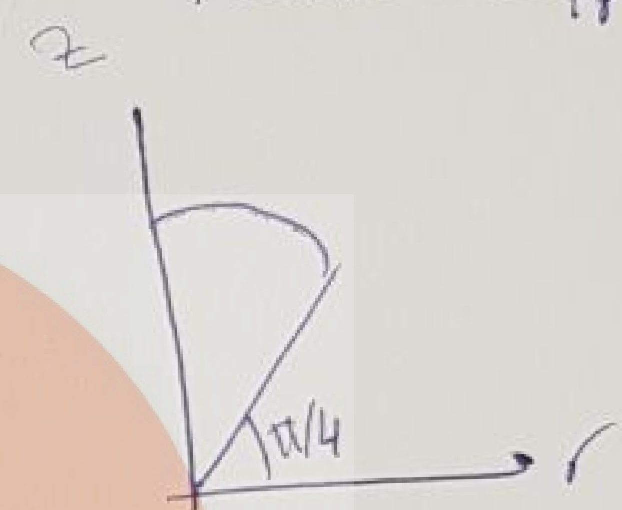
Problem 1 (20 pts)

Use the divergence theorem to find the outward flux of the field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ across the boundary of the region D which is bounded from above by the sphere $\rho = 1$ and from below by the cone $\phi = \frac{\pi}{4}$.

$$\operatorname{div}(F) = 3x^2 + 3y^2 + 3z^2 = 3\rho^2.$$

the surface is a closed surface, outward flux, divergence theorem is applicable



$$\text{flux} = \iiint_R 3\rho^2 \, dV$$

$$= \iiint_R 3\rho^2 \, d\varphi \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\rho=0}^{1}\int_{\varphi=0}^{\pi/4} 3\rho^4 \sin\varphi \, d\varphi \, d\rho \, d\theta$$

$$= \int_0^{2\pi}\int_0^{1} -3\rho^4 \left[\cos\varphi\right]_0^{\pi/4} d\rho \, d\theta$$

$$= \int_0^{2\pi}\int_0^{1} \frac{3\sqrt{2}}{2}\rho^4 \, d\rho \, d\theta$$

$$= \int_0^{2\pi} \frac{3\sqrt{2}}{10}\left[\rho^5\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{3\sqrt{2}}{10} \, d\theta = \frac{6\pi\sqrt{2}}{10} = \frac{3\pi\sqrt{2}}{5}.$$

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Problem 2 (18 pts)
Solve the following IVP.

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{\left(x + ye^{y/x}\right)}{xe^{y/x}} \\ y(1) = 0 \end{cases}$$

Please make sure to write your solution in explicit form, and to indicate the interval where your solution is defined.

$$y' = \frac{(x + y^{e^{y/x}})}{xe^{y/x}} = \frac{\left(1 + \frac{y}{x}e^{\frac{y}{x}}\right)}{e^{\frac{y}{x}}}$$

let $u = \frac{y}{x}$ $\quad y = ux$

$y' = u'x + u.$

$$xu' + u = \frac{1 + ue^{u}}{e^{u}}$$

$$xu' + u = e^{-u} + u$$

$$xu' = e^{-u}$$

$$e^{u} u' = \frac{1}{x}$$

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$$\int e^{u} du = \int \frac{1}{x} dx$$

$$e^{u} \cancel{+C} = \ln|x| + c$$

$$u = \ln(\ln|x| + c)$$

$$y = x \ln(\ln|x| + c)$$

$y(1) = 0$

$0 = 1 \ln(\ln 1 + c)$ $\quad 0 = 1 \ln c$ $\quad \ln c = 0$

$c = 1$

so $y = x \ln(\ln|x| + 1)$

defined over ~~$\mathbb{R}$~~ $x \neq 0$,

~~I =]-∞~~ and $\ln|x| + 1 > 0$

$\ln|x| > -1$

$|x| > e^{-1}$

$x < -1$ or $x > 1$

so $I =]-\infty, -1[\cup]1, +\infty[$

Problem 3 (14 points)

Use the substitution $x = e^{3t} + 1$ to transform the given differential equation into an equation with constant coefficients and then solve.

$$(x-1)^2 y'' + \frac{2}{9} y = x - 1 \qquad x > 1$$

Please make sure to write your solution in explicit form. You do **NOT** have to specify the interval in this problem.

