7

Problem 1: (10%) Evaluate the following limits

(a) $\lim_{x\to\infty} \sin^{-1}\left(\frac{x^2}{1+2x^2}\right)$

5pts

$$\lim_{x\to\infty} \frac{x^2}{1+2x^2} = \lim_{x\to\infty} \frac{\cancel{x^2}}{\cancel{x^2}\left(\frac{1}{x^2}+2\right)} = 1/2$$

$\sin^{-1}$ is a continuous function near $1/2$ $\Rightarrow$

$$\lim_{x\to\infty} \sin^{-1}\left(\frac{x^2}{1+2x^2}\right) = \boxed{\sin^{-1} 1/2} = \underline{\pi/6}$$

$\leftarrow$ But this value does not matter

(b) $\lim_{x\to\infty} \frac{\tan^{-1}(e^x)}{e^{2x}+x}$

Note first: this is NOT the form for L'Hôpital's rule.

$\lim_{x\to\infty} \tan^{-1}(e^x) = \pi/2$ since $\lim_{x\to\infty} e^x = \infty$ & $\lim_{x\to\infty} \tan^{-1} x = \pi/2$

$$\lim_{x\to\infty} e^{2x}+x = \infty$$

$$\Rightarrow \lim_{x\to\infty} \frac{\tan^{-1}(e^x)}{e^{2x}+x} = 0$$

5pts

Problem 3: (20%) Evaluate the following integrals

(a) $\int \tan^{-1}(4x)\,dx$

Integration by parts

$$u = \tan^{-1}(4x) \qquad dv = 1$$

$$du = \frac{4}{1+16x^2} \qquad v = x$$

10 pts

$$\int \tan^{-1}(4x)\,dx = x\tan^{-1}(4x) - 4\underbrace{\int \frac{x}{1+16x^2}\,dx}_{\text{substitution}}$$

substitution ← let $z = 1+16x^2$

$dz = 32x\,dx$

$$= x\tan^{-1}(4x) - \frac{4}{32}\int \frac{dz}{z}$$

$$= x\tan^{-1}(4x) - \frac{1}{8}\ln|z| + C \overset{\text{back to } x}{=} \boxed{x\tan^{-1}(4x) - \frac{1}{8}\ln(\underbrace{1+16x^2}_{>0}) + C}$$

(b) $\int \frac{1}{x^2+4x+5}\,dx$

Complete the square in the denominator to reach a known form:

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{x^2+4x+4-4+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx$$

$$= \tan^{-1}(x+2) + C$$

10 pts

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10 pts

7

Problem 4: (14%) Evaluate the following improper integrals

(a) $\int_2^4 \frac{1}{x^2-x-2}\,dx$

$$\int_2^4 \frac{1}{x^2-x-2}\,dx = \int_2^4 \frac{dx}{(x-2)(x+1)} \longrightarrow$$ denominator is zero at 2, integrand has an asymptote there

$$= \lim_{b\to 2^+} \int_b^4 \frac{dx}{(x-2)(x+1)}$$

7pts

Now do $\int_b^4 \frac{dx}{(x-2)(x+1)}$ by partial fractions

$$\int_b^4 \frac{dx}{(x-2)(x+2)} = \int_b^4 \frac{1/3}{x-2}\,dx + \int_b^4 \frac{-2/3}{x+1}\,dx = \tfrac{1}{3}\ln\underset{\text{+ve for } x>2}{\underset{\downarrow}{(x-2)}}\Big]_b^4 - \tfrac{2}{3}\ln(x+1)\Big]_b^4$$

$$= \tfrac{1}{3}(\ln 2 - \ln(b-2)) - \tfrac{2}{3}(\ln 5 - \ln(b+1))$$

$$\lim_{b\to 2^+}\int_b^4 \frac{dx}{(x-2)(x+1)} = \lim_{b\to 2^+} -\ln(b-2) + \text{constants} = +\infty$$

integral is Divergent

(b) $\int_{-\infty}^{\infty} \frac{1}{(e^x+e^{-x})}\,dx$

$$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx = 2\int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx \quad \text{(even integrand)}$$

$$= 2\lim_{b\to\infty}\int_0^b \frac{1}{e^x+e^{-x}}\,dx$$

7pts

$$\int \frac{1}{e^x+e^{-x}}\,dx = \int \frac{e^x\,dx}{(1+e^{+2x})} \qquad \text{Let } u = e^x,\ du = e^x\,dx$$

$$= \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1} e^x + C$$

$$\Rightarrow \lim_{b\to\infty}\int_0^b \frac{1}{e^x+e^{-x}}\,dx = \lim_{b\to\infty} \tan^{-1} e^x\Big]_0^b = \pi/2 - \tan^{-1} e^0$$

$$= \pi/2 - \pi/4$$

$$= \pi/4$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx = 2\cdot\pi/4 = \boxed{\pi/2}$$

Obviously Convergent

Problem 5: (1~~2~~4%) Determine the convergence or divergence of the following improper integrals. Justify your answers.

(a) $\int_0^\infty \frac{1}{(e^x+1)^2}\,dx$

MANY WAYS TO DO THIS, one of which:

$$(e^x+1)^2 \geq (e^x)^2 = e^{2x} \quad \forall\, x$$

$$\Rightarrow \int_0^\infty \frac{1}{(e^x+1)^2}\,dx \leq \int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty} -\frac{e^{-2x}}{2}\Big]_0^b = 1/2$$

By DCT, original integral converges.

