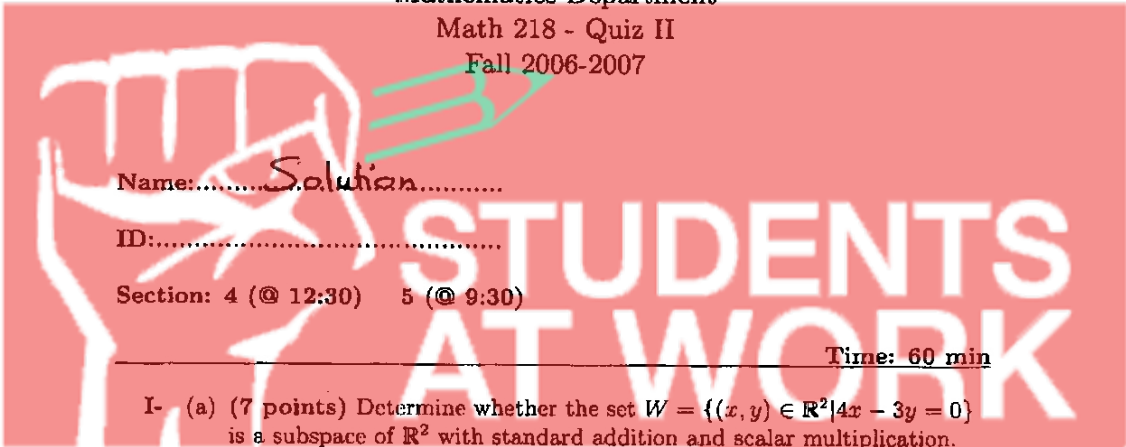


AMERICAN UNIVERSITY OF BEIRUT

Mathematics Department

Math 218 - Quiz II

Fall 2006-2007



Name:.....Solution.....

ID:.....

Section: 4 (@ 12:30) 5 (@ 9:30)

Time: 60 min

I- (a) (7 points) Determine whether the set  $W = \{(x, y) \in \mathbb{R}^2 \mid 4x - 3y = 0\}$  is a subspace of  $\mathbb{R}^2$  with standard addition and scalar multiplication.

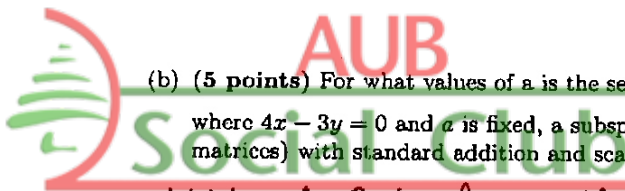
Let  $u = (x, y) \in W \quad 4x - 3y = 0$  ... Together At Work  
 $v = (x', y') \in W \quad 4x' - 3y' = 0$

①  $u+v = (x+x', y+y')$   $4(x+x') - 3(y+y')$   
 $= 4x - 3y + 4x' - 3y' = 0 + 0 = 0$

so  $u+v \in W$

②  $ku = (kx, ky) \quad 4(kx) - 3(ky) = k(4x - 3y) = 0$

so  $ku \in W$



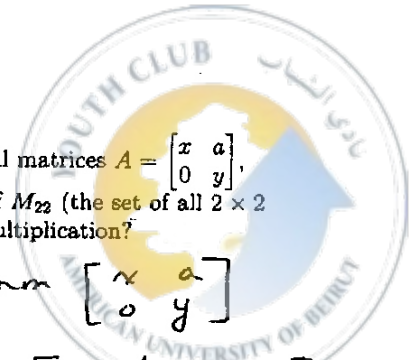
(b) (5 points) For what values of  $a$  is the set of all matrices  $A = \begin{bmatrix} x & a \\ 0 & y \end{bmatrix}$ , where  $4x - 3y = 0$  and  $a$  is fixed, a subspace of  $M_{22}$  (the set of all  $2 \times 2$  matrices) with standard addition and scalar multiplication?

