-x^2}}$; $\begin{bmatrix} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{bmatrix}$ $\rightarrow \int \frac{3\cos\theta}{3\cos\theta}\,d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$

37. $\int \sin^3 x\cos^4 x\,dx = \int \cos^4 x(1-\cos^2 x)\sin x\,dx = \int \cos^4 x\sin x\,dx - \int \cos^6 x\sin x\,dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

38. $\int \cos^5 x\sin^5 x\,dx = \int \sin^5 x\cos^4 x\cos x\,dx = \int \sin^5 x\,(1-\sin^2 x)^2\cos x\,dx$
$= \int \sin^5 x\cos x\,dx - 2\int \sin^7 x\cos x\,dx + \int \sin^9 x\cos x\,dx = \frac{\sin^6 x}{6} - \frac{2\sin^8 x}{8} + \frac{\sin^{10} x}{10} + C$

39. $\int \tan^4 x\sec^2 x\,dx = \frac{\tan^5 x}{5} + C$

40. $\int \tan^3 x\sec^3 x\,dx = \int (\sec^2 x - 1)\sec^2 x \cdot \sec x \cdot \tan x\,dx = \int \sec^4 x \cdot \sec x \cdot \tan x\,dx - \int \sec^2 x \cdot \sec x \cdot \tan x\,dx$
$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

41. $\int \sin 5\theta \cos 6\theta \, d\theta = \frac{1}{2}\int (\sin(-\theta) + \sin(11\theta))\, d\theta = \frac{1}{2}\int \sin(-\theta)\, d\theta + \frac{1}{2}\int \sin(11\theta)\, d\theta = \frac{1}{2}\cos(-\theta) - \frac{1}{22}\cos 11\theta + C$
$= \frac{1}{2}\cos\theta - \frac{1}{22}\cos 11\theta + C$

42. $\int \cos 3\theta \cos 3\theta \, d\theta = \frac{1}{2}\int \left(\cos 0 + \cos 6\theta\right) d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 6\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{12}\sin 6\theta + C$

43. $\int \sqrt{1 + \cos\left(\frac{t}{2}\right)}\, dt = \int \sqrt{2}\left|\cos\ \frac{t}{4}\right| dt = 4\sqrt{2}\ \left|\sin\ \frac{t}{4}\right| + C$

44. $\int e^t \sqrt{\tan^2 e^t + 1}\, dt = \int |\sec e^t|\, e^t\, dt = \ln|\sec e^t + \tan e^t| + C$

45. $|E_s| \le \frac{3-1}{180}(\triangle x)^4 M$ where $\triangle x = \frac{3-1}{n} = \frac{2}{n}$; $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$
$\Rightarrow f^{(4)}(x) = 24x^{-5}$ which is decreasing on $[1,3] \Rightarrow$ maximum of $f^{(4)}(x)$ on $[1,3]$ is $f^{(4)}(1) = 24 \Rightarrow M = 24$. Then
$|E_s| \le 0.0001 \Rightarrow \left(\frac{3-1}{180}\right)\left(\frac{2}{n}\right)^4(24) \le 0.0001 \Rightarrow \left(\frac{768}{180}\right)\left(\frac{1}{n^4}\right) \le 0.0001 \Rightarrow \frac{1}{n^4} \le (0.0001)\left(\frac{180}{768}\right) \Rightarrow n^4 \ge 10{,}000\left(\frac{768}{180}\right)$
$\Rightarrow n \ge 14.37 \Rightarrow n \ge 16$ (n must be even)

46. $|E_T| \le \frac{1-0}{12}(\triangle x)^2 M$ where $\triangle x = \frac{1-0}{n} = \frac{1}{n}$; $0 \le f''(x) \le 8 \Rightarrow M = 8$. Then $|E_T| \le 10^{-3} \Rightarrow \frac{1}{12}\left(\frac{1}{n}\right)^2(8) \le 10^{-3}$
$\Rightarrow \frac{2}{3n^2} \le 10^{-3} \Rightarrow \frac{3n^2}{2} \ge 1000 \Rightarrow n^2 \ge \frac{2000}{3} \Rightarrow n \ge 25.82 \Rightarrow n \ge 26$

47. $\triangle x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6} \Rightarrow \frac{\triangle x}{2} = \frac{\pi}{12}$;
$\sum_{i=0}^{6} mf(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{12}\right)(12) = \pi$;

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	2	1
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	2	4
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	2	1
x_6	π	0	1	0

$\sum_{i=0}^{6} mf(x_i) = 18$ and $\frac{\triangle x}{3} = \frac{\pi}{18} \Rightarrow$
$S = \left(\frac{\pi}{18}\right)(18) = \pi$.

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	$\pi/6$	1/2	4	2
x_2	$\pi/3$	3/2	2	3
x_3	$\pi/2$	2	4	8
x_4	$2\pi/3$	3/2	2	3
x_5	$5\pi/6$	1/2	4	2
x_6	π	0	1	0

48. $\left|f^{(4)}(x)\right| \le 3 \Rightarrow M = 3$; $\triangle x = \frac{2-1}{n} = \frac{1}{n}$. Hence $|E_s| \le 10^{-5} \Rightarrow \left(\frac{2-1}{180}\right)\left(\frac{1}{n}\right)^4(3) \le 10^{-5} \Rightarrow \frac{1}{60n^4} \le 10^{-5} \Rightarrow n^4 \ge \frac{10^5}{60}$
$\Rightarrow n \ge 6.38 \Rightarrow n \ge 8$ (n must be even)

49. $y_{av} = \frac{1}{365-0}\int_0^{365}\left[37\sin\left(\frac{2\pi}{365}(x-101)\right) + 25\right] dx = \frac{1}{365}\left[-37\left(\frac{365}{2\pi}\cos\left(\frac{2\pi}{365}(x-101)\right) + 25x\right)\right]_0^{365}$
$= \frac{1}{365}\left[\left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(365-101)\right] + 25(365)\right) - \left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(0-101)\right] + 25(0)\right)\right]$
$= -\frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(264)\right) + 25 + \frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(-101)\right) = -\frac{37}{2\pi}\left(\cos\left(\frac{2\pi}{365}(264)\right) - \cos\left(\frac{2\pi}{365}(-101)\right)\right) + 25$
$\approx -\frac{37}{2\pi}(0.16705 - 0.16705) + 25 = 25°$ F

50. $av(C_v) = \frac{1}{675-20}\int_{20}^{675}\left[8.27 + 10^{-5}(26T - 1.87T^2)\right] dT = \frac{1}{655}\left[8.27T + \frac{13}{10^5}T^2 - \frac{0.62333}{10^5}T^3\right]_{20}^{675}$
$\approx \frac{1}{655}[(5582.25 + 59.23125 - 1917.03194) - (165.4 + 0.052 - 0.04987)] \approx 5.434$;
$8.27 + 10^{-5}(26T - 1.87T^2) = 5.434 \Rightarrow 1.87T^2 - 26T - 283{,}600 = 0 \Rightarrow T \approx \frac{26+\sqrt{676 + 4(1.87)(283{,}600)}}{2(1.87)} \approx 396.45°$ C

51. (a) Each interval is 5 min $= \frac{1}{12}$ hour.

$\frac{1}{24}\left[2.5 + 2(2.4) + 2(2.3) + \dots + 2(2.4) + 2.3\right] = \frac{29}{12} \approx 2.42$ gal

(b) $(60 \text{ mph})\left(\frac{12}{29} \text{ hours/gal}\right) \approx 24.83$ mi/gal

52. Using the Simpson's rule, $\triangle x = 15 \Rightarrow \frac{\triangle x}{3} = 5$;
$\sum mf(x_i) = 1211.8 \Rightarrow$ Area $\approx (1211.8)(5) = 6059$ ft^2;
The cost is Area $\cdot$ (\$2.10/ft^2) $\approx$ (6059 ft^2)(\$2.10/ft^2)
$=$ \$12,723.90 $\Rightarrow$ the job cannot be done for \$11,000.

	x_i	$f(x_i)$	m	$mf(x_i)$
x_0	0	0	1	0
x_1	15	36	4	144
x_2	30	54	2	108
x_3	45	51	4	204
x_4	60	49.5	2	99
x_5	75	54	4	216
x_6	90	64.4	2	128.8
x_7	105	67.5	4	270
x_8	120	42	1	42

53. $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{b \to 3^-} \int_0^b \frac{dx}{\sqrt{9-x^2}} = \lim_{b \to 3^-} \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b = \lim_{b \to 3^-} \sin^{-1}\left(\frac{b}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