$$x = e^{3t} + 1$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}(3e^{3t}) = 3e^{3t}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d\left(\frac{dy}{dx}3e^{3t}\right)}{dx}\frac{dx}{dt}$$

$$= e^{3t}\left(\frac{d^2y}{dx^2}3e^{3t} + 9\frac{dy}{dx}e^{3t}\right)$$

$$= 3e^{6t}\frac{d^2y}{dt^2} + 9e^{6t}\frac{dy}{dt}$$

$$(x-1)^2 y'' + \frac{2}{9} y = x - 1$$

$$e^{6t} y$$

the solution is guaranteed to converge for $x > 1$ since there are no singular points on the interval.

$$x = e^{3t} + 1$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y'(3e^{3t})$$

$$y'' = \frac{d(y' 3e^{3t})}{dt} = 3e^{3t}\frac{dy'}{dt} + 9e^{3t}y'$$

$$= 3e^{3t}\frac{dy'}{dx}\frac{dx}{dt} + 9e^{3t}y' = (3e^{3t})^2 y'' + 9e^{3t}y'$$

$$= 9e^{6t}y'' + 9e^{3t}y'$$

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$(x-1)^2 y'' + \frac{2}{9} y = x-1.$

$9(x-1)^2 y'' + 2y = e^{3t}$

$9e^{6t} y'' + 2y = e^{3t}$

$\cancel{9e^{6t}} y_t'' - 9e^{3t} y' + 2y = e^{3t}$

$y_T'' - 3y_T' + 2y_T = e^{3t}$

$y_c'' - 3y_c' + 2y_c = 0$

$m^2 - 3m + 2 = 0$

$m = 2, m = 1$

$\text{so } y_c = c_1 e^{2t} + c_2 e^{t}.$

$y_p = Ae^{3t}$

$y_p'' - 3y_p' + 2y_p = e^{3t}$

$9Ae^{3T} - 9Ae^{3T} + 2Ae^{3T} = e^{3t}, \quad A = \frac{1}{2}$

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$\text{so } y_T = c_1 e^{2T} + c_2 e^{t} + \frac{1}{2} e^{3t}.$

$\text{put } T = \frac{\ln(x-1)}{3}$

$\text{so } y_{(t)} = c_1 e^{\frac{2}{3}\ln(x-1)} + c_2 e^{\frac{\ln(x-1)}{3}} + \frac{1}{2} e^{3\ln(x-1)}$

$y_{(x)} = c_1 (x-1)^{\frac{2}{3}} + c_2 (x-1)^{\frac{1}{3}} + \frac{1}{2}(x-1).$

Differential Equations

Spring 2017

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7b	8a	8b	Total
		16	200

Problem 4 (14 pts each)

Find the general solution for each of the following ODE's.

a. $y^{(8)} - 3y^{(6)} - 4y^{(4)} = 0$

constant coefficients,

corresponding equation: $y = e^{mx}$

$m^8 - 3m^6 - 4m^4 = 0$

$m^4(m^4 - 3m^2 - 4) = 0$

$m^4(m^2+1)(m^2-4) = 0$

$m = 0$ mult 4 $\rightarrow y_1 = c_1 + c_2 x + c_2 x^2 + c_3 x^3$

$\left.\begin{array}{l} m = i \\ m = -i \end{array}\right\} \rightarrow y_2 = A\cos t + B\sin t$

$m = 2 \longrightarrow y_3 = c_4 e^{2t}$

$m = -2 \longrightarrow y_4 = c_5 e^{-2T}$

so $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 \cos t + c_5 \sin t + c_6 e^{2t} + c_7 e^{-2t}$.

solution converges always

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b. $y^{(4)} + 3y^{(3)} + 4y'' - 8y' = 8 + 16x$

constant coefficients; $y_c^{(4)} + 3y_c^{(3)} + 4y_c'' - 8y_c' = 0$

$y = e^{mx}$

$m^4 + 3m^3 + 4m^2 - 8m = 0$

$m(m^3 + 3m^2 + 4m - 8) = 0$

Interval of convergence $= \mathbb{R}$

$m = 0$ 1st root

$(1)^3 + 3(1) + 4(1) - 8 = 0$ so $\boxed{m = 1}$ root.

$$\begin{array}{r|l} m^3 + 3m^2 + 4m - 8 & m - 1 \\ -\ m^3 - m^2 \qquad\qquad & \overline{m^2 + 4m + 8} \\ 4m^2 + 4m - 8 & \\ -\ 4m^2 - 4m \quad & \\ 8m - 8 & \end{array}$$

$m(m-1)(m^2 + 4m + 8) = 0$

$\Delta = 16 - 32 = -16$

$m_1 = \frac{-4 - 4i}{2} = -2 - 2i$

$m_2 = -\frac{4 + 4i}{2} = -2 + 2i$

$m(m-1)(m - (-2-2i))(m - (-2+2i)) = 0$

$m = 0 \rightarrow y_1 = c_0$

$m = 1 \rightarrow y_2 = c_1 e^t$

$m = -2-2i$, $m = -2+2i$ $\rightarrow y_3 = e^{-2t}(c_2 \cos 2t + c_3 \sin 2t)$

$y_c = c_0 + c_1 e^t + e^{-2t}(c_2 \cos 2t + c_3 \sin 2t)$.