Want  $A+B$  to have the form  $\begin{bmatrix} x & a \\ 0 & y \end{bmatrix}$   
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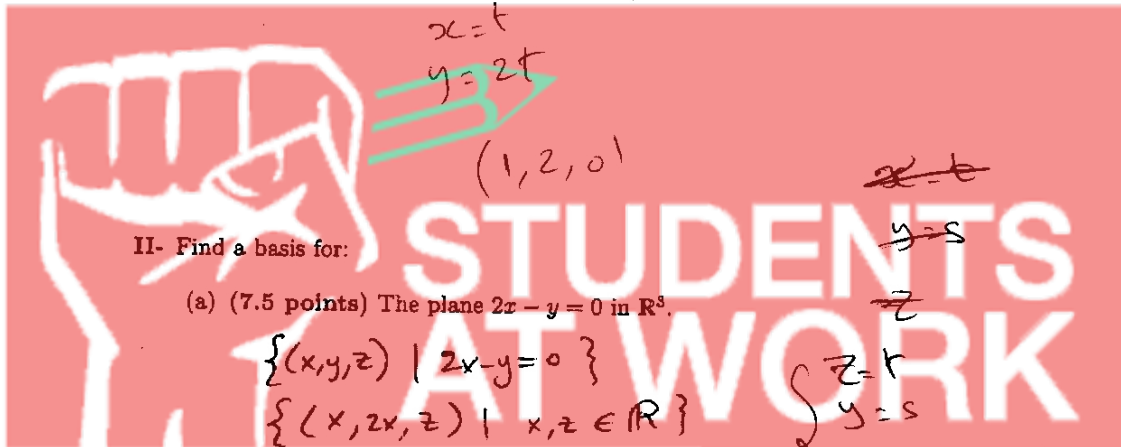
$$\begin{bmatrix} x & a \\ 0 & y \end{bmatrix} + \begin{bmatrix} x' & a \\ 0 & y' \end{bmatrix} = \begin{bmatrix} x+x' & a+a \\ 0 & y+y' \end{bmatrix}$$

so  $a+a = a$ , i.e.  $a=0$

(Note this is the only thing we need to check because from part (a) satisfy the above properties)



(Continue your answer here)



II- Find a basis for:

(a) (7.5 points) The plane  $2x - y = 0$  in  $\mathbb{R}^3$ .

$$\{(x, y, z) \mid 2x - y = 0\}$$

$$\{(x, 2x, z) \mid x, z \in \mathbb{R}\}$$

$$\{x(1, 2, 0) + z(0, 0, 1)\}$$

~~$$x=t$$~~  

$$y=s$$
  

$$z=t$$

$$x = \frac{1}{2}s$$
  

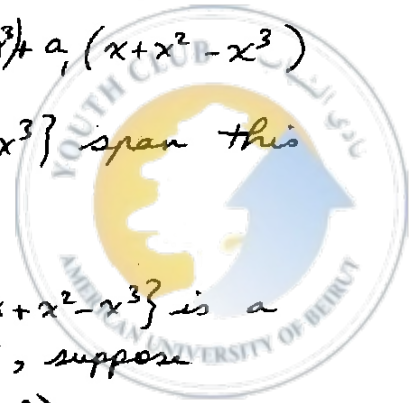
$$(x, y, z) = (x, s, t)$$

Therefore,  $\{(1, 2, 0), (0, 0, 1)\}$  is a basis since  $(\frac{1}{2}s, s, t)$  it is clearly a linearly independent set that spans the plane.

(b) (7.5 points) The subspace of  $P_3$  consisting of all polynomials of the form  $a_0 + a_1x + a_2x^2 + a_3x^3$  for which  $a_2 = a_0 + a_1$  and  $a_3 = a_0 - a_1$ .

$$a_0 + a_1x + a_2x^2 + a_3x^3 = a_0 + a_1x + (a_0 + a_1)x^2 + (a_0 - a_1)x^3$$

$$= a_0(1 + x^2 + x^3) + a_1(x + x^2 - x^3)$$



Therefore  $\{1 + x^2 + x^3, x + x^2 - x^3\}$  span this subspace.

To check that  $\{1 + x^2 + x^3, x + x^2 - x^3\}$  is a linearly independent set, suppose

$$k_1(1 + x^2 + x^3) + k_2(x + x^2 - x^3) = 0$$

$$k_1 + k_2x + (k_1 + k_2)x^2 + (k_1 - k_2)x^3 = 0$$

$$k_1 = 0$$

$$k_2 = 0$$

$$k_1 + k_2 = 0$$

$$k_1 - k_2 = 0 \text{ so } k_1 = k_2 = 0$$

So  $\{1 + x^2 + x^3, x + x^2 - x^3\}$  is a basis.

not included

III- (8 points) Find a quadratic polynomial interpolant to the data  $(-1, 2)$ ,  $(0, 0)$ , and  $(1, 2)$ . (Use either Vandermonde system approach or Newton approach)

$$\begin{matrix} 4 \\ 3 \end{matrix}
 \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}
 \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}
 =
 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}
 \rightarrow
 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & 0 & 0 \end{bmatrix}
 \rightarrow
 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ . Therefore,  $a_2 = 2$ . Together At Work  
 $a_1 = 0$   $a_0 = 0$   
 so the polynomial is  $2x^2$ .  
 ①

IV- (a) (8 points) Given

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -2 \\ 3 & 0 & 18 \\ 0 & -5 & 10 \end{bmatrix}$$

Find a basis for the nullspace of  $A$ .

$$\begin{matrix} 2 \\ 3 \end{matrix}
 \left[ \begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 0 & 18 & 0 \\ 0 & -5 & 10 & 0 \end{array} \right]
 \rightarrow
 \left[ \begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & -5 & 10 & 0 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

so

$$\begin{aligned}
 x_3 &= t \\
 x_2 &= 2t \\
 x_1 &= -x_2 - 4x_3 \\
 &= -2t - 4t \\
 &= -6t
 \end{aligned}$$

$$\begin{aligned}
 \mathfrak{B}(x_1, x_2, x_3) &= (-6t, 2t, t) \\
 &= t(-6, 2, 1)
 \end{aligned}$$

3 so  $\{(-6, 2, 1)\}$  is a basis

(b) (5 points) Find a basis for the row space of  $A$  that consists entirely of row vectors of  $A$ .

