LEBANESE AMERICAN UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND MATHEMATICS
MTH 201 - CALCULUS 3
EXAM 1 – FALL 2013

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Duration: 70 minutes

Name: KEY **ID#:**

- **This exam consists of 8 pages and 5 problems.**
- **Answer the questions below on the space provided. You can use the back pages for scratch or for more space for your answers. Please specify.**
- **Make sure you justify all your answers.**

Question Number	Grade
1. 14%	
2. 20%	
3. 36%	
4. 12%	
5. 18%	
TOTAL	

Problem 1: (14%) Evaluate the following proper integrals

(a) $\displaystyle\int \cosh(\ln x)\, dx$

$$= \int \frac{e^{\ln x} + e^{-\ln x}}{2}\, dx$$

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$$= \int \frac{x + \frac{1}{x}}{2}\, dx = \frac{1}{2}\left[\frac{x^2}{2} - \cancel{\frac{1}{x^2}}\ \ln|x|\right] + C$$

(b) $\displaystyle\int \frac{1}{e^x + e^{-x}}\, dx$ $\displaystyle = \int \frac{1}{e^x + e^{-x}}\, dx = \int \frac{e^x}{e^{2x} + 1}\, dx$

$$= \tan^{-1}(e^x) + C$$

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Problem 2: (20%) Evaluate the following improper integrals

(a) $\int_3^\infty \frac{1}{x^2-6x+8}\,dx$ $= \cancel{\lim} \int_3^\infty \frac{1}{(x-2)(x-3)}\,dx$

$$= \int_3^4 \frac{1}{(x-2)(x-3)}\,dx + \int_4^\infty \frac{1}{(x-2)(x-3)}\,dx \; ; \; \frac{1}{(x-2)(x-3)} = \frac{-\frac{1}{2}}{x-2} + \frac{\frac{1}{2}}{x-3}$$

$$\int_3^4 \frac{1}{(x-2)(x-3)}\,dx = \lim_{l\to 3^+} \frac{1}{2}\int_l^4 \left(\frac{-1}{x-2} + \frac{1}{x-3}\right)dx = \lim_{l\to 3^+} \frac{1}{2}\Big[-\ln|x-2| + \ln|x-3|\Big]_l^4$$

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$$= \frac{1}{2}\Big[-\ln|4-2| + \ln|4-3|\Big] - \lim_{l\to 3^+} \frac{1}{2}\Big[-\ln|l-2| + \ln|l-3|\Big]$$

→ undetermined

(b) $\int_1^\infty \frac{1}{\sqrt{x}(9+x)}\,dx$ $= \lim_{l\to\infty} \int_1^l \frac{1}{9\sqrt{x}\left[1+\left(\frac{\sqrt{x}}{3}\right)^2\right]}$; let $u = \frac{\sqrt{x}}{3}$

$du = \frac{1}{3\sqrt{x}}$

$$= \lim_{l\to\infty} \frac{1}{3}\int_{\frac{1}{3}}^{l} \frac{du}{1+u^2}$$

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$$= \lim_{l\to+\infty} \frac{1}{3}\tan^{-1}(u)\Big|_{1/3}^{l}$$

$$= \lim_{l\to+\infty} \left[\frac{1}{3}\tan^{-1}(l) - \frac{1}{3}\tan^{-1}(1/3)\right]$$

$$= \frac{1}{3}\times\frac{\pi}{2} - \frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right).$$

(c) $\int_e^\infty \frac{1}{(\ln x)^3 x}\,dx$ $= \lim_{\ell \to \infty} \int_e^\ell \frac{1/x}{(\ln x)^3}\,dx = \lim_{\ell \to \infty} \left.\frac{(\ln x)^{-2}}{-2}\right|_e^\ell$

$= \lim_{\ell \to \infty} \frac{(\ln \ell)^{-2}}{-2} + \frac{(\ln e)^{-2}}{2} = \boxed{\frac{1}{2}}$

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Problem 3: (36%) Determine the convergence or divergence of the following improper integrals. Justify your answers.

1. $\int_0^\infty \frac{x}{(x^3+1)^2}\,dx$ $= \underbrace{\int_0^1 \frac{x}{(x^3+1)^2}\,dx}_{\text{not improper} \therefore \text{converges.}} + \int_1^\infty \frac{x}{(x^3+1)^2}\,dx$

$\int_1^\infty \frac{x}{(x^3+1)^2}\,dx$: LCT with $\frac{1}{x^5}$.

$\lim_{x \to \infty} \frac{\frac{x}{(x^3+1)^2}}{\frac{1}{x^5}} = \lim_{x \to \infty} \frac{x^6}{(x^3+1)^2} = 1$

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$\therefore$ both converge.