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$L(y_p) = 8 + 16x$

$y_{p_1} = Ax^2 + Bx$ (c_0 is a root of $L[y] = 0$)

$L(A x) = -8A$

$-8A = 8$

$\boxed{A = -1}$

$L(y_{p_2}) = 16x$ $\quad y_{p_2} = Bx^2$ (e^{0x} is a solut.)

$L(Bx^2) = \cancel{-8 \times 2x}$

$-8.2.$

$L(Ax^2) = 8A - 16Ax$

$L(Bx) = -8B$

$L(Ax^2 + Bx) = -16Ax + 8A - 8B = 8 + 16x$

$\boxed{A = -1}$ $\quad -8 - 8B = 8$

$-1 - B = 1$, $B = -2$

$y_p = -x^2 - 2x$

so $y = y_p + y_c$

$= c_0 + c_1 e^t + e^{-2t}(c_3 \cos 2t + c_4 \sin 2t) - x^2 - 2x$.

Problem 5 (18 pts)
Find two linearly independent series solutions of the ODE

$$y'' - 2xy' - 2y = 0$$

about the ordinary point $x = 0$. (It is enough to find the first four nonzero terms of each series.)

let $y = \sum_{n=0}^{\infty} c_n x^n$.

No singular points
Converges on all of R.

$y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$

$y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$

$-2xy' = -2\sum_{n=0}^{\infty} n c_n x^n$

$-2y = -2\sum_{n=0}^{\infty} c_n x^n$.

so; $\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - 2\sum_{n=0}^{\infty} n c_n x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$

$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - 2\sum_{n=0}^{\infty} (n+1) c_n x^n = 0$

$K = n$ $\quad k-2 = n$, $K = n-2$, $n = K+2$.

$\sum_{K=0}^{\infty} K(K-1) c_K x^{K-2} - 2\sum_{K=2}^{\infty} (K-1) c_{K-2} x^{K-2} = 0$

$0c_0 + 0c_1 + \sum_{K=2}^{\infty} x^{K-2}\left[K(K-1) c_K - 2(K-1) c_{K-2} \right] = 0$

c_0 anything
c_1 anything

$$c_K = \frac{2(K-1) c_{K-2}}{K(K-1)} = \frac{2c_{K-2}}{K}.$$

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$$CK = \frac{2CK-2}{K}$$

for $c_0 = 1,\ c_1 = 0$

$c_2 = \frac{2c_0}{2} = c_0$

$c_3 = \frac{2c_1}{3} = 0$ $\qquad c_3 = c_5 = c_7 \cdots = 0.$

$c_4 = \frac{2c_2}{4} = \frac{c_2}{2} = \frac{c_0}{2}$

$c_6 = \frac{2c_4}{6} = \frac{c_0}{6}$

$c_8 = \frac{2c_6}{8} = \frac{c_0}{24}$

$c_{10} = \frac{2c_8}{10} = \frac{c_0}{120}.$

so $y_1 = c_0\left(1 + \frac{x^2}{\cancel{2}} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right)$

$y_1 = c_0 (\cancel{\cos}\ e^{x^2})$

linearly independen

for $c_1 = 1,\ c_0 = 0$

$c_0 = c_2 = c_4 = \cdots = 0$

$c_3 = \frac{2c_1}{3}$

$c_5 = \frac{2}{5}c_3 = \frac{4}{3.5}c_1$ $\qquad y_2 = c_1\left(x + \frac{2}{3}x^3 + \frac{2^2}{3.5}x^5 + \frac{2^3}{3.5.7}x^7 + \frac{2^4}{3.5.7.9}x^9 + \cdots\right)$

$c_7 = \frac{2}{7}c_5 = \frac{2^3}{3.5.7}$

$= c_1 \sum_{n=0}^{\infty} x^{2n+1} \times \frac{2^n}{\prod_{i=0}^{n} (2i+1)}.$

$c_9 = \frac{2}{9}c_7 = \frac{2^4}{3.5.7.9}$

$\vdots$

Problem 6 (30 pts)
Use the method of Frobenius to find ONE series solutions of the ODE

$$(x^2 - x)y'' - xy' + y = 0$$

about the regular singular point $x = 0$. Then use reduction of order to find the second solution. (You may use the formula for reduction of order if you wish without the proof.)