54. $\int_0^1 \ln x \, dx = \lim_{b \to 0^+} \left[x \ln x - x\right]_b^1 = (1 \cdot \ln 1 - 1) - \lim_{b \to 0^+} \left[b \ln b - b\right] = -1 - \lim_{b \to 0^+} \frac{\ln b}{\left(\frac{1}{b}\right)} = -1 - \lim_{b \to 0^+} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$
$= -1 + 0 = -1$

55. $\int_{-1}^1 \frac{dy}{y^{2/3}} = \int_{-1}^0 \frac{dy}{y^{2/3}} + \int_0^1 \frac{dy}{y^{2/3}} = 2\int_0^1 \frac{dy}{y^{2/3}} = 2 \cdot 3 \lim_{b \to 0^+} \left[y^{1/3}\right]_b^1 = 6\left(1 - \lim_{b \to 0^+} b^{1/3}\right) = 6$

56. $\int_{-2}^{\infty} \frac{d\theta}{(\theta+1)^{3/5}} = \int_{-2}^{-1} \frac{d\theta}{(\theta+1)^{3/5}} + \int_{-1}^{2} \frac{d\theta}{(\theta+1)^{3/5}} + \int_{2}^{\infty} \frac{d\theta}{(\theta+1)^{3/5}}$ converges if each integral converges, but
$\lim_{\theta \to \infty} \frac{\theta^{3/5}}{(\theta+1)^{3/5}} = 1$ and $\int_2^{\infty} \frac{d\theta}{\theta^{3/5}}$ diverges $\Rightarrow \int_{-2}^{\infty} \frac{d\theta}{(\theta+1)^{3/5}}$ diverges

57. $\int_3^{\infty} \frac{2\,du}{u^2-2u} = \int_3^{\infty} \frac{du}{u-2} - \int_3^{\infty} \frac{du}{u} = \lim_{b \to \infty} \left[\ln\left|\frac{u-2}{u}\right|\right]_3^b = \lim_{b \to \infty} \left[\ln\left|\frac{b-2}{b}\right|\right] - \ln\left|\frac{3-2}{3}\right| = 0 - \ln\left(\frac{1}{3}\right) = \ln 3$

58. $\int_1^{\infty} \frac{3v-1}{4v^3-v^2} \, dv = \int_1^{\infty} \left(\frac{1}{v} + \frac{1}{v^2} - \frac{4}{4v-1}\right) dv = \lim_{b \to \infty} \left[\ln v - \frac{1}{v} - \ln(4v-1)\right]_1^b$
$= \lim_{b \to \infty} \left[\ln\left(\frac{b}{4b-1}\right) - \frac{1}{b}\right] - (\ln 1 - 1 - \ln 3) = \ln \frac{1}{4} + 1 + \ln 3 = 1 + \ln \frac{3}{4}$

59. $\int_0^{\infty} x^2 e^{-x} \, dx = \lim_{b \to \infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = \lim_{b \to \infty} \left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (-2) = 0 + 2 = 2$

60. $\int_{-\infty}^0 xe^{3x} \, dx = \lim_{b \to -\infty} \left[\frac{x}{3} e^{3x} - \frac{1}{9} e^{3x}\right]_b^0 = -\frac{1}{9} - \lim_{b \to -\infty} \left(\frac{b}{3} e^{3b} - \frac{1}{9} e^{3b}\right) = -\frac{1}{9} - 0 = -\frac{1}{9}$

61. $\int_{-\infty}^{\infty} \frac{dx}{4x^2+9} = 2\int_0^{\infty} \frac{dx}{4x^2+9} = \frac{1}{2}\int_0^{\infty} \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2} \lim_{b \to \infty} \left[\frac{2}{3} \tan^{-1}\left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2} \lim_{b \to \infty} \left[\frac{2}{3} \tan^{-1}\left(\frac{2b}{3}\right)\right] - \frac{1}{3} \tan^{-1}(0)$
$= \frac{1}{2}\left(\frac{2}{3} \cdot \frac{\pi}{2}\right) - 0 = \frac{\pi}{6}$

62. $\int_{-\infty}^{\infty} \frac{4\,dx}{x^2+16} = 2\int_0^{\infty} \frac{4\,dx}{x^2+16} = 2 \lim_{b \to \infty} \left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^b = 2\left(\lim_{b \to \infty} \left[\tan^{-1}\left(\frac{b}{4}\right)\right] - \tan^{-1}(0)\right) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$

63. $\lim_{\theta \to \infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^{\infty} \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^{\infty} \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

64. $I = \int_0^\infty e^{-u} \cos u \, du = \lim_{b \to \infty} \left[-e^{-u} \cos u\right]_0^b - \int_0^\infty e^{-u} \sin u \, du = 1 + \lim_{b \to \infty} \left[e^{-u} \sin u\right]_0^b - \int_0^\infty (e^{-u}) \cos u \, du$
$\Rightarrow I = 1 + 0 - I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges

65. $\int_1^\infty \frac{\ln z}{z} \, dz = \int_1^e \frac{\ln z}{z} \, dz + \int_e^\infty \frac{\ln z}{z} \, dz = \left[\frac{(\ln z)^2}{2}\right]_1^e + \lim_{b \to \infty} \left[\frac{(\ln z)^2}{2}\right]_e^b = \left(\frac{1^2}{2} - 0\right) + \lim_{b \to \infty} \left[\frac{(\ln b)^2}{2} - \frac{1}{2}\right] = \infty$
$\Rightarrow$ diverges

66. $0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$ and $\int_1^\infty e^{-t} \, dt$ converges $\Rightarrow \int_1^\infty \frac{e^{-t}}{\sqrt{t}} \, dt$ converges

67. $\int_{-\infty}^\infty \frac{2 \, dx}{e^x + e^{-x}} = 2\int_0^\infty \frac{2 \, dx}{e^x + e^{-x}} < \int_0^\infty \frac{4 \, dx}{e^x}$ converges $\Rightarrow \int_{-\infty}^\infty \frac{2 \, dx}{e^x + e^{-x}}$ converges

68. $\int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)} = \int_{-\infty}^{-1} \frac{dx}{x^2(1+e^x)} + \int_{-1}^0 \frac{dx}{x^2(1+e^x)} + \int_0^1 \frac{dx}{x^2(1+e^x)} + \int_1^\infty \frac{dx}{x^2(1+e^x)}$;
$\lim_{x \to 0} \frac{\left(\frac{1}{x^2}\right)}{\left[\frac{1}{x^2(1+e^x)}\right]} = \lim_{x \to 0} \frac{x^2(1+e^x)}{x^2} = \lim_{x \to 0} (1+e^x) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2(1+e^x)}$ diverges
$\Rightarrow \int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)}$ diverges

69. $\int \frac{x \, dx}{1+\sqrt{x}}$; $\left[\begin{array}{c} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{array}\right] \to \int \frac{u^2 \cdot 2u \, du}{1+u} = \int \left(2u^2 - 2u + 2 - \frac{2}{1+u}\right) du = \frac{2}{3}u^3 - u^2 + 2u - 2\ln|1+u| + C$
$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

70. $\int \frac{x^3+2}{4-x^2} \, dx = -\int \left(x + \frac{4x+2}{x^2-4}\right) dx = -\int x \, dx - \frac{3}{2}\int \frac{dx}{x+2} - \frac{5}{2}\int \frac{dx}{x-2} = -\frac{x^2}{2} - \frac{3}{2}\ln|x+2| - \frac{5}{2}\ln|x-2| + C$

71. $\int \frac{dx}{x(x^2+1)^2}$; $\left[\begin{array}{c} x = \tan\theta \\ dx = \sec^2\theta \, d\theta \end{array}\right] \to \int \frac{\sec^2\theta \, d\theta}{\tan\theta \sec^4\theta} = \int \frac{\cos^3\theta \, d\theta}{\sin\theta} = \int \left(\frac{1-\sin^2\theta}{\sin\theta}\right) d(\sin\theta)$
$= \ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)^2 + C$

72. $\int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{d(x+1)}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$

73. $\int \frac{2 - \cos x + \sin x}{\sin^2 x} \, dx = \int 2\csc^2 x \, dx - \int \frac{\cos x \, dx}{\sin^2 x} + \int \csc x \, dx = -2\cot x + \frac{1}{\sin x} - \ln|\csc x + \cot x| + C$
$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$

74. $\int \frac{\sin^2\theta}{\cos^2\theta} \, d\theta = \int \frac{1-\cos^2\theta}{\cos^2\theta} \, d\theta = \int \sec^2\theta \, d\theta - \int d\theta = \tan\theta - \theta + C$

75. $\int \frac{9 \, dv}{81 - v^4} = \frac{1}{2}\int \frac{dv}{v^2+9} + \frac{1}{12}\int \frac{dv}{3-v} + \frac{1}{12}\int \frac{dv}{3+v} = \frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$

76. $\int_2^\infty \frac{dx}{(x-1)^2} = \lim_{b \to \infty} \left[\frac{1}{1-x}\right]_2^b = \lim_{b \to \infty} \left[\frac{1}{1-b} - (-1)\right] = 0 + 1 = 1$

77.

