* Problem I:

- The Surface is a portion of the paraboloid we call it a (level curve) where for $z=3$ we get $x^2+y^2=3$ we thus parametrize it:

$$\left\{ \vec{r} = \sqrt{3}\cos(t)\vec{i} - \sqrt{3}\sin(t)\vec{j} + 3\vec{k} \right\}$$

- The circulation is then:

$$\oint F(\vec{r(t)}) \cdot r'(t)\,dt = (-\sqrt{3}\sin(t)\vec{i} + 3\vec{k}) \cdot (-\sqrt{3}\sin(t)\vec{i} - \sqrt{3}\cos(t)\vec{j})$$

$$\oint_{\theta=0}^{\theta=2\pi} (-\sqrt{3}\sin(t))(-\sqrt{3}\sin(t))\,dt = 3\int_0^{2\pi} \sin^2(t)\,dt = \{3\pi\}$$

- On the other hand, we will now show that stokes theorem will yield to same value:

- To find ∇f, apply "Implicit differentiation"

$$\left\{ \nabla f = -2x\vec{i} - 2y\vec{j} + \vec{k} \right\}$$

where n points to the opposite direction _so, we would want the "minus sign" from flux integral, we will now find $\vec{\nabla} \wedge \vec{F}$

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & z \end{vmatrix} = -\vec{k}$$

Then: $$\iint_S (\vec{\nabla} \wedge \vec{F}) \cdot \vec{n} \, d\sigma = \iint_R F \cdot \frac{-\nabla f}{|k \cdot \nabla f|} \, dA$$

$$= \iint_R (-\vec{k}) \cdot (2x\vec{i} + 2y\vec{j} - \vec{k}) \, dA$$

$$= \iint_R dA = \text{area of "}R\text{"}; \ \pi (\sqrt{3})^2 = \{3\pi\}$$

$$\therefore \left\{ \iint_S (\vec{\nabla} \wedge \vec{F}) \cdot \vec{n} \, d\sigma = \oint_C F \cdot dr \right\}$$

* Problem II:

- $x^2+y^2+z^2 \le 1 \Rightarrow (u^2)^2 + (\sqrt{2}\,uv)^2 + (v^2)^2 \le 1$

Then $u^4 + 2u^2v^2 + v^4 \le 1 \Rightarrow \{(u^2+v^2)^2 \le 1\}$

which means: $\{u^2+v^2 \le 1\}$

- now find $\vec{r_u}$ and $\vec{r_v}$ and $|\vec{r_u} \wedge \vec{r_v}|$

$\{\vec{r_u} = 2u\vec{i} + \sqrt{2}\,v\vec{j}\}$ (1)

$\{\vec{r_v} = \sqrt{2}\,u\vec{j} + 2v\vec{k}\}$ (2)

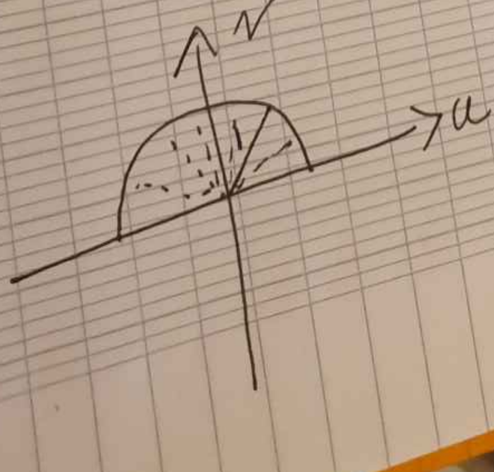
$\vec{r_u} \wedge \vec{r_v} = \dots = 2\sqrt{2}\,v^2\vec{i} - 4uv\vec{j} + 2\sqrt{2}\,u^2\vec{k}$

$\{|\vec{r_u} \wedge \vec{r_v}| = 2\sqrt{2}\,(u^2+v^2)\}$ (3)

Then: Surface Area $= \iint_R |\vec{r_u} \wedge \vec{r_v}|\,du\,dv$

$= 2\sqrt{2} \iint_R (u^2+v^2)\,du\,dv = \int_0^{\pi}\int_0^1 2\sqrt{2}\cdot r^2 \cdot r\,dr\,d\theta$

$= \int_0^{\pi} \frac{1}{\sqrt{2}}\,d\theta = \left\{\frac{\pi}{\sqrt{2}}\right\}$



Take $(dx)^2$ common

* Problem III:

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\left(\frac{dy}{dx}\right)^2 = 4x^2$$

$$\text{- mass} = \int_{(P)} x\,ds = \int_0^1 x\sqrt{(dx)^2+(dy)^2} = \int_0^1 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\text{- mass} = \int_0^1 x\sqrt{1+4x^2}\,dx = \left\{\frac{1}{12}(5\sqrt{5}-1)\right\}$$