7pts

<u>Remark</u>: It's NOT true that $(e^x+1)^2 \geq e^{-x^2}$

(b) $\int_1^\infty \frac{\ln x}{e^x}\,dx$

Also MANY WAYS to do this.

7pts

$$\ln x \leq e^{x/2} \quad x \geq 1$$

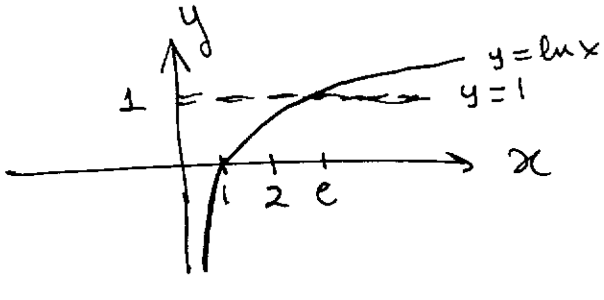
$$\int_1^\infty \frac{\ln x}{e^x}\,dx \leq \int_1^\infty \frac{e^{x/2}}{e^x}\,dx = \int_1^\infty e^{-x/2}\,dx = \lim_{b\to\infty} -2e^{-x/2}\Big]_1^b = 2e^{-1/2}$$

finite
Convergent

$\rightsquigarrow$ By DCT, original integral converges

- can also compare top & bottom to x^p (chosen carefully)

- Remark: $\ln x$ does not lie below 1 for all x!



10%.

Problem 6: (10%) Show that

$$\int_1^\infty \frac{\sin x + 2}{x^2}\, dx$$

converges, whereas

$$\int_1^\infty \frac{\sin x + 2}{x}\, dx$$

diverges.

First integral

5pts

$$0 \le \sin x \le 1$$

$$\Rightarrow 2 \le \sin x + 2 \le 3$$

$$\Rightarrow \quad 0 \le \int_1^\infty \frac{\sin x+2}{x^2}\, dx \le \underbrace{\int_1^\infty \frac{3}{x^2}\, dx}_{\text{this is a p-integral } p=2 \Rightarrow \text{ it converges}}$$

By DCT, 1st integral converges

NOTE: You can't conclude from SANDWICH THEOREM here applied to $\frac{\sin x + 2}{x^2}$ that the integral converges!

Second integral:

5pts

$$-1 \le \sin x$$

$$\Rightarrow 1 \le \sin x + 2$$

$$\Rightarrow \underbrace{\int_1^\infty \frac{1}{x}\, dx}_{\text{Divergent p-integral}} \le \int_1^\infty \frac{\sin x+2}{x}\, dx$$

By DCT, second integral diverges

NOTE: You can't bound $\int \frac{\sin x + 2}{x}dx$ above and conclude it's divergent!

Problem 7: (12%)

Find the values of p for which

$$\int_1^\infty \frac{x}{\sqrt{x^p+1}}\,dx$$

converges.

Justify your answer.

the integrand behaves like $\frac{x}{\sqrt{x^p}}$ so we can apply LCT with

$f(x) = \frac{x}{\sqrt{x^p+1}}$ & $g(x) = \frac{x}{\sqrt{x^p}} = x^{1-p/2}$

- $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{\sqrt{x^p}}{\sqrt{x^p+1}} = \lim\limits_{x\to\infty} \sqrt{\frac{x^p}{x^p+1}} = 1$

this means the integrals behave <u>EXACTLY</u> Alike (convergence or divergence)

Now $\int_1^\infty g(x)\,dx = \int_1^\infty x^{1-p/2}\,dx = \int_1^\infty \frac{1}{x^{p/2-1}}\,dx$

The last integral converges if $p/2 - 1 > 1 \Rightarrow p > 4$

diverges if $p/2 - 1 \le 1 \Rightarrow p \le 4$

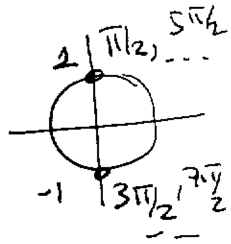
So we can say that for $p > 4$, the original integral converges

<u>NOTE:</u> If you use DCT here, you have to do double effort by comparing $\int f \le \underline{\int g} \xleftarrow{\text{for } p>4}$ convergent and $\underline{\int h} < \int f$ to conclude divergence for $p \le 4$

Problem 8: (12%) Determine if the following sequences converge or diverge. Justify your answers.

(a) $a_n = \dfrac{n \sin\left((2n-1)\frac{\pi}{2}\right)}{n+1}$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$



6pts $$\lim_{n\to\infty} \sin\{(2n-1)\pi/2\} = \begin{cases} +1 & \text{if } n \text{ odd} \\ -1 & \text{if } n \text{ even} \end{cases}$$

$\Rightarrow$ Overall limit is ± 1 $\Rightarrow$ limit does not exist (oscillates)

(b) $a_n = (1+\frac{2}{n})^n \frac{1}{\sqrt[n]{n^2}}$

$$\lim_{n\to\infty} (1+2/n)^n = e^2$$

- either table / theorem

or $e^{\ln(1+2/n)^n} = e^{\overbrace{n\ln(1+2/n)}^{\to 2 \text{ by L'Hopital}}} \to e^2$

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n^2}} = \lim_{n\to\infty} \frac{1}{n^{2/n}}$$

6pts $$\lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} e^{\frac{2}{n}\ln n} = e^0 = 1$$

$\frac{2}{n}\ln n \to 0$ by L'Hop

$\Rightarrow$ Overall limit is e^2.