		$\cos(2\theta+1)$
θ	$\xrightarrow{(+)}$	$\frac{1}{2}\sin(2\theta+1)$
1	$\xrightarrow{(-)}$	$-\frac{1}{4}\cos(2\theta+1)$
0		

$\Rightarrow \int \theta\cos(2\theta+1) \, d\theta = \frac{\theta}{2}\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$

78. $\int \frac{x^3\,dx}{x^2-2x+1} = \int \left(x+2+\frac{3x-2}{x^2-2x+1}\right) dx = \int (x+2)\,dx + 3\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$
$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

79. $\int \frac{\sin 2\theta\, d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2}\int \frac{d(1+\cos 2\theta)}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4}\sec^2\theta + C$

80. $\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx = -\sqrt{2}\int_{\pi/4}^{\pi/2} \cos 2x\,dx = \left[-\frac{\sqrt{2}}{2}\sin 2x\right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2}$

81. $\int \frac{x\,dx}{\sqrt{2-x}}\,;\ \begin{bmatrix} y = 2-x \\ dy = -dx \end{bmatrix} \rightarrow -\int \frac{(2-y)\,dy}{\sqrt{y}} = \frac{2}{3}y^{3/2} - 4y^{1/2} + C = \frac{2}{3}(2-x)^{3/2} - 4(2-x)^{1/2} + C$
$= 2\left[\frac{\left(\sqrt{2-x}\right)^3}{3} - 2\sqrt{2-x}\right] + C$

82. $\int \frac{\sqrt{1-v^2}}{v^2}\,dv;\ [v=\sin\theta] \rightarrow \int \frac{\cos\theta\cdot\cos\theta\,d\theta}{\sin^2\theta} = \int \frac{(1-\sin^2\theta)\,d\theta}{\sin^2\theta} = \int \csc^2\theta\,d\theta - \int d\theta = \cot\theta - \theta + C$
$= -\sin^{-1}v - \frac{\sqrt{1-v^2}}{v} + C$

83. $\int \frac{dy}{y^2-2y+2} = \int \frac{d(y-1)}{(y-1)^2+1} = \tan^{-1}(y-1) + C$

84. $\int \frac{x\,dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{9-(x^2+1)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right) + C$

85. $\int \frac{z+1}{z^2(z^2+4)}\,dz = \frac{1}{4}\int \left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right) dz = \frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{8}\ln(z^2+4) - \frac{1}{8}\tan^{-1}\frac{z}{2} + C$

86. $\int x^3 e^{x^2}\,dx = \frac{1}{2}\int x^2 e^{x^2}\,d(x^2) = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C = \frac{(x^2-1)\,e^{x^2}}{2} + C$

87. $\int \frac{t\,dt}{\sqrt{9-4t^2}} = -\frac{1}{8}\int \frac{d(9-4t^2)}{\sqrt{9-4t^2}} = -\frac{1}{4}\sqrt{9-4t^2} + C$

88. $u = \tan^{-1}x,\ du = \frac{dx}{1+x^2}\,;\ dv = \frac{dx}{x^2},\ v = -\frac{1}{x}\,;$
$\int \frac{\tan^{-1}x\,dx}{x^2} = -\frac{1}{x}\tan^{-1}x + \int \frac{dx}{x(1+x^2)} = -\frac{1}{x}\tan^{-1}x + \int \frac{dx}{x} - \int \frac{x\,dx}{1+x^2}$
$= -\frac{1}{x}\tan^{-1}x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C = -\frac{\tan^{-1}x}{x} + \ln|x| - \ln\sqrt{1+x^2} + C$

89. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}\,;\ [e^t = x] \rightarrow \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C = \ln\left|\frac{x+1}{x+2}\right| + C$
$= \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

90. $\int \tan^3 t\,dt = \int (\tan t)(\sec^2 t - 1)\,dt = \frac{\tan^2 t}{2} - \int \tan t\,dt = \frac{\tan^2 t}{2} - \ln|\sec t| + C$

91. $\int_1^\infty \frac{\ln y\,dy}{y^3}\,;\ \begin{bmatrix} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x\,dx \end{bmatrix} \rightarrow \int_0^\infty \frac{x\cdot e^x}{e^{3x}}\,dx = \int_0^\infty xe^{-2x}\,dx = \lim_{b\to\infty}\left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$
$= \lim_{b\to\infty}\left(\frac{-b}{2e^{2b}} - \frac{1}{4e^{2b}}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

92. $\int \frac{\cot v\,dv}{\ln(\sin v)} = \int \frac{\cos v\,dv}{(\sin v)\ln(\sin v)}\,;\ \begin{bmatrix} u = \ln(\sin v) \\ du = \frac{\cos v\,dv}{\sin v} \end{bmatrix} \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\ln(\sin v)| + C$

93. $\int e^{\ln\sqrt{x}}\,dx = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

94. $\int e^{\theta}\sqrt{3+4e^{\theta}}\,d\theta;\ \begin{bmatrix} u = 4e^{\theta} \\ du = 4e^{\theta}\,d\theta \end{bmatrix} \to \frac{1}{4}\int \sqrt{3+u}\,du = \frac{1}{4}\cdot\frac{2}{3}(3+u)^{3/2} + C = \frac{1}{6}\left(3+4e^{\theta}\right)^{3/2} + C$

95. $\int \frac{\sin 5t\,dt}{1+(\cos 5t)^2};\ \begin{bmatrix} u = \cos 5t \\ du = -5\sin 5t\,dt \end{bmatrix} \to -\frac{1}{5}\int \frac{du}{1+u^2} = -\frac{1}{5}\tan^{-1} u + C = -\frac{1}{5}\tan^{-1}(\cos 5t) + C$

96. $\int \frac{dv}{\sqrt{e^{2v}-1}};\ \begin{bmatrix} x = e^{v} \\ dx = e^{v}\,dv \end{bmatrix} \to \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C = \sec^{-1}(e^{v}) + C$

97. $\int \frac{dr}{1+\sqrt{r}};\ \begin{bmatrix} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{bmatrix} \to \int \frac{2u\,du}{1+u} = \int \left(2 - \frac{2}{1+u}\right) du = 2u - 2\ln|1+u| + C = 2\sqrt{r} - 2\ln\left(1+\sqrt{r}\right) + C$

98. $\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx = \int \frac{d\left(x^4-10x^2+9\right)}{x^4-10x^2+9} = \ln|x^4-10x^2+9| + C$

99. $\int \frac{x^3}{1+x^2}\,dx = \int \left(x - \frac{x}{1+x^2}\right) dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2) + C$

100. $\int \frac{x^2}{1+x^3}\,dx = 3\int \frac{3x^2}{1+x^3}\,dx = 3\ln|1+x^3| + C$

101. $\int \frac{1+x^2}{1+x^3}\,dx;\ \frac{1+x^2}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2} \Rightarrow 1+x^2 = A(1-x+x^2) + (Bx+C)(1+x)$
$= (A+B)x^2 + (-A+B+C)x + (A+C) \Rightarrow A+B = 1,\ -A+B+C = 0,\ A+C = 1 \Rightarrow A = \frac{2}{3}, B = \frac{1}{3}, C = \frac{1}{3};$
$\int \frac{1+x^2}{1+x^3}\,dx = \int \left(\frac{2/3}{1+x} + \frac{(1/3)x+1/3}{1-x+x^2}\right) dx = \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{1-x+x^2}\,dx = \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{\frac{3}{4}+\left(x-\frac{1}{2}\right)^2}\,dx;$
$\begin{bmatrix} u = x - \frac{1}{2} \\ du = dx \end{bmatrix} \to \frac{1}{3}\int \frac{u+\frac{3}{2}}{\frac{3}{4}+u^2}\,du = \frac{1}{3}\int \frac{u}{\frac{3}{4}+u^2}\,du + \frac{1}{2}\int \frac{1}{\frac{3}{4}+u^2}\,du = \frac{1}{6}\ln\left|\frac{3}{4} + u^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right)$
$= \frac{1}{6}\ln\left|\frac{3}{4} + \left(x-\frac{1}{2}\right)^2\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right) = \frac{1}{6}\ln|1 - x + x^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$
$\Rightarrow \frac{2}{3}\int \frac{1}{1+x}\,dx + \frac{1}{3}\int \frac{x+1}{1-x+x^2}\,dx = \frac{2}{3}\ln|1+x| + \frac{1}{6}\ln|1 - x + x^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

