AMERICAN UNIVERSITY OF BEIRUT

Mathematics Department

Math 218 - Quiz II

Fall 2006-2007

Name:.........Solution..........

ID:..

Section: 4 (@ 12:30)   5 (@ 9:30)

Time: 60 min

I- (a) (7 points) Determine whether the set $W = \{(x, y) \in \mathbb{R}^2 | 4x - 3y = 0\}$ is a subspace of $\mathbb{R}^2$ with standard addition and scalar multiplication.

Let $u = (x,y) \in W$ $\quad 4x - 3y = 0$

$v = (x', y') \in W$ $\quad 4x' - 3y' = 0$

① $u + v = (x + x', y + y')$ $\quad 4(x+x') - 3(y+y')$

$= 4x - 3y + 4x' - 3y' = 0 + 0 = 0$

$\boxed{\text{so } u+v \in W}$

② $ku = (kx, ky)$ $\quad 4(kx) - 3(ky) = k(4x - 3y) = 0$

$\boxed{\text{so } ku \in W}$

(b) (5 points) For what values of a is the set of all matrices $A = \begin{bmatrix} x & a \\ 0 & y \end{bmatrix}$, where $4x - 3y = 0$ and a is fixed, a subspace of M_{22} (the set of all 2×2 matrices) with standard addition and scalar multiplication?

Want $A+B$ to have the form $\begin{bmatrix} x & a \\ 0 & y \end{bmatrix}$

$$\begin{bmatrix} x & a \\ 0 & y \end{bmatrix} + \begin{bmatrix} x' & a \\ 0 & y' \end{bmatrix} = \begin{bmatrix} x+x' & a+a \\ 0 & y+y' \end{bmatrix}$$

so $a + a = a$, i.e $\boxed{a = 0}$

(Note this is the only thing we need to check because from part (a) satisfy the above properties)

(Continue your answer here)

$x=t$
$y=2t$
$(1,2,0)$

~~$x=t$~~
~~$y=s$~~
~~z~~

II- Find a basis for:

(a) **(7.5 points)** The plane $2x-y=0$ in $\mathbf{R}^3$.

$$\{(x,y,z) \mid 2x-y=0\}$$

$$\{(x,2x,z) \mid x,z\in\mathbb{R}\}$$

$$\{x(1,2,0)+z(0,0,1)\}$$

$z=t$
$y=s$
$x=\frac{1}{2}s$

$(x,y,z)=(x,s,t)$

$(\frac{1}{2}s,s,t)$

$s(\frac{1}{2},1,0)+t(0,0,1)$

Therefore, $\{(1,2,0),(0,0,1)\}$ is a basis since it is clearly a linearly independent set that spans the plane.

(b) **(7.5 points)** The subspace of P_3 consisting of all polynomials of the form $a_0+a_1x+a_2x^2+a_3x^3$ for which $a_2=a_0+a_1$ and $a_3=a_0-a_1$.

$$a_0+a_1x+a_2x^2+a_3x^3 = a_0+a_1x+(a_0+a_1)x^2+(a_0-a_1)x^3$$

$$= a_0(1+x^2+x^3)+a_1(x+x^2-x^3)$$

Therefore $\{1+x^2+x^3,\ x+x^2-x^3\}$ span this subspace.

To check that $\{1+x^2+x^3,\ x+x^2-x^3\}$ is a linearly independent set, suppose

$$k_1(1+x^2+x^3)+k_2(x+x^2-x^3)=0$$

$$k_1+k_2x+(k_1+k_2)x^2+(k_1-k_2)x^3=0$$

$k_1=0$
$k_2=0$
$k_1+k_2=0$
$k_1-k_2=0$ so $k_1=k_2=0$

So $\{1+x^2+x^3,\ x+x^2-x^3\}$ is a basis.

not included

III- **(8 points)** Find a quadratic polynomial interpolant to the data $(-1, 2)$, $(0, 0)$, and $(1, 2)$. (Use either Vandermonde system approach or Newton approach)

4

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

3

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Therefore, $a_2 = 2$ $a_1 = 0$ $a_0 = 0$

so the polynomial is $2x^2$.

1

IV- (a) **(8 points)** Given

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -2 \\ 3 & 0 & 18 \\ 0 & -5 & 10 \end{bmatrix}$$

Find a basis for the nullspace of A.

2

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 0 & 18 & 0 \\ 0 & -5 & 10 & 0 \end{array}\right] \to \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & -5 & 10 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = t$

$x_2 = 2t$

$x_1 = -x_2 - 4x_3$

$= -2t - 4t$

$= -6t$

3 $(x_1, x_2, x_3) = (-6t, 2t, t)$

$= t(-6, 2, 1)$

3 so $\{(-6, 2, 1)\}$ is a basis

(b) **(5 points)** Find a basis for the row space of A that consists entirely of row vectors of A.

Take $A^T = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & -5 \\ 4 & -2 & 18 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & -2 & 6 & 10 \end{bmatrix}$

2 $\to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $\left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a bas for the column space of A^T

Therefore, $\{ [1 \;\; 1 \;\; 4], [0 \;\; 1 \;\; -2] \}$ is a basis for the row space of A.
2

(c) **(5 points)** Define $nullity(A)$ and $rank(A)$ then find both of them.

