AMERICAN UNIVERSITY OF BEIRUT

Mathematics Department

MATH 218 - QUIZ III

Fall 2008-2009

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Directions:

- This is a partial assessment of what you are learning in this course. Read carefully and write down what you know.
- The exam is meant to detect any gaps in your knowledge so that they are corrected. Show your thought process in the minimum number of sentences.
- You have 70 minutes. Finish as much as possible. Work quickly but accurately.
- Do not forget to write down your name and circle your section number.

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Name:....................................... **ID:..**

Section: 5 (@ 9:30) 4 (@ 12:30)

Question	Number of parts and subparts	Grade
I	2	/24
II	3	/25
III	1	/12
IV	3	/24
V	3	/15

Total Grade = /100

I- **(a)** **(12 points)** Determine whether the set of all 2×2 matrices, M_{22}, is a vector space under the following addition and scalar multiplication:

$$A + B = A + B + I_2$$

and

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (k+1)a & (k+2)b \\ (k+3)c & (k+4)d \end{bmatrix}$$

Note: In the above definition of addition, the plus sign on the left is the new addition operation, and the one on the right is the standard addition operation.

Let $u \in M_{22}$

$$u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$1u = \begin{bmatrix} (1+1)a & (1+2)b \\ (1+3)c & (1+4)d \end{bmatrix} = \begin{bmatrix} 2a & 3b \\ 4c & 5d \end{bmatrix} \neq \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

So $1u \neq u$

Therefore M_{22} with the above two operations is not a vector space.

(b) **(12 points)** Let W be the set of all polynomials of degree **exactly** 4. Determine whether W is a subspace of the set of polynomials of degree less than or equal to 4, $\mathcal{P}_4$.

$W = \{ p(x) \in P_4 \mid p(x) \text{ has degree } 4 \}$

$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ such that $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$ and $a_4 \neq 0 \}$

$1 + x + x^4 \in W$

$2 - x^3 - x^4 \in W$

$(1 + x + x^4) + (2 - x^3 - x^4) = 3 + x - x^3 \notin W$

5 points for correct application of the rule

II- (a) (i) **(10 points)** Find a basis for the space spanned by the vectors $v_1 = (1,3,5,9,7)$, $v_2 = (2,4,6,12,-2)$, $v_3 = (3,7,11,21,5)$, $v_4 = (2,8,5,1,0)$ which is a subset of $\{v_1, v_2, v_3, v_4\}$.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 4 & 7 & 8 \\ 5 & 6 & 11 & 5 \\ 9 & 12 & 21 & 1 \\ 7 & -2 & 5 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -2 & -2 & 2 \\ 0 & -4 & -4 & -5 \\ 0 & -6 & -6 & -17 \\ 0 & -16 & -16 & -14 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 4 & 4 & 5 \\ 0 & 6 & 6 & 17 \\ 0 & 16 & 16 & 14 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 23 \\ 0 & 0 & 0 & 30 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ u_1 & u_2 & u_3 & u_4 \end{array} \end{array}$$

$\{v_1, v_2, v_4\}$ is a basis for $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$

(ii) **(5 points)** Write any vector that is not in the basis as a linear combination of vectors in the basis.

$u_3 = u_1 + u_2$

So $v_3 = v_1 + v_2$

(b) (10 points) Find a basis for the space W consisting of vectors (a, b, c) such that (a, b) is a solution of the system
$$\begin{cases} 2x_1 & +16\,x_2 = 0 \\ 3x_1 & +24\,x_2 = 0 \\ -4x_1 & -32\,x_2 = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 16 & 0 \\ 3 & 24 & 0 \\ -4 & -32 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 8 & 0 \\ 1 & 8 & 0 \\ 1 & 8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 8 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

3

$$x_1 = -8x_2 = -8t \qquad t \in \mathbb{R}$$
$$x_2 = t$$

2 $(a,b) = (-8t, t)$

$W = \{(a,b,c)$ where $a = -8t$, $b = t$ and $t \in \mathbb{R}\}$

$(a,b,c) = (-8t, t, c)$

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$= t(-8,1,0) + c\,(0,0,1)$

So $\{(-8,1,0), (0,0,1)\}$ spans W

None of the two vectors is a multiple of
1 the other so

$\{(-8,1,0), (0,0,1)\}$ is a basis for W

III- (12 points)

Prove that if the rank of an $m \times n$ matrix A is m then $ColumnSpace_A = \mathbb{R}^m$.

$$\text{rank}(A) = \dim(\text{Row Space}_A)$$
$$= \dim(\text{Column Space}_A) = m$$

But Column Space_A is a subspace of $\mathbb{R}^m$ and it has the same dimension as $\mathbb{R}^m$.