102. $\int \frac{1+x^2}{(1+x)^3}\,dx;\ \begin{bmatrix} u = 1+x \\ du = dx \end{bmatrix} \to \int \frac{1+(u-1)^2}{u^3}\,du = \int \frac{u^2-2u+2}{u^3}\,du = \int \frac{1}{u}\,du - \int \frac{2}{u^2}\,du + \int \frac{2}{u^3}\,du = \ln|u| + \frac{2}{u} - \frac{1}{u^2} + C$
$= \ln|1+x| + \frac{2}{1+x} - \frac{1}{(1+x)^2} + C$

103. $\int \sqrt{x}\sqrt{1+\sqrt{x}}\,dx;\ \begin{bmatrix} w = \sqrt{x} \Rightarrow w^2 = x \\ 2w\,dw = dx \end{bmatrix} \to \int 2w^2\sqrt{1+w}\,dw$

		$\sqrt{1+w}$
$2w^2$	$\xrightarrow{(+)}$	$\frac{2}{3}(1+w)^{3/2}$
$4w$	$\xrightarrow{(-)}$	$\frac{4}{15}(1+w)^{5/2}$
4	$\xrightarrow{(+)}$	$\frac{8}{105}(1+w)^{7/2}$
0		

$\Rightarrow \int 2w^2\sqrt{1+w}\,dw = \frac{4}{3}w^2(1+w)^{3/2} - \frac{16}{15}w(1+w)^{5/2} + \frac{32}{105}(1+w)^{7/2} + C$
$= \frac{4}{3}x\left(1+\sqrt{x}\right)^{3/2} - \frac{16}{15}\sqrt{x}\left(1+\sqrt{x}\right)^{5/2} + \frac{32}{105}\left(1+\sqrt{x}\right)^{7/2} + C$

104. $\int \sqrt{1+\sqrt{1+x}}\,dx$; $\left[\begin{matrix} w=\sqrt{1+x} \Rightarrow w^2 = 1+x \\ 2w\,dw = dx \end{matrix}\right] \to \int 2w\sqrt{1+w}\,dw$;

$\left[u = 2w,\ du = 2\,dw,\ dv = \sqrt{1+w}\,dw,\ v = \frac{2}{3}(1+w)^{3/2}\right]$

$\int 2w\sqrt{1+w}\,dw = \frac{4}{3}w(1+w)^{3/2} - \int \frac{4}{3}(1+w)^{3/2}\,dw = \frac{4}{3}w(1+w)^{3/2} - \frac{8}{15}(1+w)^{5/2} + C$

$= \frac{4}{3}\sqrt{1+x}\left(1+\sqrt{1+x}\right)^{3/2} - \frac{8}{15}\left(1+\sqrt{1+x}\right)^{5/2} + C$

105. $\int \frac{1}{\sqrt{x}\sqrt{1+x}}\,dx$; $\left[\begin{matrix} u=\sqrt{x} \Rightarrow u^2 = x \\ 2u\,du = dx \end{matrix}\right] \to \int \frac{2}{\sqrt{1+u^2}}\,du$; $\left[u = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ du = \sec^2\theta\,d\theta,\ \sqrt{1+u^2} = \sec\theta\right]$

$\int \frac{2}{\sqrt{1+u^2}}\,du = \int \frac{2\sec^2\theta}{\sec\theta}\,d\theta = \int 2\sec\theta\,d\theta = 2\ln|\sec\theta + \tan\theta| + C = 2\ln\left|\sqrt{1+u^2} + u\right| + C$

$= 2\ln\left|\sqrt{1+x} + \sqrt{x}\right| + C$

106. $\int_0^{1/2} \sqrt{1+\sqrt{1-x^2}}\,dx$;

$\left[x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ \sqrt{1-x^2} = \cos\theta,\ x = 0 = \sin\theta \Rightarrow \theta = 0,\ x = \frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}\right]$

$\to \int_0^{\pi/6} \sqrt{1+\cos\theta}\cos\theta\,d\theta = \int_0^{\pi/6} \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1-\cos\theta}}\cos\theta\,d\theta = \int_0^{\pi/6} \frac{\sin\theta\cos\theta}{\sqrt{1-\cos\theta}}\,d\theta = \lim_{c\to0^+}\int_c^{\pi/6} \frac{\sin\theta\cos\theta}{\sqrt{1-\cos\theta}}\,d\theta$;

$\left[u = \cos\theta,\ du = -\sin\theta\,d\theta,\ dv = \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,d\theta,\ v = 2(1-\cos\theta)^{1/2}\right]$

$= \lim_{c\to0^+}\left[\left[2\cos\theta\,(1-\cos\theta)^{1/2}\right]_c^{\pi/6} + \int_c^{\pi/6} 2(1-\cos\theta)^{1/2}\sin\theta\,d\theta\right]$

$= \lim_{c\to0^+}\left[\left(2\cos\left(\frac{\pi}{6}\right)\left(1-\cos\left(\frac{\pi}{6}\right)\right)^{1/2} - 2\cos c\,(1-\cos c)^{1/2}\right) + \left[\frac{4}{3}(1-\cos\theta)^{3/2}\right]_c^{\pi/6}\right]$

$= \lim_{c\to0^+}\left[\sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} - 2\cos c\,(1-\cos c)^{1/2} + \left(\frac{4}{3}\left(1-\cos\left(\frac{\pi}{6}\right)\right)^{3/2} - \frac{4}{3}(1-\cos c)^{3/2}\right)\right]$

$= \lim_{c\to0^+}\left[\sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} - 2\cos c\,(1-\cos c)^{1/2} + \frac{4}{3}\left(1-\frac{\sqrt{3}}{2}\right)^{3/2} - \frac{4}{3}(1-\cos c)^{3/2}\right]$

$= \sqrt{3}\left(1-\frac{\sqrt{3}}{2}\right)^{1/2} + \frac{4}{3}\left(1-\frac{\sqrt{3}}{2}\right)^{3/2} = \left(1-\frac{\sqrt{3}}{2}\right)^{1/2}\left(\frac{4+\sqrt{3}}{3}\right) = \frac{\left(4+\sqrt{3}\right)\sqrt{2-\sqrt{3}}}{3\sqrt{2}}$

107. $\int \frac{\ln x}{x + x\ln x}\,dx = \int \frac{\ln x}{x(1+\ln x)}\,dx$; $\left[\begin{matrix} u = 1+\ln x \\ du = \frac{1}{x}dx \end{matrix}\right] \to \int \frac{u-1}{u}\,du = \int du - \int \frac{1}{u}\,du = u - \ln|u| + C$

$= (1+\ln x) - \ln|1+\ln x| + C = \ln x - \ln|1+\ln x| + C$

108. $\int \frac{1}{x\ln x\cdot\ln(\ln x)}\,dx$; $\left[\begin{matrix} u = \ln(\ln x) \\ du = \frac{1}{x\ln x}dx \end{matrix}\right] \to \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(\ln x)| + C$

109. $\int \frac{x^{\ln x}\ln x}{x}dx$; $\left[u = x^{\ln x} \Rightarrow \ln u = \ln x^{\ln x} = (\ln x)^2 \Rightarrow \frac{1}{u}du = \frac{2\ln x}{x}\,dx \Rightarrow du = \frac{2u\ln x}{x}\,dx = \frac{2x^{\ln x}\ln x}{x}\,dx\right] \to \frac{1}{2}\int du$

$= \frac{1}{2}u + C = \frac{1}{2}x^{\ln x} + C$

110. $\int (\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx$; $\left[u = (\ln x)^{\ln x} \Rightarrow \ln u = \ln(\ln x)^{\ln x} = (\ln x)\ln(\ln x) \Rightarrow \frac{1}{u}du = \left(\frac{(\ln x)}{x\ln x} + \frac{\ln(\ln x)}{x}\right)dx\right.$

$\left.\Rightarrow du = u\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx = (\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]dx\right] \to \int du = u + C = (\ln x)^{\ln x} + C$

