

Lecture 4 : Conditional Probability (§ 2.4)

Conditional probability provides us with a way to reason about the outcome of an experiment based on partial information.

Ex. What is the probability of drawing a queen from a deck of cards? $4/52 = 1/13$

What is the probability of drawing a queen if someone tells me the card drawn is a face card (J, Q, K, A)?

$$4J, 4Q, 4K, 4A \rightarrow P(Q) = \frac{4}{16} = \frac{1}{4}$$

We define the probability of an event A given some other event B as $P(A|B)$ \rightarrow This is the conditional probability of event A occurring given that event B has occurred.

$P(A|B)$ for different events A should constitute a legitimate probability law that satisfies the axioms.

Ex. Consider rolling a die $\Rightarrow \Omega = \{1, 2, 3, 4, 5, 6\}$.

If the die is not weighted, all outcomes are equally likely.

If we are told that the outcome is even, then we are left with $\{2, 4, 6\}$. Since these were equally likely to begin with, they should remain equally likely.

Therefore, $P(\text{outcome is } 4 | \text{outcome is even}) = \frac{1}{3}$.

This suggests that an appropriate definition of conditional probability when all outcomes are equally likely is given by :

$$P(A|B) = \frac{\text{number of elements in } A \cap B}{\text{number of elements in } B}$$

Generally :

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{with } P(B) > 0$$

Note: Conditional probabilities constitute a probability law.

$$1. P(A|B) \geq 0$$

$$2. P(\neg A|B) = \frac{P(\neg A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

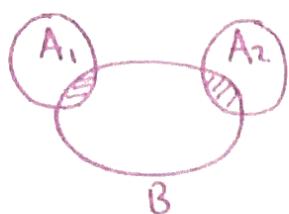
$$3. \underbrace{P(A_1 \cup A_2 | B)}_{A_1 \text{ and } A_2 \text{ are disjoint}} = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

note that $A_1 \cap B$ and $A_2 \cap B$ are disjoint

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 | B) + P(A_2 | B)$$



Conditional probabilities can be viewed as a probability law on a new universe B .

Ex. We toss a fair coin three times.

$$A = \{\text{more heads than tails come up}\}$$

$$B = \{1^{\text{st}} \text{ toss is head}\}$$

Find $P(A|B)$.

$$\Omega = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

$$A = \{\text{HHH, HHT, HTH, THH}\}$$

$$B = \{\text{HHH, HHT, HTH, HTT}\}$$

$$P(A) = 4/8, P(B) = 4/8, P(A \cap B) = 3/8$$

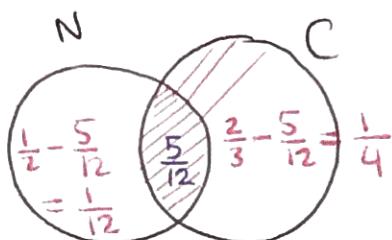
$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/8}{4/8} = 3/4$$

Ex. A conservative design team , call it C, and an innovative design team, call it N, are asked to separately design a new product within a month. From past experience we know that :

- $P(C \text{ is successful}) = 2/3$
- $P(N \text{ is successful}) = 1/2$
- $P(\text{at least one team is successful}) = 3/4 \leftarrow P(NUC)$

IF both C and N are successful \Rightarrow adopt N.

Assuming exactly one successful design is produced, what is the probability that it was designed by team N?



$$\begin{aligned} P(N \cap C) &= P(N) + P(C) - P(NUC) \\ &= \frac{1}{2} + \frac{2}{3} - \frac{3}{4} \\ &= \frac{5}{12} \end{aligned}$$

$$P(\text{exactly one team successful}) \leq \left\{ \begin{array}{l} P(\text{only } N \text{ successful}) = \frac{1}{12} \\ P(\text{only } C \text{ successful}) = \frac{1}{4} \end{array} \right. \\ = \frac{1}{12} + \frac{1}{4} = \frac{4}{12}$$