Therefore, $\text{Column Space}_A = \mathbb{R}^m$ (by a theorem)

IV- Answer each of the following questions. **Give justifications for your answers.**

(a) **(8 points)** Consider $\mathbb{R}^2$ with the inner product $< u, v >= 3u_1v_1 + 5u_1v_1$. Use the Gram Schmidt process to transform the given basis $u_1 = (2, -1)$, $u_2 = (1, 1)$ into an **orthogonal** basis.

1 $v_1 = (2, -1)$

2 $v_2 = u_2 - \text{Proj}_W u_2$ where $W = \text{Span}\{v_1\}$

$$= u_2 - \frac{<u_2, v_1>}{\|v_1\|^2} v_1$$

1 $$= (1,1) - \frac{3\times 2\times 1 + 5\times(-1)\times 1}{3\times 2\times 2 + 5\times(-1)\times(-1)}(2,-1)$$

$$= (1,1) - \frac{1}{17}(2,-1)$$

1 $$= \left(\frac{15}{17}, \frac{18}{17}\right)$$

(b) **(8 points)** Given a space V with the usual Euclidean inner product and a subspace W of V with the basis $\{(1, 2, 3), (3, 0, -1)\}$. Find the projection of $(2, -7, 3)$ on W.

$\{(1,2,3), (3,0,-1)\}$ is othogonal

3 $$\text{Proj}_W (2,-7,3) = \frac{2-14+9}{1+4+9}(1,2,3) + \frac{6+0-3}{9+0+1}(3,0,-1)$$

$$= \frac{-3}{14}(1,2,3) + \frac{3}{10}(3,0,-1)$$

1 $$= \left(\frac{-3}{14} + \frac{3}{10}, \frac{-3}{7}, \frac{-9}{14} - \frac{3}{10}\right) =$$

(c) **(8 points)** Find the orthogonal complement of the space spanned by the vectors $(1,2,0,0,3)$, $(0,0,0,1,1)$, $(0,0,3,0,1)$, $(0,0,0,0,\sqrt{2})$

Let $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$

$(\text{RowSpace}_A)^{\perp} = \text{NullSpace}_A$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_5 = 0$

$x_4 = 0$

$x_3 = 0$

$x_2 = t$

$x_1 = -2t$

so $\text{NullSpace}_A = \{x \in \mathbb{R}^5 \; x = (-2t, t, 0, 0, 0)$ where $t \in \mathbb{R}\}$

V- Give a short answer to each of the following. JUSTIFY.

1. **(5 points for each answer)** The nullity of an $n \times n$ matrix whose entries are all ones is:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ & \ddots & & \\ 1 & & \ddots & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \vdots & \cdots & 0 \end{bmatrix}$$

there are $n-1$ free variables

dim (Null Space) = nullity of the matrix

$= n-1$

2. **(5 points for each answer)** Can an overdetermined system $Ax = b$ be consistent for every possible b?

$Ax = b$ is consistent for all b

$\iff$ Column vectors of A span $\mathbb{R}^m$

overdetermined system has more equations than unknowns so the number of column vectors is less than m therefore, the column vectors can not span $\mathbb{R}^m$.

So such a system can't be consistent $\forall b$.

3. **(5 points for each answer)** If the vectors u, v, w are three vectors orthogonal to a space W, can you conclude that the space spanned by u, v, and w is orthogonal to W?

u, v, w are orthogonal to W

Any element in $\mathrm{Span}\{u, v, w\}$ has the form

$$k_1 u + k_2 v + k_3 w$$

$$\begin{aligned}\langle k_1 u + k_2 v + k_3 w , w' \rangle &= k_1 \langle u, w' \rangle + k_2 \langle v, w' \rangle + k_3 \langle w, w' \rangle \\ &= 0 + 0 + 0 \\ &= 0\end{aligned}$$

for all $w' \in W$

So any element in $\mathrm{Span}\{u, v, w\}$ is orthogonal to W.

GOOD LUCK