**LEBANESE AMERICAN UNIVERSITY**
**Division of Computer Science and Mathematics**

# Calculus III

## Exam I

**Fall 2007 (November 2, 2007)**

**Name:** Solutions **ID:**

**Circle the name of your instructor:** **Dr. Habre** **Dr. Hamdan** **Dr. Touma**

| Question Number | Grade |
|---|---|
| 1. 15% | |
| 2. 18% | |
| 3. 15% | |
| 4. 15% | |
| 5. 5% | |
| 6. 5% | |
| 7. 21% | |
| 8. 6% | |
| Total | |

1. (15%) Evaluate the following integrals:

(a) $\int \frac{1}{\sqrt{16-x^2}}dx$

$\sin^{-1}\left(\frac{x}{4}\right)$

(b) $\int \sinh(\ln x)dx$ $= \int \frac{e^{\ln x} - e^{-\ln x}}{2} dx$

$\int \frac{x - \frac{1}{x}}{2} dx = \frac{x^2}{4} - \frac{\ln x}{2} + C$

(c) $\int \underbrace{\tan^{-1}x}_{u}\,\underbrace{dx}_{dv}$ by parts. $= x\tan^{-1}x - \int \frac{x}{1+x^2}dx$

$= x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + C$

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2. (18%) Evaluate the following improper integrals.

(a) $\int_0^\infty \frac{x}{x^4+1}dx$ $\quad u = x^2$

$$\frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}(x^2)\Big|_0^\infty = \frac{1}{2}(\pi/2) = \pi/4$$

(b) $\int_0^\infty \underbrace{x}_{u}\underbrace{e^{-x}dx}_{dv}$ $= \text{by parts.}$

$$-xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x}\Big|_0^\infty = -\frac{x}{e^x} - \frac{1}{e^x}\Big|_0^\infty$$

$$= 0 - 0 - (0 - 1)$$

$$= 1 \quad \text{Converges to } 1.$$

$$\lim_{x\to\infty}\frac{x}{e^x} = 0$$

(c) $\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}dx$

$$u = 1 - 2\sin x$$
$$du = -2\cos x\, dx$$

$$= -\frac{1}{2}\int\frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-1/2}du = -1/2\ \frac{u^{1/2}}{1/2} = -\sqrt{1-2\sin}$$

$$= -\sqrt{1-2\sin x}\Big|_0^{\pi/6} = \sqrt{1} = 1$$

3. (15%) Test the following improper integrals for convergence or divergence..

(a) $\int_1^\infty \frac{1}{e^{(x^2)}} dx$ $< \int_1^\infty \frac{1}{x^2}\, dx \Rightarrow$

converges by DCT

(b) $\int_2^\infty \frac{x^3}{(x^4+5x)(\ln x)^{30}} dx$ $\approx \int_2^\infty \frac{dx}{x(\ln x)^{30}}$

$= \int^\infty \frac{du}{u^{30}}$ converge p: int $p > 1$

(c) $\int_{-\infty}^\infty \frac{1}{e^x - e^{-x}} dx$ $=$

$2\int_0 \frac{1}{e^x - e^{-x}}\, dx = 2\int_0^1 + \int_1^\infty$

Consider $\int_1^\infty \frac{1}{e^x - e^{-x}}\, dx \approx \int_1^\infty \frac{dx}{e^x}$ conv.

Now Consider $\int_0^1 \frac{1}{e^x - e^{-x}}\, dx = \int \frac{dx}{e^x - \frac{1}{e^x}} = \int \frac{e^x\, dx}{e^{2x}-1} = \int \frac{du}{(u-1)(u+1)}$

$= \int \frac{1}{2(u-1)} - \frac{1}{2(u+1)} = \frac{1}{2} \ln\left|\frac{u-1}{u+1}\right| = \frac{1}{2}\ln\left(\frac{e^x-1}{e^x+1}\right)\Big|_0^1$ : converges.

4. (15%) Consider the following sequences and check if they converge or diverge. In case they converge, find the limit:

(a) $a_n = (1 + \ln n)^{1/n}$

$$y = (1 + \ln n)^{1/n}$$

$$\ln y = \frac{1}{n} \ln(1 + \ln n) \to 0$$

$$y \to 1$$

(b) $a_n = \left(\dfrac{n}{n+3}\right)^n$ $= \left(\dfrac{n+3-3}{n+3}\right)^n$

or

$$a_n = \frac{1}{\left(\frac{n+3}{n}\right)^n} = \frac{1}{\left(1 + \frac{3}{n}\right)^n} \longrightarrow \frac{1}{e^3}$$

(c) $a_n = (-1)^n e^{\cos(\pi/n)}$

$$\boxed{\begin{array}{l} \frac{\pi}{n} \to 0 \\ \cos\left(\frac{\pi}{n}\right) \to 1 \end{array}}$$

$$a_n \to (-1)^n e^{\cos 0} = (-1)^n$$

$$a_n \not\to (-1)^n \Rightarrow \text{no limit}$$

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5. (5%) Find a sequence which is bounded, converging and neither increasing nor decreasing. JUSTIFY.

$$a_n = \frac{(-1)^n}{n}$$

6. (5%) Find a sequence that is bounded and converging to 3, and either increasing or decreasing. JUSTIFY.

$$a_n = \frac{3n}{n}$$

7. (21%) Consider the following series and determine if they converge or diverge. In case of convergence, find the limit.

Geom. series

(a) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} 5^{-n}$ $= \sum 2 \frac{2^n}{3^n \cdot 5^{n}} = \sum 2 \left(\frac{2}{15}\right)^n$

$r = \frac{2}{15} < 1 \Rightarrow$ conv.

$$\frac{\text{first term}}{1-r} = \frac{2(2/15)}{1 - 2/15} = \frac{4/15}{13/15}$$

$$\rightarrow \frac{4}{13}$$

(b) $\sum_{n=2}^{\infty} \frac{2}{n(n-1)}$ telescoping.

$\frac{2}{n(n-1)} = \frac{A}{n} + \frac{B}{(n-1)} = (A+B)n - A = 2$

$A = -2$

$B = 2$

$= \sum \left( -\frac{2}{n} + \frac{2}{n-1} \right)$

$S_n = \left( \cancel{-\frac{2}{2}} + \frac{2}{1} \right) + \left( -\cancel{\frac{2}{3}} + \cancel{\frac{2}{2}} \right) + \left( -\frac{2}{4} + \cancel{\frac{2}{3}} \right) + \cdots \left( -\frac{2}{n} + \frac{2}{n-1} \right)$

$S_n \to 2 - \frac{2}{n} \to 2$

(c) $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^n$

$a_n \to e^2 \Rightarrow$ Series diverges by $n$th term test.

8. (6%) Express the rational number 1.9121212... as a ratio $\frac{m}{n}$ of two integers $m$ and $n$. Show your work

$A = 1.9 + 0.012 + 0.00012 + 0.0000012 + \cdots$

$= 1.9 + \frac{12}{(10)^3} + \frac{12}{10^5} + \frac{12}{10^7} + \cdots$ $\boxed{r = 1/100}$

$= \frac{19}{10} + \sum_{n=1}^{\infty} \frac{12/10}{(100)^n} = \frac{19}{10} + \frac{12/10^3}{1 - 1/100}$

$= \frac{19}{10} + \frac{12}{990} = \frac{(99)(19) + 12}{990}$

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