111. $\int \frac{1}{x\sqrt{1-x^4}}\,dx = \int \frac{x}{x^2\sqrt{1-x^4}}\,dx;\ \left[x^2 = \sin\theta,\ 0 \le \theta < \frac{\pi}{2},\ 2x\,dx = \cos\theta\,d\theta,\ \sqrt{1-x^4} = \cos\theta\right] \to \frac{1}{2}\int \frac{\cos\theta}{\sin\theta\cos\theta}\,d\theta$
$= \frac{1}{2}\int \csc\theta\,d\theta = -\frac{1}{2}\ln|\csc\theta + \cot\theta| + C = -\frac{1}{2}\ln\left|\frac{1}{x^2} + \frac{\sqrt{1-x^4}}{x^2}\right| + C = -\frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^4}}{x^2}\right| + C$

112. $\int \frac{\sqrt{1-x}}{x}\,dx;\ \left[u = \sqrt{1-x} \Rightarrow u^2 = 1-x \Rightarrow 2u\,du = -dx\right] \to \int \frac{-2u^2}{1-u^2}\,du = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right) du;$
$\frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} \Rightarrow 2 = A(u+1) + B(u-1) = (A+B)u + A - B \Rightarrow A + B = 0,\ A - B = 2$
$\Rightarrow A = 1 \Rightarrow B = -1;\ \int \left(2 + \frac{2}{u^2-1}\right) du = \int 2\,du + \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$
$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1-x} + \frac{1}{2}\ln\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| + C$

113. (a) $\int_0^a f(a-x)\,dx;\ \left[u = a - x \Rightarrow du = -dx,\ x = 0 \Rightarrow u = a,\ x = a \Rightarrow u = 0\right] \to -\int_a^0 f(u)\,du = \int_0^a f(u)\,du$, which is the same integral as $\int_0^a f(x)\,dx$.

(b) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x}{\sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x + \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x}\,dx$
$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx \Rightarrow 2\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} dx$
$= \left[x\right]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow 2\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$

114. $\int \frac{\sin x}{\sin x + \cos x}\,dx = \int \frac{\sin x + \cos x - \cos x + \sin x - \sin x}{\sin x + \cos x}\,dx = \int \frac{\sin x + \cos x}{\sin x + \cos x}\,dx + \int \frac{-\cos x + \sin x}{\sin x + \cos x}\,dx + \int \frac{-\sin x}{\sin x + \cos x}\,dx$
$= \int dx - \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx - \int \frac{\sin x}{\sin x + \cos x}\,dx = x - \ln|\sin x + \cos x| - \int \frac{\sin x}{\sin x + \cos x}\,dx$
$\Rightarrow 2\int \frac{\sin x}{\sin x + \cos x}\,dx = x - \ln|\sin x + \cos x| \Rightarrow \int \frac{\sin x}{\sin x + \cos x}\,dx = \frac{x}{2} - \frac{1}{2}\ln|\sin x + \cos x| + C$

115. $\int \frac{\sin^2 x}{1 + \sin^2 x}\,dx = \int \frac{\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}}\,dx = \int \frac{\tan^2 x}{\sec^2 x + \tan^2 x}\,dx = \int \frac{\tan^2 x + \sec^2 x - \sec^2 x}{\sec^2 x + \tan^2 x}\,dx = \int \frac{\tan^2 x + \sec^2 x}{\sec^2 x + \tan^2 x}\,dx - \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx$
$= \int dx - \int \frac{\sec^2 x}{1 + 2\tan^2 x}\,dx = x - \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$

116. $\int \frac{1 - \cos x}{1 + \cos x}\,dx = \int \frac{(1 - \cos x)^2}{1 - \cos^2 x}\,dx = \int \frac{1 - 2\cos x + \cos^2 x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx - \int \frac{2\cos x}{\sin^2 x}\,dx + \int \frac{\cos^2 x}{\sin^2 x}\,dx$
$= \int \csc^2 x\,dx - 2\int \csc x \cot x\,dx + \int \cot^2 x\,dx = -\cot x + 2\csc x + \int \left(\csc^2 x - 1\right)dx = \ = -2\cot x + 2\csc x - x + C$

CHAPTER 8 ADDITIONAL AND ADVANCED EXERCISES

1. $u = \left(\sin^{-1} x\right)^2,\ du = \frac{2\sin^{-1} x\,dx}{\sqrt{1-x^2}};\ dv = dx,\ v = x;$
$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 - \int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}};$
$u = \sin^{-1} x,\ du = \frac{dx}{\sqrt{1-x^2}};\ dv = -\frac{2x\,dx}{\sqrt{1-x^2}},\ v = 2\sqrt{1-x^2};$
$-\int \frac{2x\sin^{-1} x\,dx}{\sqrt{1-x^2}} = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - \int 2\,dx = 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C$; therefore
$\int \left(\sin^{-1} x\right)^2 dx = x\left(\sin^{-1} x\right)^2 + 2\left(\sin^{-1} x\right)\sqrt{1-x^2} - 2x + C$

2. $\frac{1}{x} = \frac{1}{x},$
$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1},$
$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)},$

$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,

$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)}$ $\Rightarrow$ the following pattern:

$\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$; therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

$= \sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln|x+k|\right] + C$

3. $u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = x\,dx$, $v = \frac{x^2}{2}$;

$\int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2\,dx}{2\sqrt{1-x^2}}$; $\begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{bmatrix} \rightarrow \int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{\sin^2\theta\cos\theta\,d\theta}{2\cos\theta}$

$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2}\sin^{-1} x + \frac{\sin\theta\cos\theta - \theta}{4} + C$

$= \frac{x^2}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$

4. $\int \sin^{-1}\sqrt{y}\,dy$; $\begin{bmatrix} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{bmatrix} \rightarrow \int 2z \sin^{-1} z\,dz$; from Exercise 3, $\int z \sin^{-1} z\,dz$

$= \frac{z^2 \sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C \Rightarrow \int \sin^{-1}\sqrt{y}\,dy = y \sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C$

$= y \sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$

5. $\int \frac{dt}{t - \sqrt{1-t^2}}$; $\begin{bmatrix} t = \sin\theta \\ dt = \cos\theta\,d\theta \end{bmatrix} \rightarrow \int \frac{\cos\theta\,d\theta}{\sin\theta - \cos\theta} = \int \frac{d\theta}{\tan\theta - 1}$; $\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \\ d\theta = \frac{du}{u^2+1} \end{bmatrix} \rightarrow \int \frac{du}{(u-1)(u^2+1)}$

$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{2}\int \frac{u\,du}{u^2+1} = \frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right| - \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\ln\left|\frac{\tan\theta - 1}{\sec\theta}\right| - \frac{1}{2}\theta + C$

$= \frac{1}{2}\ln\left(t - \sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1} t + C$

6. $\int \frac{1}{x^4+4}\,dx = \int \frac{1}{(x^2+2)^2 - 4x^2}\,dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}\,dx$

$= \frac{1}{16}\int \left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right] dx$

$= \frac{1}{16}\ln\left|\frac{x^2+2x+2}{x^2-2x+2}\right| + \frac{1}{8}\left[\tan^{-1}(x+1) + \tan^{-1}(x-1)\right] + C$

7. $\lim_{x\to\infty} \int_{-x}^{x} \sin t\,dt = \lim_{x\to\infty} \left[-\cos t\right]_{-x}^{x} = \lim_{x\to\infty} \left[-\cos x + \cos(-x)\right] = \lim_{x\to\infty} (-\cos x + \cos x) = \lim_{x\to\infty} 0 = 0$

8. $\lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$; $\lim_{t\to 0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} = \lim_{t\to 0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim_{x\to 0^+} \int_x^1 \frac{\cos t}{t^2}\,dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus

$\lim_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt$ is an indeterminate $0 \cdot \infty$ form and we apply l'Hôpital's rule:

$\lim_{x\to 0^+} x\int_x^1 \frac{\cos t}{t^2}\,dt = \lim_{x\to 0^+} \frac{-\int_1^x \frac{\cos t}{t^2}\,dt}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \cos x = 1$

9. $\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1+\frac{k}{n}} = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + k\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}\right) = \int_0^1 \ln(1+x)\,dx$; $\begin{bmatrix} u = 1+x,\ du = dx \\ x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 2 \end{bmatrix}$

$\rightarrow \int_1^2 \ln u\,du = \left[u \ln u - u\right]_1^2 = (2\ln 2 - 2) - (\ln 1 - 1) = 2\ln 2 - 1 = \ln 4 - 1$

10. $\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(\frac{n}{\sqrt{n^2-k^2}}\right)\left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(\frac{1}{\sqrt{1-\left[k\left(\frac{1}{n}\right)\right]^2}}\right)\left(\frac{1}{n}\right)$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2}$

11. $\frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x = 2\cos^2 x$; $L = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos 2t}\right)^2}\,dt = \sqrt{2}\int_0^{\pi/4} \sqrt{\cos^2 t}\,dt$