1½ nullity (A) = dimension of nullspace of A
1½ rank (A) = dimension of row space of A
= dimension of column space of A

1 nullity (A) = 1   1 rank (A) = 2

(d) **(5 points)** Is the column space of A all of $\mathbb{R}^4$? Explain why in one statement.

No, since the column space is the span of 3 vectors and dim $(\mathbb{R}^4) = 4$.

nullity : dim (nullspace of A)
rank = dim (col space of A)

V- (10 points) Let T: $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be a linear transformation such that

$T(1,3,0) = (2,5,7,-1)$,

$T(3,1,1) = (4,1,3,2)$,

$T(3,0,1) = (5,2,-2,3)$.

Find the standard matrix for T.

$T(1,3,0) = b_1$

$(1,3,0) = e_1 + 3e_2 + 0$

so $T(e_1) + 3T(e_2) = b_1$

$T(1,3,0) = b_1$

$(1,3,0) = e_1 + 3e_2$ so $T(e_1) + 3\,T(e_2) = b_1$

$T(3,1,1) = b_2$

$(3,1,1) = 3e_1 + e_2 + e_3$ so $3T(e_1) + T(e_2) + T(e_3) = b_2$

$T(3,0,1) = b_3$

$(3,0,1) = 3e_1 + e_3$ so $3T(e_1) + T(e_3) = b_3$

Solve the system,

$$\begin{bmatrix} 1 & 3 & 0 & b_1 \\ 3 & 1 & 1 & b_2 \\ 3 & 0 & 1 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & b_1 \\ 0 & -8 & 1 & -3b_1 + b_2 \\ 0 & -9 & 1 & -3b_1 + b_3 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & b_1 \\ 0 & 1 & 0 & b_2 - b_3 \\ 0 & 0 & 1 & 9b_2 - 8b_3 - 3b_1 \end{array}\right]$$

$T(e_3) = 9b_2 - 8b_3 - 3b_1 = (36, 9, 2, 18) + (-40, -16, 16, -24) + (-6, -15, -21, 3)$

$= (-10, -22, 22, -3)$

$T(e_2) = b_2 - b_3 = (4,1,3,2) - (5,2,-2,3) = (-1,-1,5,-1)$

$T(e_1) = b_1 - 3T(e_2) = (2,5,7,-1) - 3(-1,-1,5,-1)$

$= (2,5,7,-1) + (3,3,-15,3)$

$= (5, 8, -8, 2)$

Therefore the standard matrix is $\begin{bmatrix} 5 & -1 & -10 \\ 8 & -1 & -22 \\ -8 & 5 & 22 \\ 2 & -1 & -3 \end{bmatrix}$

$\longrightarrow$

The previous method is the standard way for such questions but here there is a much easier way

$(0,1,0) = (3,1,1) - (3,0,1)$

$T(0,1,0) = T\big((3,1,1) - (3,0,1)\big) = T(3,1,1) - T(3,0,1)$

$= (4,1,3,2) - (5,2,-2,3) = (-1,-1,5,-1)$

$(1,0,0) = (1,3,0) - 3(0,1,0)$

$T(1,0,0) = T(1,3,0) - 3T(0,1,0) = (2,5,7,-1) - 3(-1,-1,5,-1)$

$= (5,8,-8,2)$

$(0,0,1) = (3,0,1) - 3(1,0,0)$

$T(0,0,1) = T(3,0,1) - 3T(1,0,0) = (5,2,-2,3) - 3(5,8,-8,2)$

$= (-10,-22,22,-3)$

Therefore the standard matrix is $\begin{bmatrix} 5 & -1 & -10 \\ 8 & -1 & -22 \\ -8 & 5 & 22 \\ 2 & -1 & -3 \end{bmatrix}$

VI- DO NOT PROVE ANY OF YOUR ANSWERS IN THIS QUESTION UNLESS YOU ARE ASKED TO DO SO.

a- **(24 points)** Write **TRUE** or **FALSE**.

1- $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if and only if $T(ku+v) = kT(u) + T(v)$ for all scalars k and vectors u and v in $\mathbb{R}^n$. T

2- If the row vectors and the column vectors of a matrix A are linearly independent then A is square. T

3- For $n \times n$ invertible matrices A and B, nullity(A+B)= nullity(A) + nullity(B). F

4- If $\{v_1, v_2, ..., v_n\}$ is linearly independent and a matrix A is nonzero then $\{Av_1, Av_2, ..., Av_n\}$ is linearly independent (provided that such multiplication is defined). F

5- If a set $\{v_1, v_2, ..., v_n\}$ spans a vector space V with dimension m then $n \geq m$. T

6- If W is a subspace of a vector space V and S is a subspace of W, then S is a subspace of V. T

7- $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x,y) = (-x, 2x+1)$ is a linear transformation. F

8- If A is a 4×5 matrix then the largest value for rank(A) is 4. T

b- **(8 points)** If A is an $n \times n$ matrix. For a nonzero scalar k, is $rank(kA) = k.rank(A)$? If yes, prove this. If no, give a counter example.

No

Take $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad rank(A) = 2$

$2A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad rank(2A) = 2$

Therefore $rank(2A) \neq 2\, rank(A)$

GOOD LUCK