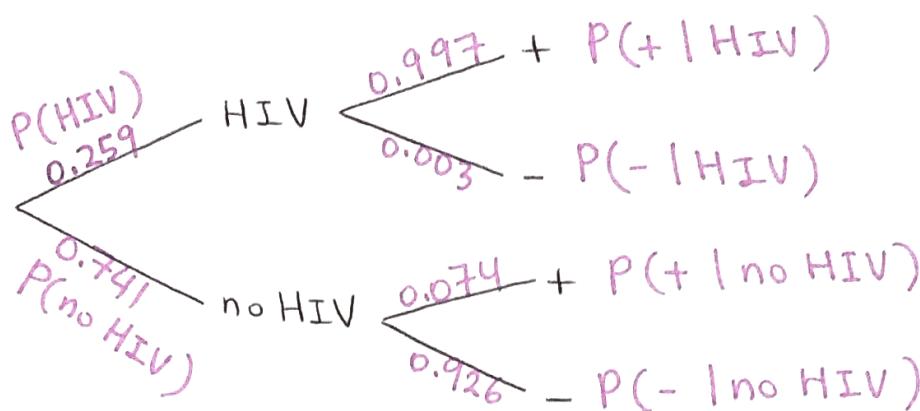
$$\begin{aligned} \text{Therefore, } P(N \text{ successful} | \text{exactly one team successful}) \\ = \frac{P(N \text{ successful AND exactly one team successful})}{P(\text{exactly one team successful})} \end{aligned}$$

$$= \frac{\frac{1}{12}}{\frac{4}{12}} = \frac{1}{4}$$

Note: Since $P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(B)P(A|B)$

Ex. HIV prevalence in Swaziland is 25.9% (in 2011).

ELISA test : 99.7% accuracy for those who carry HIV
92.6% accuracy for those who do not



What is the probability a person from Swaziland tests positive?

$$\begin{aligned}
 P(+) &= P(+ \cap \text{HIV}) + P(+ \cap \text{no HIV}) \\
 &= P(\text{HIV})P(+ | \text{HIV}) + P(\text{no HIV})P(+ | \text{no HIV}) \\
 &= 0.259(0.997) + 0.741(0.074) \\
 &= 0.258 + 0.055 = 0.313
 \end{aligned}$$

If we are dealing with an event A which occurs if and only if A_1, A_2, \dots, A_n has occurred, i.e., $A = A_1 \cap \dots \cap A_n$, then the probability of A is given by :

$$\begin{aligned}
 P(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n) &= P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2) \dots P(A_n | \bigcap_{i=1}^{n-1} A_i) \\
 &= P(A_1) \cdot \frac{P(A_2 \cap A_1)}{P(A_1)} \cdot \dots \cdot \frac{P(\bigcap_{i=1}^n A_i)}{P(\bigcap_{i=1}^{n-1} A_i)}
 \end{aligned}$$

Ex. Deck of cards \rightarrow 3 cards drawn without replacement. Find $P(\text{none is a heart})$.

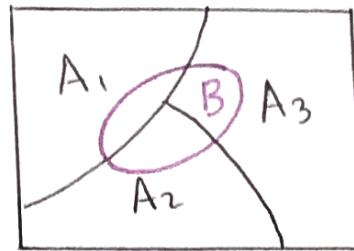
$$\begin{aligned}
 A_1 &= \{\text{card 1 not heart}\} \Rightarrow P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2) \\
 A_2 &= \{\text{card 2 not heart}\} \\
 A_3 &= \{\text{card 3 not heart}\} \\
 &= \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50}
 \end{aligned}$$

Total Probability theorem and Bayes' Rule

Total probability theorem

Let A_1, A_2, \dots, A_n be disjoint events that form a partition of the sample space Ω , and assume $P(A_i) > 0$. Then, for any event B

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + \dots + P(A_n \cap B) \\ &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \dots + P(A_n)P(B|A_n) \end{aligned}$$



Ex. You enter a chess tournament.

$$P(\text{winning against half the players}) = 0.3$$

type 1

$$P(\text{winning against quarter of the players}) = 0.4$$

type 2

$$P(\text{winning against the remaining quarter}) = 0.5$$

type 3

You play a game against a randomly chosen opponent. What is the probability of winning?