Take  $A^T = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & -5 \\ 4 & -2 & 18 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & -2 & 6 & 10 \end{bmatrix}$

2  $\rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$  so  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$  is a basis for the column space of  $A^T$

Therefore,  $\{ [1 \ 1 \ 4], [0 \ 1 \ -2] \}$  is a basis for the row space of  $A$ .

(c) (5 points) Define  $\text{nullity}(A)$  and  $\text{rank}(A)$  then find both of them.

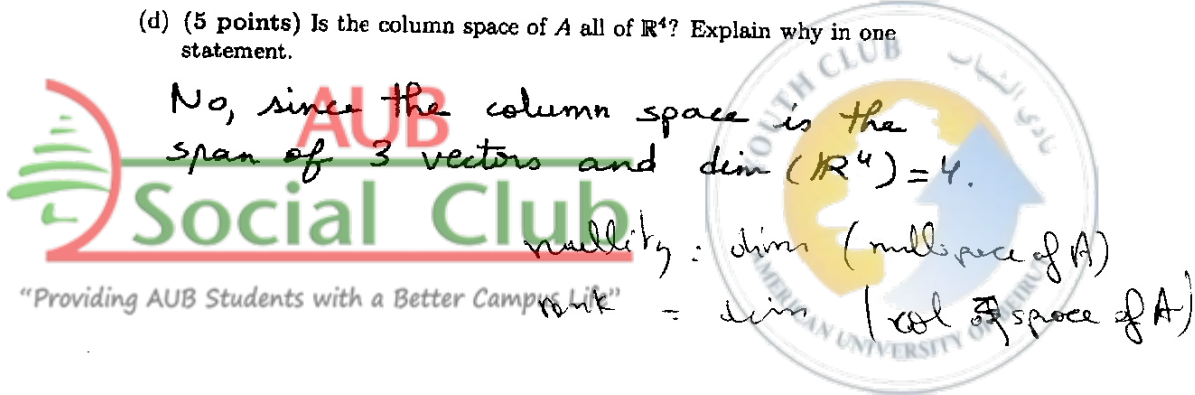
1  $\text{nullity}(A) = \text{dimension of nullspace of } A$   
 1  $\text{rank}(A) = \text{dimension of row space of } A$   
 = dimension of column space of  $A$

1  $\text{nullity}(A) = 1$       1  $\text{rank}(A) = 2$

(d) (5 points) Is the column space of  $A$  all of  $\mathbb{R}^4$ ? Explain why in one statement.

No, since the column space is the span of 3 vectors and  $\dim(\mathbb{R}^4) = 4$ .

$\text{nullity} = \dim(\text{nullspace of } A)$   
 $\text{rank} = \dim(\text{col space of } A)$



V- (10 points) Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$  be a linear transformation such that

$$T(1, 3, 0) = (2, 5, 7, -1),$$

$$T(3, 1, 1) = (4, 1, 3, 2),$$

$$T(3, 0, 1) = (5, 2, -2, 3).$$

Find the standard matrix for  $T$ .

$$T(1, 3, 0) = b_1$$

$$(1, 3, 0) = e_1 + 3e_2 + 0$$

$$\text{so } T(e_1) + 3T(e_2) = b_1$$

$$T(1, 3, 0) = b_1$$

$$(1, 3, 0) = e_1 + 3e_2$$

$$\text{so } T(e_1) + 3T(e_2) = b_1$$

$$T(3, 1, 1) = b_2$$

$$(3, 1, 1) = 3e_1 + e_2 + e_3$$

$$\text{so } 3T(e_1) + T(e_2) + T(e_3) = b_2$$

$$T(3, 0, 1) = b_3$$

$$(3, 0, 1) = 3e_1 + e_3$$

$$\text{so } 3T(e_1) + T(e_3) = b_3$$

Solve the system,

$$\left[ \begin{array}{ccc|c} 1 & 3 & 0 & b_1 \\ 3 & 1 & 1 & b_2 \\ 3 & 0 & 1 & b_3 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 3 & 0 & b_1 \\ 0 & -8 & 1 & -3b_1 + b_2 \\ 0 & -9 & 1 & -3b_1 + b_3 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 3 & 0 & b_1 \\ 0 & 1 & 0 & b_2 - b_3 \\ 0 & 0 & 1 & 9b_1 - 8b_3 - 3b_2 \end{array} \right]$$