$= \sqrt{2}\,[\sin t]_0^{\pi/4} = 1$

12. $\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{(1-x^2)^2+4x^2}{(1-x^2)^2} = \frac{1+2x^2+x^4}{(1-x^2)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$; $L = \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_0^{1/2} \left(\frac{1+x^2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right) dx = \int_0^{1/2} \left(-1 + \frac{1}{1+x} + \frac{1}{1-x}\right) dx = \left[-x + \ln\left|\frac{1+x}{1-x}\right|\right]_0^{1/2}$

$= \left(-\frac{1}{2} + \ln 3\right) - (0 + \ln 1) = \ln 3 - \frac{1}{2}$

13. $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi xy\,dx$

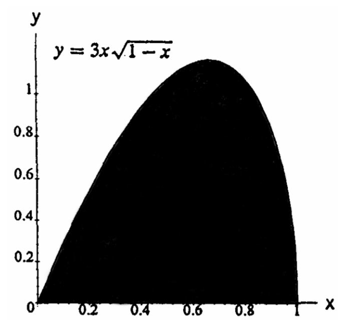
$= 6\pi \int_0^1 x^2\sqrt{1-x}\,dx$; $\begin{bmatrix} u = 1-x \\ du = -\,dx \\ x^2 = (1-u)^2 \end{bmatrix}$

$\rightarrow\ -6\pi \int_1^0 (1-u)^2\sqrt{u}\,du$

$= -6\pi \int_1^0 \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right) du$

$= -6\pi \left[\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_1^0 = 6\pi\left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right)$

$= 6\pi\left(\frac{70-84+30}{105}\right) = 6\pi\left(\frac{16}{105}\right) = \frac{32\pi}{35}$

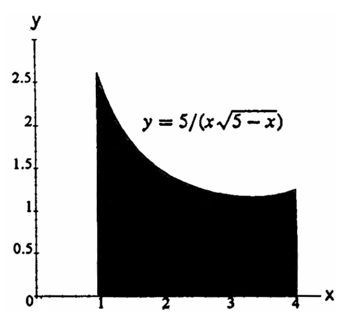


14. $V = \int_a^b \pi y^2\,dx = \pi \int_1^4 \frac{25\,dx}{x^2(5-x)}$

$= \pi \int_1^4 \left(\frac{dx}{x} + \frac{5\,dx}{x^2} + \frac{dx}{5-x}\right)$

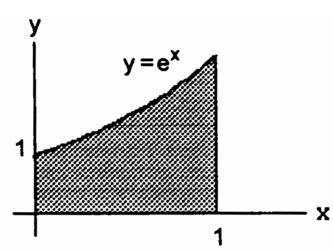
$= \pi\left[\ln\left|\frac{x}{5-x}\right| - \frac{5}{x}\right]_1^4 = \pi\left(\ln 4 - \frac{5}{4}\right) - \pi\left(\ln\frac{1}{4} - 5\right)$

$= \frac{15\pi}{4} + 2\pi\ln 4$

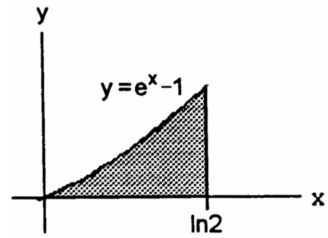


15. $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x e^x\,dx$

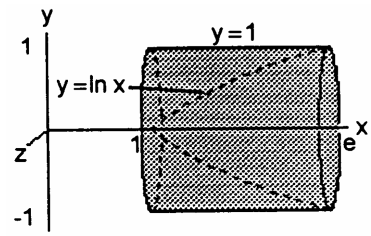
$= 2\pi\left[xe^x - e^x\right]_0^1 = 2\pi$



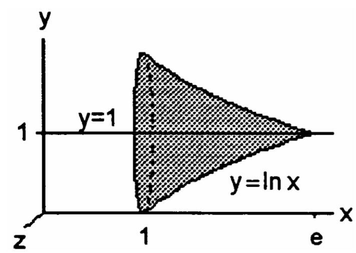
16. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)(e^x - 1)\,dx$
$= 2\pi \int_0^{\ln 2} [(\ln 2)\,e^x - \ln 2 - xe^x + x]\,dx$
$= 2\pi \left[(\ln 2)\,e^x - (\ln 2)x - xe^x + e^x + \frac{x^2}{2}\right]_0^{\ln 2}$
$= 2\pi \left[2\ln 2 - (\ln 2)^2 - 2\ln 2 + 2 + \frac{(\ln 2)^2}{2}\right] - 2\pi(\ln 2 + 1)$
$= 2\pi \left[-\frac{(\ln 2)^2}{2} - \ln 2 + 1\right]$



17. (a) $V = \int_1^e \pi\,[1 - (\ln x)^2]\,dx$
$= \pi\,[x - x(\ln x)^2]_1^e + 2\pi \int_1^e \ln x\,dx$
(FORMULA 110)
$= \pi\,[x - x(\ln x)^2 + 2(x\ln x - x)]_1^e$
$= \pi\,[-x - x(\ln x)^2 + 2x\ln x]_1^e$
$= \pi\,[-e - e + 2e - (-1)] = \pi$



(b) $V = \int_1^e \pi(1 - \ln x)^2\,dx = \pi \int_1^e [1 - 2\ln x + (\ln x)^2]\,dx$
$= \pi\,[x - 2(x\ln x - x) + x(\ln x)^2]_1^e - 2\pi \int_1^e \ln x\,dx$
$= \pi\,[x - 2(x\ln x - x) + x(\ln x)^2 - 2(x\ln x - x)]_1^e$
$= \pi\,[5x - 4x\ln x + x(\ln x)^2]_1^e$
$= \pi\,[(5e - 4e + e) - (5)] = \pi(2e - 5)$



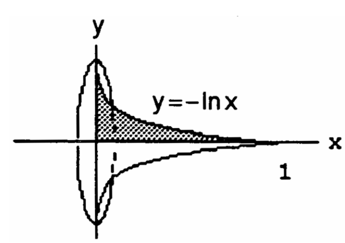
18. (a) $V = \pi \int_0^1 \left[(e^y)^2 - 1\right] dy = \pi \int_0^1 (e^{2y} - 1)\,dy = \pi\left[\frac{e^{2y}}{2} - y\right]_0^1 = \pi\left[\frac{e^2}{2} - 1 - \left(\frac{1}{2}\right)\right] = \frac{\pi\,(e^2 - 3)}{2}$

(b) $V = \pi \int_0^1 (e^y - 1)^2\,dy = \pi \int_0^1 (e^{2y} - 2e^y + 1)\,dy = \pi\left[\frac{e^{2y}}{2} - 2e^y + y\right]_0^1 = \pi\left[\left(\frac{e^2}{2} - 2e + 1\right) - \left(\frac{1}{2} - 2\right)\right]$
$= \pi\left(\frac{e^2}{2} - 2e + \frac{5}{2}\right) = \frac{\pi\,(e^2 - 4e + 5)}{2}$

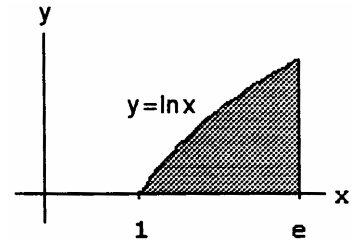
19. (a) $\lim_{x \to 0^+} x\ln x = 0 \Rightarrow \lim_{x \to 0^+} f(x) = 0 = f(0) \Rightarrow$ f is continuous

(b) $V = \int_0^2 \pi x^2(\ln x)^2\,dx;\ \begin{bmatrix} u = (\ln x)^2 \\ du = (2\ln x)\,\frac{dx}{x} \\ dv = x^2 dx \\ v = \frac{x^3}{3} \end{bmatrix} \to \pi\left(\lim_{b \to 0^+} \left[\frac{x^3}{3}(\ln x)^2\right]_b^2 - \int_0^2 \left(\frac{x^3}{3}\right)(2\ln x)\,\frac{dx}{x}\right)$
$= \pi\left[\left(\frac{8}{3}\right)(\ln 2)^2 - \left(\frac{2}{3}\right) \lim_{b \to 0^+} \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_b^2\right] = \pi\left[\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27}\right]$

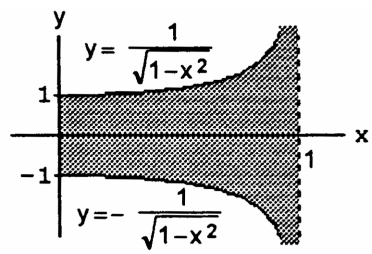
20. $V = \int_0^1 \pi(-\ln x)^2\,dx$
$= \pi\left(\lim_{b \to 0^+} [x(\ln x)^2]_b^1 - 2\int_0^1 \ln x\,dx\right)$
$= -2\pi \lim_{b \to 0^+} [x\ln x - x]_b^1 = 2\pi$