since $x=1$ is a singular

converges for

$|x-d|<1$

$|x|<1$

$x \in]-1,1[$.

at least.

let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$

$(x^2 - \cancel{y}) y'' + \sum_{n=0}^{\infty} n+r \quad - xy' + y = 0$

$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$

$\sum_{n=0}^{\infty} c_n x^{n+r} \left((n+r)(n+r-1) - (n+r) + 1 \right) - \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} (n+r-1) = 0$

$\sum_{n=0}^{\infty} c_n x^{n+r} \left((n+r-1)(n+r-1) \right) - \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} = 0.$

$n-1 = k$

$n = k+1$

$\sum_{k=0}^{\infty} c_k x^{k+r} (k+r-1)^2 - \sum_{k=-1}^{\infty} (k+r+1)(k+r) c_{k+1} x^{k+r} = 0$

$x^r \left[\sum_{k=0}^{\infty} (k+r-1)^2 c_k x^k - \sum_{k=0}^{\infty} x^k (k+r+1)(k+r) c_{k+1} x^k - r(r-1) c_0 x^{-1+r} \right] = 0$

so $r(r-1) = 0$, $r = 1$, or $r = 0$

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	8b	Total

$$x^r \sum_{k=0}^{\infty} x^k \left[(k+r-1)^2 c_k - (k+r+1)(k+r)c_{k+1}\right] = 0$$

so for $r=1$

$$k^2 c_k - (k+2)(k+1)c_{k+1} = 0$$

$$c_{k+1} = \frac{k^2}{(k+2)(k+1)} c_k \quad \text{for } k \geq 0.$$

for $c_0 = 1$

for $k=0$, $c_1 = \frac{0^2}{(k+2)(k+1)} c_0 = 0$

$$c_2 = \frac{1^2}{3.2} c_1 = 0$$

$c_3 = c_4 \cdots = 0$ since $(k+2)(k+1)$ is never 0.

so $y_1 = x^1(c_0) = x^1 = x$,

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx \qquad P(x) = \frac{-x}{x^2-x} = \frac{-1}{x-1}$$

$$-\int P(x)\,dx = \int \frac{1}{x-1}\,dx = \ln(|x-1|)$$

$$e^{-\int P(x)\,dx} = e^{\ln(x-1)} = |x-1|.$$

$= 1-x$ around 0.

$$y_2 = y_1 \int \frac{1-x}{x^2}\,dx$$

$$= y_1\left[\frac{-1}{x} - \ln|x|\right] = -1 - x\ln|x|$$

$$\text{so } y = c_1 x + c_2(-1 - x\ln|x|)$$

$$= c_1 x + c_3(1 + x\ln|x|).$$

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Problem 7 (20 pts each)

Use Laplace transforms to solve the following IVP's.

a. $\begin{cases} y'' - 2y' + 2y = \delta(t-1) \\ y(0)=0,\ y'(0)=1 \end{cases}$

$y'' - 2y' + 2y = \delta(t-1)$

$s^2 F(s) - 0 - 1 - 2sF(s) + 2F(s) = e^{-s}$

$F(s)(s^2 - 2s + 2) = 1 + e^{-s}$

$F(s) = \dfrac{1+e^{-s}}{s^2-2s+2} = \dfrac{1-e^{-s}}{(s-1)^2+1}$

$f(t) = \mathcal{L}^{-1}\left(\dfrac{1}{(s-1)^2+1}\right) + \mathcal{L}^{-1}\left(\dfrac{1}{(s-1)^2+1}\right)\Big|_{\text{at } t = t-1} u(t-1)$

$= e^t \sin t + e^t \sin t \Big|_{t=t-1} u(t-1)$

$= e^t \sin t + e^{t-1} \sin(t-1)\, u(t-1)$

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b. $\begin{cases} y'' + 4y' + 3y = f(t) \\ y(0) = 0, y'(0) = 0 \end{cases}$ where $f(t) = \begin{cases} 0 & t < 3 \\ t & t \geq 3 \end{cases}$

$f(t) = t\,u(t-3)$

$s^2 F(s) - 0s - 0 + 4sF(s) + 3F(s) = e^{-3s} \mathcal{L}(t+3)$

$F(s)(s^2+4s+3) = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

$F(s) = e^{-3s}\left(\frac{1}{s^2(s+1)(s+3)} + \frac{3}{s(s+1)(s+3)}\right)$

$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)(s+3)} + \frac{3}{s(s+1)(s+3)}\right)\Bigg|_{T=t-3} u(t-3)$