* Problem IV :

a) $\oint_C F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$

- $\vec{r}(t) = \cos(\theta)\,\vec{i} + \sin(\theta)\,\vec{j}$
- $\vec{r}'(t) = -\sin(\theta)\,\vec{i} + \cos(\theta)\,\vec{j}$
- $F(\vec{r}(t)) = \cos^2(\theta)\sin(\theta)\,\vec{i}$

$$\int_0^{2\pi} \left(\cos^2(\theta)\sin(\theta)\right) \cdot \left(-\sin(\theta)\right) \cdot (1)\, dt$$

$$= -\int_0^{2\pi} \cos^2(\theta)\sin^2(\theta) = \frac{1}{8}\theta - \frac{1}{32}\sin(4\theta)\Big|_0^{2\pi}$$

$$= \left\{\frac{\pi}{4}\right\}$$

b) Green's Theorem :

$$\oint \vec{F} \cdot d\vec{r} = \iint_R (M_y - N_x)\, dA \Big/ \begin{cases} M_y = x^2 \\ N_x = 0 \end{cases}$$

$$\rightarrow \iint_R x^2\, dA \rightarrow \int_0^{2\pi}\int_0^1 r\,\cos^2(\theta)\, dr\, d\theta = \frac{\pi}{4}$$

c) Stokes's Theorem:

$$\iint_S (\nabla \wedge \vec{F}) \cdot \frac{\nabla f}{|\nabla f \cdot \rho|}$$

$$(\Delta \wedge \vec{F}) = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 2y^3 z & 3z \end{vmatrix} = \ldots$$

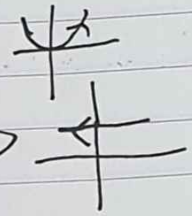
$$\nabla f = \nabla f\left(z + \sqrt{x^2+y^2} - 1 = 0\right) = \begin{cases} f_x = \ldots \\ f_y = \ldots \\ f_z = \ldots \end{cases}$$

$$\rho = \vec{K}$$

$$\ldots = \frac{\pi}{4}$$

* Problem V :

a) write $C = C_1 + C_2$ / $C_1 \to$ $C_2 \to$



$$\begin{cases} (C_1) \equiv \vec{r}(t) = t\vec{i} + t^2\vec{j} \\ (C_2) \equiv \vec{r}(t) = -t\vec{i} + 4\vec{j} \end{cases}$$ knowing that $-2 < t \leq 2$

- Note : other paramatrzations are possible

then: $$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{t=-2}^{t=2} (y+1, x+e^y)\cdot dr$$

(1) $$\int_{-2}^{2} \left[(t^2+1)\vec{i} + (t+e^{t^2})\vec{j}\right]\cdot(1\vec{i} + 2t\vec{j})\,dt$$

$+$

(2) $$\int_{-2}^{2} \left(5\vec{i} + (e^4 - t)\vec{j}\right)\cdot -\vec{i}\,dt$$

$$(1)+(2) ,,,, = \int_{-2}^{2} (t^2 + 1 + 2t^2 + \underbrace{2te^{t^2}}_{du \to u})\,dt + \int_{-2}^{2} -5\,dt$$

$$,,,, = \boxed{0}$$

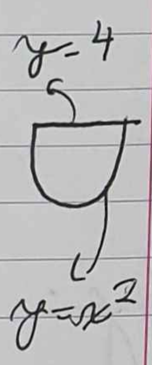
b) Since:

i) C is traversed counterclockwise

ii) C is a simple closed curve

. Green's Theorem can be applied:

$$\int_C \vec{G}\cdot d\vec{r} = \iint_R (N_x - M_y)\,dA = \iint (0-1)\,dA$$



$$= \int_{-2}^{2}\int_{x^2}^{4} (-1)\,dy\,dx = \left\{-\frac{32}{3}\right\}$$

c) From {b} $\vec{G}$ satisfies $N_x - M_y = -1 \neq 0$ so $\vec{G}$ is not conservative, thus $\vec{F}$ is conservative

or $\int_C \vec{G}\cdot d\vec{r} = -32/3 \neq 0$ $\therefore$ not conservative

$$\vec{\nabla} f = \vec{F} = (y+1,\, x+e^y) \rightarrow \begin{cases} \frac{\partial f}{\partial x} = y+1 \\ \frac{\partial f}{\partial y} = x+e^y \end{cases}$$

Integrate wrt x :

$$f = \int (y+1)\,dx = xy + x + K(y)$$

$$\frac{\partial f}{\partial y} = \cancel{x} + e^y = \cancel{x} + 0 + \frac{\partial k}{\partial y}$$

$$\Rightarrow \frac{\partial k}{\partial y} = e^y \Rightarrow K = e^y + C$$

$$\left\{\left\{ \therefore f = xy + x + e^y + C \right\}\right\}$$

d) For $\vec{G} = (M, N)$, Green Theorem Say:

$$\oint_C \vec{G}\cdot\vec{n}\,ds = \iint_{R'} (M_x + N_y)\,dA = \iint_{R'} (1+e^y)\,dA$$

since $1 + e^y > 0 \ \forall y \in \mathbb{R}$. Then :

$$\iint_{R'} (1+e^y)\,dA > 0$$

{Thus G will always have (+ve) flux}

* Problem VI :

- Voltage : $V = \oint_c E \cdot dr$

- Flux : $\Phi = \oiint_s H \cdot n \cdot d\sigma$

- Maxwell's Equations : $\nabla \times E = -\frac{\partial H}{\partial t}$

- Faraday's Law : $V = -\frac{\partial \Phi}{\partial t}$

$$\Longrightarrow \oint_c E dr = -\frac{\partial \left[\oiint_s H \cdot n \cdot d\sigma \right]}{\partial t}$$

$$\Longrightarrow \oint_c E dr = \oiint_s -\underbrace{\frac{\partial H}{\partial t}} \cdot n \cdot d\sigma$$

$$\Longrightarrow \oint_c E dr = \oiint_s (\nabla \times E) \cdot n \cdot d\sigma$$

$$\therefore \left\{ \oiint_s (\nabla \times E) \cdot n \cdot d\sigma \right\}$$

is Faradays law where $\nabla \times E$ follows from maxwell's equation of $\left\{ \nabla \times E = -\frac{\partial H}{\partial t} \right\}$.