21. $M = \int_1^e \ln x\, dx = [x \ln x - x]_1^e = (e - e) - (0 - 1) = 1;$
$M_x = \int_1^e (\ln x)\left(\frac{\ln x}{2}\right) dx = \frac{1}{2}\int_1^e (\ln x)^2\, dx$
$= \frac{1}{2}\left([x(\ln x)^2]_1^e - 2\int_1^e \ln x\, dx\right) = \frac{1}{2}(e - 2);$
$M_y = \int_1^e x \ln x\, dx = \left[\frac{x^2 \ln x}{2}\right]_1^e - \frac{1}{2}\int_1^e x\, dx$
$= \frac{1}{2}\left[x^2 \ln x - \frac{x^2}{2}\right]_1^e = \frac{1}{2}\left[\left(e^2 - \frac{e^2}{2}\right) + \frac{1}{2}\right] = \frac{1}{4}(e^2 + 1);$
therefore, $\overline{x} = \frac{M_y}{M} = \frac{e^2+1}{4}$ and $\overline{y} = \frac{M_x}{M} = \frac{e-2}{2}$



22. $M = \int_0^1 \frac{2\, dx}{\sqrt{1 - x^2}} = 2\left[\sin^{-1} x\right]_0^1 = \pi;$
$M_y = \int_0^1 \frac{2x\, dx}{\sqrt{1 - x^2}} = 2\left[-\sqrt{1 - x^2}\right]_0^1 = 2;$
therefore, $\overline{x} = \frac{M_y}{M} = \frac{2}{\pi}$ and $\overline{y} = 0$ by symmetry



23. $L = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\, dx; \begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{bmatrix} \rightarrow L = \int_{\pi/4}^{\tan^{-1} e} \frac{\sec\theta \cdot \sec^2\theta\, d\theta}{\tan\theta}$
$= \int_{\pi/4}^{\tan^{-1} e} \frac{(\sec\theta)(\tan^2\theta + 1)}{\tan\theta}\, d\theta = \int_{\pi/4}^{\tan^{-1} e} (\tan\theta \sec\theta + \csc\theta)\, d\theta = [\sec\theta - \ln|\csc\theta + \cot\theta|]_{\pi/4}^{\tan^{-1} e}$
$= \left(\sqrt{1 + e^2} - \ln\left|\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right|\right) - \left[\sqrt{2} - \ln\left(1 + \sqrt{2}\right)\right] = \sqrt{1 + e^2} - \ln\left(\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$

24. $y = \ln x \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + x^2 \Rightarrow S = 2\pi\int_c^d x\sqrt{1 + x^2}\, dy \Rightarrow S = 2\pi\int_0^1 e^y\sqrt{1 + e^{2y}}\, dy; \begin{bmatrix} u = e^y \\ du = e^y\, dy \end{bmatrix}$
$\rightarrow S = 2\pi\int_1^e \sqrt{1 + u^2}\, du; \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \rightarrow 2\pi\int_{\pi/4}^{\tan^{-1} e} \sec\theta \cdot \sec^2\theta\, d\theta$
$= 2\pi\left(\frac{1}{2}\right)[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_{\pi/4}^{\tan^{-1} e} = \pi\left[\left(\sqrt{1 + e^2}\right)e + \ln\left|\sqrt{1 + e^2} + e\right|\right] - \pi\left[\sqrt{2} \cdot 1 + \ln\left(\sqrt{2} + 1\right)\right]$
$= \pi\left[e\sqrt{1 + e^2} + \ln\left(\frac{\sqrt{1+e^2}+e}{\sqrt{2}+1}\right) - \sqrt{2}\right]$

25. $S = 2\pi\int_{-1}^1 f(x)\sqrt{1 + [f'(x)]^2}\, dx; f(x) = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow [f'(x)]^2 + 1 = \frac{1}{x^{2/3}} \Rightarrow S = 2\pi\int_{-1}^1 \left(1 - x^{2/3}\right)^{3/2} \cdot \frac{dx}{\sqrt{x^{2/3}}}$
$= 4\pi\int_0^1 \left(1 - x^{2/3}\right)^{3/2}\left(\frac{1}{x^{1/3}}\right) dx; \begin{bmatrix} u = x^{2/3} \\ du = \frac{2}{3}\frac{dx}{x^{1/3}} \end{bmatrix} \rightarrow 4 \cdot \frac{3}{2}\pi\int_0^1 (1 - u)^{3/2}\, du = -6\pi\int_0^1 (1 - u)^{3/2}\, d(1 - u)$
$= -6\pi \cdot \frac{2}{5}\left[(1 - u)^{5/2}\right]_0^1 = \frac{12\pi}{5}$

26. $y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \Rightarrow \frac{dy}{dx} = \sqrt{\sqrt{x} - 1} \Rightarrow L = \int_1^{16}\sqrt{1 + \left(\sqrt{\sqrt{x} - 1}\right)^2}\, dx = \int_1^{16}\sqrt{1 + \sqrt{x} - 1}\, dx$
$= \int_1^{16} \sqrt[4]{x}\, dx = \left[\frac{4}{5}x^{5/4}\right]_1^{16} = \frac{4}{5}(16)^{5/4} - \frac{4}{5}(1)^{5/4} = \frac{124}{5}$

27. $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) dx = \lim_{b \to \infty} \int_1^b \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) dx = \lim_{b \to \infty} \left[\frac{a}{2}\ln(x^2 + 1) - \frac{1}{2}\ln x\right]_1^b = \lim_{b \to \infty} \left[\frac{1}{2}\ln\frac{(x^2+1)^a}{x}\right]_1^b$
$= \lim_{b \to \infty} \frac{1}{2}\left[\ln\frac{(b^2+1)^a}{b} - \ln 2^a\right]; \lim_{b \to \infty} \frac{(b^2+1)^a}{b} > \lim_{b \to \infty} \frac{b^{2a}}{b} = \lim_{b \to \infty} b^{2\left(a - \frac{1}{2}\right)} = \infty$ if $a > \frac{1}{2} \Rightarrow$ the improper
integral diverges if $a > \frac{1}{2}$; for $a = \frac{1}{2}$: $\lim_{b \to \infty} \frac{\sqrt{b^2+1}}{b} = \lim_{b \to \infty} \sqrt{1 + \frac{1}{b^2}} = 1 \Rightarrow \lim_{b \to \infty} \frac{1}{2}\left[\ln\frac{(b^2+1)^{1/2}}{b} - \ln 2^{1/2}\right]$

$= \frac{1}{2}\left(\ln 1 - \frac{1}{2}\ln 2\right) = -\frac{\ln 2}{4}$; if $a < \frac{1}{2}$: $0 \le \lim\limits_{b\to\infty} \frac{(b^2+1)^a}{b} < \lim\limits_{b\to\infty} \frac{(b+1)^{2a}}{b+1} = \lim\limits_{b\to\infty} (b+1)^{2a-1} = 0$

$\Rightarrow \lim\limits_{b\to\infty} \ln \frac{(b^2+1)^a}{b} = -\infty \Rightarrow$ the improper integral diverges if $a < \frac{1}{2}$; in summary, the improper integral $\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right) dx$ converges only when $a = \frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

28. $G(x) = \lim\limits_{b\to\infty} \int_0^b e^{-xt}\,dt = \lim\limits_{b\to\infty} \left[-\frac{1}{x}e^{-xt}\right]_0^b = \lim\limits_{b\to\infty} \left(\frac{1-e^{-xb}}{x}\right) = \frac{1-0}{x} = \frac{1}{x}$ if $x > 0 \Rightarrow xG(x) = x\left(\frac{1}{x}\right) = 1$ if $x > 0$

29. $A = \int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$. Thus, $p \le 1$ for infinite area. The volume of the solid of revolution about the x-axis is $V = \int_1^\infty \pi\left(\frac{1}{x^p}\right)^2 dx = \pi \int_1^\infty \frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \le 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \le 1$.

30. The area is given by the integral $A = \int_0^1 \frac{dx}{x^p}$;

$p = 1$: $A = \lim\limits_{b\to 0^+} [\ln x]_b^1 = -\lim\limits_{b\to 0^+} \ln b = \infty$, diverges;

$p > 1$: $A = \lim\limits_{b\to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim\limits_{b\to 0^+} b^{1-p} = -\infty$, diverges;

$p < 1$: $A = \lim\limits_{b\to 0^+} \left[x^{1-p}\right]_b^1 = 1 - \lim\limits_{b\to 0^+} b^{1-p} = 1 - 0$, converges; thus, $p \ge 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V_x = \pi\int_0^1 \frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \ge \frac{1}{2}$. Thus, V_x is infinite whenever the area is infinite ($p \ge 1$).