2. $\int_{-\infty}^{\infty} \frac{1}{e^{(x^2)}}\,dx$ $= 2\int_0^\infty e^{-x^2}dx$

$= 2\Big[\underbrace{\int_0^1 e^{-x^2}dx}_{\text{not improper}} + \int_1^\infty e^{-x^2}dx \Big]$

$\int_1^\infty e^{-x^2}dx < \int_1^\infty e^{-x}dx$ which converges ✓

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3. $\int_1^\infty \frac{|\sin x|}{x^2}\,dx$

$\frac{|\sin x|}{x^2} < \frac{1}{x^2}$

and $\int_1^\infty \frac{1}{x^2}$ converges

$\therefore$ by DCT, $\int_1^\infty \frac{|\sin x|}{x^2}dx$ converges

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4. $\int_1^\infty \frac{(\ln x)^{100}}{x^{0.2}}\,dx$ $= \underbrace{\int_1^e \frac{(\ln x)^{100}}{x^{0.2}}\,dx}_{\text{not improper}} + \int_e^\infty \frac{(\ln x)^{100}}{x^{0.2}}\,dx$

$\ln x > 1 \Rightarrow \frac{(\ln x)^{100}}{x^{0.2}} > \frac{1}{x^{0.2}}$ and $\int_e^\infty \frac{1}{x^{0.2}}\,dx$ diverges

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$\therefore$ by Comparison, $\int_e^\infty \frac{(\ln x)^{100}}{x^{0.2}}\,dx$ diverges

5. $\int_1^\infty \frac{x^3+1}{\sin x + 2 + x^3\sqrt{x}}\,dx$

$\sin x < 1 \Rightarrow \sin x + 2 + x^3\sqrt{x} < 3 + x^3\sqrt{x}$

$\Rightarrow \frac{x^3+1}{\sin x + 2 + x^3\sqrt{x}} > \frac{x^3+1}{x^{3.5}+3}$

But $\int_1^\infty \frac{x^3+1}{x^{3.5}+3}\,dx$ is like $\int_1^\infty \frac{1}{x^{0.5}}$ which

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diverges $\Rightarrow \int_1^\infty \frac{x^3+1}{x^{3.5}+3}\,dx$ can be shown to diverge

$\therefore$ by Direct Comparison, $\int_1^\infty \frac{x^3+1}{\sin x + 2 + x^3\sqrt{x}}\,dx$ diverges

6. $\displaystyle\int_1^\infty \frac{e^x}{\sqrt{1+x^2}}\,dx$

$$\lim_{x\to\infty} \frac{e^x}{\sqrt{1+x^2}} = \infty \quad (\neq 0) \quad \therefore \text{ improper integral diverges}$$

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Problem 4: (12%)

Find the values of p for which

$$\int_0^\infty \frac{1}{(x^2+1)^p}\,dx = \underbrace{\int_0^1 \frac{1}{(x^2+1)^p}\,dx}_{\text{not improper}} + \int_1^\infty \frac{1}{(x^2+1)^p}\,dx$$

converges. Justify your answer.

LCT with $\displaystyle\int_1^\infty \frac{1}{x^{2p}}\,dx$

$$\lim_{x\to\infty} \frac{\frac{1}{x^{2p}}}{\frac{1}{(x^2+1)^p}} = 1 \quad \therefore \text{ both converge or both diverge.}$$

Now $\displaystyle\int_1^\infty \frac{1}{x^{2p}}\,dx$ converges if $2p<1 \Rightarrow 0<p<\frac{1}{2}$

diverges if $2p\geq 1 \Rightarrow p\geq\frac{1}{2}$.

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Problem 5: (18%) Find the following limits if they exist. Justify your answers.

(a) $\lim_{n\to\infty} (-1)^n \frac{n}{7+n}$ $\longrightarrow$ oscillates between (-1) and (1)

$\therefore$ diverges $\left(\lim_{n\to\infty} \frac{n}{7+n} = 1 \right)$

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(b) $\lim_{n\to\infty} (1 + \frac{1}{\sqrt{n}})^n$ = ~~lim~~ Let $y = \left(1 + \frac{1}{\sqrt{n}}\right)^n \Rightarrow \ln y = n \ln\left(1 + \frac{1}{\sqrt{n}}\right)$

$\ln y = \frac{\ln(1 + 1/\sqrt{n})}{\frac{1}{n}}$ $\therefore \lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\frac{+1/2\sqrt{n}}{1+1/\sqrt{n}}}{-1/n^2}$

6 $= \lim_{n\to\infty} - \frac{n^2}{2\sqrt{n}(1+\frac{1}{\sqrt{n}})} = \lim_{n\to\infty} \frac{-n^2}{2\sqrt{n}+2} = -\infty$

$\therefore y \to e^{-\infty} = \boxed{0}$.

(c) $\lim_{n\to\infty} \frac{n!}{6^{n+2}}$

$L = \lim_{n\to\infty} \frac{n!}{6^n} \cdot \frac{1}{6^2}$

6 But $\lim_{n\to\infty} \frac{6^n}{n!} = 0$ $\therefore \lim_{n\to\infty} \frac{n!}{6^n} = \infty$

and the sequence diverges.