$$T(e_3) = 9b_2 - 8b_3 - 3b_1 = (36, 9, 9, 18) + (-40, -16, 16, -24)$$

$$+ (-6, -15, -21, 3)$$

$$= (-10, -22, 22, -3)$$

$$T(e_3) = b_2 - b_3 = (4, 1, 3, 2) - (5, 2, -2, 3) = (-1, -1, 5, -1)$$

$$T(e_1) = b_1 - 3T(e_2) = (2, 5, 7, -1) - 3(-1, -1, 5, -1)$$

$$= (2, 5, 7, -1) + (3, 3, -15, 3)$$

$$= (5, 8, -8, 2)$$

Therefore the standard matrix is

$$\begin{bmatrix} 5 & -1 & -10 \\ 8 & -1 & -22 \\ -8 & 5 & 22 \\ 2 & -1 & -3 \end{bmatrix}$$



The previous method is the standard way for such questions but here there is a much easier way

$$(0,1,0) = (3,1,1) - (3,0,1)$$

$$T(0,1,0) = T((3,1,1) - (3,0,1)) = T(3,1,1) - T(3,0,1) \\ = (4,1,3,2) - (5,2,-2,3) = (-1,-1,5,-1)$$

$$(1,0,0) = (1,3,0) - 3(0,1,0)$$

$$T(1,0,0) = T(1,3,0) - 3T(0,1,0) = (2,5,7,-1) - 3(-1,-1,5,-1) \\ = (5,8,-8,2)$$

...Together At Work

$$(0,0,1) = (3,0,1) - 3(1,0,0)$$

$$T(0,0,1) = T(3,0,1) - 3T(1,0,0) = (5,2,-2,3) - 3(5,8,-8,2) \\ = (-10,-22,22,-3)$$

Therefore the standard matrix is

$$\begin{bmatrix} 5 & -1 & -10 \\ 8 & -1 & -22 \\ -8 & 5 & 22 \\ 2 & -1 & -3 \end{bmatrix}$$


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VI- DO NOT PROVE ANY OF YOUR ANSWERS IN THIS QUESTION UNLESS YOU ARE ASKED TO DO SO.

a- (24 points) Write TRUE or FALSE.

- 1-  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation if and only if  $T(ku + v) = kT(u) + T(v)$  for all scalars  $k$  and vectors  $u$  and  $v$  in  $\mathbb{R}^n$ . **T**
- 2- If the row vectors and the column vectors of a matrix  $A$  are linearly independent then  $A$  is square. **T**
- 3- For  $n \times n$  invertible matrices  $A$  and  $B$ ,  $\text{nullity}(A+B) = \text{nullity}(A) + \text{nullity}(B)$ . **F**
- 4- If  $\{v_1, v_2, \dots, v_n\}$  is linearly independent and a matrix  $A$  is nonzero then  $\{Av_1, Av_2, \dots, Av_n\}$  is linearly independent (provided that such multiplication is defined). **F**
- 5- If a set  $\{v_1, v_2, \dots, v_n\}$  spans a vector space  $V$  with dimension  $m$  then  $n \geq m$ . **T**
- 6- If  $W$  is a subspace of a vector space  $V$  and  $S$  is a subspace of  $W$ , then  $S$  is a subspace of  $V$ . **T**
- 7-  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T(x, y) = (-x, 2x + 1)$  is a linear transformation. **F**
- 8- If  $A$  is a  $4 \times 5$  matrix then the largest value for  $\text{rank}(A)$  is 4. **T**

b- (8 points) If  $A$  is an  $n \times n$  matrix. For a nonzero scalar  $k$ , is  $\text{rank}(kA) = k \cdot \text{rank}(A)$ ? If yes, prove this. If no, give a counter example.

No

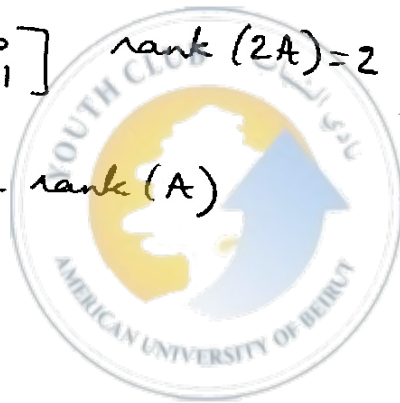
Take  $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$        $\text{rank}(A) = 2$

$2A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$        $\text{rank}(2A) = 2$

Therefore  $\text{rank}(2A) \neq 2 \cdot \text{rank}(A)$



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GOOD LUCK