The volume of the solid of revolution about the y-axis is $V_y = \pi\int_1^\infty [R(y)]^2\,dy = \pi\int_1^\infty \frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 29). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \le p < 2$, as described above.

31. (a) $\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim\limits_{b\to\infty} \int_0^b e^{-t}\,dt = \lim\limits_{b\to\infty} \left[-e^{-t}\right]_0^b = \lim\limits_{b\to\infty} \left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1$

(b) $u = t^x$, $du = xt^{x-1}\,dt$; $dv = e^{-t}\,dt$, $v = -e^{-t}$; $x =$ fixed positive real

$\Rightarrow \Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \lim\limits_{b\to\infty} \left[-t^x e^{-t}\right]_0^b + x\int_0^\infty t^{x-1}e^{-t}\,dt = \lim\limits_{b\to\infty} \left(-\frac{b^x}{e^b} + 0^x e^0\right) + x\Gamma(x) = x\Gamma(x)$

(c) $\Gamma(n+1) = n\Gamma(n) = n!$:

$n = 0$: $\Gamma(0+1) = \Gamma(1) = 0!$;		
$n = k$: Assume $\Gamma(k+1) = k!$		for some $k > 0$;
$n = k+1$: $\Gamma(k+1+1)$	$= (k+1)\,\Gamma(k+1)$	from part (b)
	$= (k+1)k!$	induction hypothesis
	$= (k+1)!$	definition of factorial

Thus, $\Gamma(n+1) = n\Gamma(n) = n!$ for every positive integer n.

32. (a) $\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}$ and $n\Gamma(n) = n! \Rightarrow n! \approx n\left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$

(b)

n	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}$	calculator
10	3598695.619	3628800
20	2.4227868×10^{18}	2.432902×10^{18}
30	2.6451710×10^{32}	2.652528×10^{32}
40	8.1421726×10^{47}	8.1591528×10^{47}
50	3.0363446×10^{64}	3.0414093×10^{64}
60	8.3094383×10^{81}	8.3209871×10^{81}

(c)

n	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}$	$\left(\frac{n}{e}\right)^n\sqrt{2n\pi}\, e^{1/12n}$	calculator
10	3598695.619	3628810.051	3628800

33.

e^{2x}	$(+)$	$\cos 3x$
$2e^{2x}$	$(-)$	$\frac{1}{3}\sin 3x$
$4e^{2x}$	$(+)$	$-\frac{1}{9}\cos 3x$

$I = \frac{e^{2x}}{3}\sin 3x + \frac{2e^{2x}}{9}\cos 3x - \frac{4}{9}I \Rightarrow \frac{13}{9}I = \frac{e^{2x}}{9}(3\sin 3x + 2\cos 3x) \Rightarrow I = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$

34.

e^{3x}	$(+)$	$\sin 4x$
$3e^{3x}$	$(-)$	$-\frac{1}{4}\cos 4x$
$9e^{3x}$	$(+)$	$-\frac{1}{16}\sin 4x$

$I = -\frac{e^{3x}}{4}\cos 4x + \frac{3e^{3x}}{16}\sin 4x - \frac{9}{16}I \Rightarrow \frac{25}{16}I = \frac{e^{3x}}{16}(3\sin 4x - 4\cos 4x) \Rightarrow I = \frac{e^{3x}}{25}(3\sin 4x - 4\cos 4x) + C$

35.

$\sin 3x$	$(+)$	$\sin x$
$3\cos 3x$	$(-)$	$-\cos x$
$-9\sin 3x$	$(+)$	$-\sin x$

$I = -\sin 3x\cos x + 3\cos 3x\sin x + 9I \Rightarrow -8I = -\sin 3x\cos x + 3\cos 3x\sin x$

$\Rightarrow I = \frac{\sin 3x\cos x - 3\cos 3x\sin x}{8} + C$

36.

$\cos 5x$	$(+)$	$\sin 4x$
$-\sin 5x$	$(-)$	$-\frac{1}{4}\cos 4x$
$-25\cos 5x$	$(+)$	$-\frac{1}{16}\sin 4$

$I = -\frac{1}{4}\cos 5x\cos 4x - \frac{5}{16}\sin 5x\sin 4x + \frac{25}{16}I \Rightarrow -\frac{9}{16}I = -\frac{1}{4}\cos 5x\cos 4x - \frac{5}{16}\sin 5x\sin 4x$

$\Rightarrow I = \frac{1}{9}(4\cos 5x\cos 4x + 5\sin 5x\sin 4x) + C$

37.

e^{ax}	$(+)$	$\sin bx$
ae^{ax}	$(-)$	$-\frac{1}{b}\cos bx$
a^2e^{ax}	$(+)$	$-\frac{1}{b^2}\sin bx$

$I = -\frac{e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2}I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

38.

e^{ax}	$(+)$	$\cos bx$
ae^{ax}	$(-)$	$\frac{1}{b}\sin bx$
a^2e^{ax}	$(+)$	$-\frac{1}{b^2}\cos bx$

$I = \frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a\cos bx + b\sin bx)$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

39.

$\ln(ax)$	$(+)$	1
$\frac{1}{x}$	$(-)$	x

$I = x\ln(ax) - \int\left(\frac{1}{x}\right)x\,dx = x\ln(ax) - x + C$

40.

$\ln(ax)$	$(+)$	x^2
$\frac{1}{x}$	$(-)$	$\frac{1}{3}x^3$

$I = \frac{1}{3}x^3\ln(ax) - \int\left(\frac{1}{x}\right)\left(\frac{x^3}{3}\right)dx = \frac{1}{3}x^3\ln(ax) - \frac{1}{9}x^3 + C$

41. $\int \frac{dx}{1-\sin x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{2z}{1+z^2}\right)} = \int \frac{2\,dz}{(1-z)^2} = \frac{2}{1-z} + C = \frac{2}{1-\tan\left(\frac{x}{2}\right)} + C$

42. $\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1+z^2+2z+1-z^2} = \int \frac{dz}{1+z} = \ln|1+z| + C$

$= \ln\left|\tan\left(\frac{x}{2}\right)+1\right| + C$

43. $\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{(1+z)^2} = -\left[\frac{2}{1+z}\right]_0^1 = -(1-2) = 1$

44. $\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x} = \int_{1/\sqrt{3}}^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int_{1/\sqrt{3}}^1 \frac{dz}{z^2} = \left[-\frac{1}{z}\right]_{1/\sqrt{3}}^1 = \sqrt{3}-1$

45. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2\,dz}{z^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$

$= \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$

46. $\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta+\sin\theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z(1-z^2)}{(1+z^2)^2}+\left(\frac{2z}{1+z^2}\right)\right]} = \int_1^{\sqrt{3}} \frac{2(1-z^2)\,dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z}\,dz$

$= \left[\frac{1}{2}\ln z - \frac{z^2}{4}\right]_1^{\sqrt{3}} = \left(\frac{1}{2}\ln\sqrt{3} - \frac{3}{4}\right) - \left(0-\frac{1}{4}\right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4}(\ln 3 - 2) = \frac{1}{2}\left(\ln\sqrt{3} - 1\right)$

47. $\int \frac{dt}{\sin t-\cos t} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{2z-1+z^2} = \int \frac{2\,dz}{(z+1)^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + C$

$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}}\right| + C$

48. $\int \frac{\cos t\,dt}{1-\cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)^2-\left(1+z^2\right)\left(1-z^2\right)} = \int \frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)\left(1+z^2-1+z^2\right)}$

$= \int \frac{\left(1-z^2\right)dz}{\left(1+z^2\right)z^2} = \int \frac{dz}{z^2\left(1+z^2\right)} - \int \frac{dz}{1+z^2} = \int \frac{dz}{z^2} - 2\int \frac{dz}{z^2+1} = -\frac{1}{z} - 2\tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C$

49. $\int \sec\theta\,d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1-z^2} = \int \frac{2\,dz}{(1+z)(1-z)} = \int \frac{dz}{1+z} + \int \frac{dz}{1-z}$

$= \ln|1+z| - \ln|1-z| + C = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$

50. $\int \csc\theta\,d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}\right)} = \int \frac{dz}{z} = \ln|z| + C = \ln\left|\tan\frac{\theta}{2}\right| + C$