$\frac{1}{s^2(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)} + \frac{Cs+D}{s^2}$

cover up, $A = \frac{1}{2}$

$B = \frac{-1}{18}$

$s^3A + s^3B + Cs^3 = 0$

$C = -A - B = \frac{-1}{2} + \frac{1}{18} = \frac{-9+1}{18} = \frac{-4}{9}$

$3D = 1$ so $\frac{1}{s^2(s+1)(s+3)} = \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{18} \cdot \frac{1}{s+3} - \frac{4}{9} \cdot \frac{1}{s} + \frac{1}{3} \cdot \frac{1}{s^2}$

$D = \frac{1}{3}$

$\mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)(s+3)}\right) = \frac{1}{2}e^{-t} - \frac{1}{18}e^{-3t} - \frac{4}{9} + \frac{1}{3}t$

page 15 $\longrightarrow$

$$\frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

Cover up, $A = \frac{1}{3}$

$B = \frac{-1}{2}$

$\mathcal{L}$ $C = \frac{1}{6}$

$$\frac{1}{s(s+1)(s+3)} = \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{6}\cdot\frac{1}{s+3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+1)(s+3)}\right) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

$$y = \frac{1}{2}e^{-t} - \frac{1}{18}e^{-3t} - \frac{4}{9} + \frac{1}{3}t + \frac{1}{3} - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

$$= \cancel{-\frac{5}{9}}\ \cancel{-e^{-t}}\ \frac{5}{9} + \frac{t}{3} - e^{-t} + \frac{4}{9}e^{-3t}$$

$t \to t-3$

-2

Problem 8 (16 points each)
Find the general solution of the following systems.

a. $X' = \begin{bmatrix} 10 & -2 \\ 18 & -2 \end{bmatrix} X$

find the eigenvalues of $\begin{pmatrix} 10 & -2 \\ 18 & -2 \end{pmatrix}$

$\begin{vmatrix} 10-\lambda & -2 \\ 18 & -2-\lambda \end{vmatrix} = 0$

$(10-\lambda)(-2-\lambda) + 36 = 0$

$(\lambda+2)(\lambda-10) + 36 = 0$

$\lambda^2 - 8\lambda - 20 + 36 = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda-4)^2 = 0$

$\lambda = 4$ double root.

let K_1 be the corresponding eigenvector.

$\begin{pmatrix} 6 & -2 \\ 18 & -6 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 0$

$6K_1 = 2K_2$

$K_2 = 3K_1$

$K_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{4t}$

find $P\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ such that: $\begin{pmatrix} 6 & -2 \\ 18 & -6 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$6P_1 - 2P_2 = 1$

$\begin{pmatrix} 10 & -2 \\ 18 & -2 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\ominus \begin{cases} 10P_1 - 2P_2 = 1 \\ 18P_1 - 2P_2 = 3 \end{cases}$

$8P_1 = 2$

$P_1 = \frac{1}{4}$.

$P_2 = 5P_1 - \frac{1}{2}$

$= \frac{5}{4} - \frac{2}{4} = \frac{3}{4}$

$P = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$ X -1

$\rightarrow$ page 17.

$$y_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} te^{4t} + \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$$

so $$y = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 1 \\ 3 \end{pmatrix} te^{4t} + \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} \right].$$

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b. $X' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} X$

find the eigenvalues of $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$(\lambda-3)(\lambda+1)+8=0$

$\lambda^2-2\lambda+5=0$

$\Delta = 4-20=-16$

$\lambda_1 = \frac{+2+4i}{2} = +1+2i$

$\lambda_2 = \bar{\lambda}_1 = +1-2i$

$K_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ $\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$

$(2-2i)K_1 - 2K_2 = 0$

$K_2 = (1-i)K_1$

$K_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

$X_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \left[e^t (\cos 2t + i\sin 2t) \right]$

$= e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] (\cos 2t + i\sin 2t) = e^t \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t + i \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t \right] \right)$

$X_1 = e^t \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t + i \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t \right) \right)$

$X_2 = \bar{X}_1 = e^t \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t - i \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t \right) \right)$

let $X_1' = \frac{X_1 + X_2}{2}$

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t$

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t$

$X_2' = \frac{X_1 - X_2}{+2i}$

$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t$

So $X = c_1 X_1' + c_2 X_2'$

$= c_1 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right) + c_2 \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t \right)$