Let $A_i = \{\text{event of playing against type } i\}$

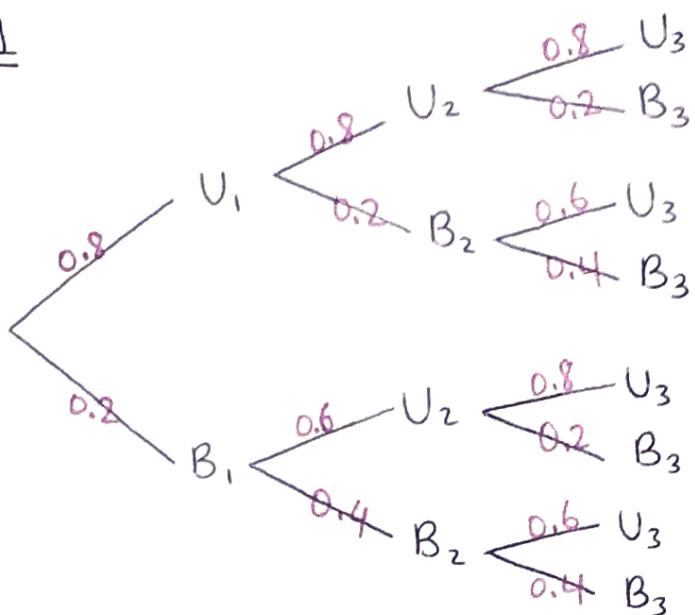
$$\Rightarrow P(A_1) = 0.5, P(A_2) = 0.25, P(A_3) = 0.25$$

Let $B = \{\text{event of winning}\}$

$$\begin{aligned} \Rightarrow P(B) &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) \\ &= 0.5(0.3) + 0.25(0.4) + 0.25(0.5) \\ &= 0.375 \end{aligned}$$

Ex. Alice is taking a probability class. At the end of each week, she can either be up to date, or she may have fallen behind. If she is up to date in a given week, then the probability she will be up to date in the following week is 0.8. If she is behind in a given week, then the probability she is up to date the next week is 0.6. Alice is up to date when she starts class. What is the probability she is up to date after three weeks?

Method 1



$$\begin{aligned}
 P(U_3) &= 0.8 \cdot 0.8 \cdot 0.8 + 0.6 \cdot 0.2 \cdot 0.8 + 0.8 \cdot 0.6 \cdot 0.2 + 0.6 \cdot 0.4 \cdot 0.2 \\
 &= 0.752
 \end{aligned}$$

Method 2

$$\begin{aligned}
 P(U_3) &= P(U_2) \underbrace{P(U_3 | U_2)}_{0.8} + P(B_2) \underbrace{P(U_3 | B_2)}_{0.6} \\
 &= [P(U_1) P(U_2 | U_1) + P(B_1) P(U_2 | B_1)] \cdot 0.8 \\
 &\quad + [P(U_1) P(B_2 | U_1) + P(B_1) P(B_2 | B_1)] \cdot 0.6 \\
 &= [0.8 \cdot 0.8 + 0.2 \cdot 0.6] \cdot 0.8 + [0.8 \cdot 0.2 + 0.2 \cdot 0.4] \cdot 0.6 \\
 &= 0.752
 \end{aligned}$$

Bayes' Rule

Recall that $P(A_i | B) = \frac{P(A_i \cap B)}{P(B)} \Rightarrow P(B|A_i) = \frac{P(B \cap A_i)}{P(A_i)}$

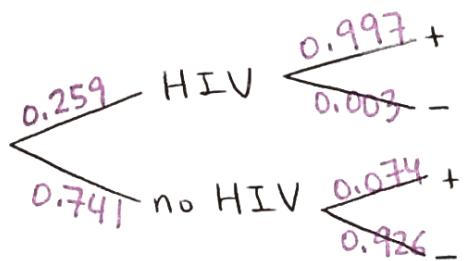
Since $P(A_i \cap B) = P(B \cap A_i) \Rightarrow P(A_i | B) = \frac{P(B|A_i)P(A_i)}{P(B)}$

Using the total probability theorem

$$\Rightarrow P(A_i | B) = \frac{P(B|A_i)P(A_i)}{P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n)}$$

where A_1, \dots, A_n are disjoint events that form a partition of Ω , with $P(A_i) > 0$.

Ex. Return to HIV example: What is the probability that a person who tested positive carries HIV?



$$\begin{aligned}
 P(\text{HIV} | +) &= \frac{P(+ | \text{HIV}) P(\text{HIV})}{P(+)} \\
 &= \frac{P(+ | \text{HIV}) P(\text{HIV})}{P(\text{HIV}) P(+ | \text{HIV}) + P(\text{no HIV}) P(+ | \text{no HIV})} \\
 &= \frac{0.997 (0.259)}{0.258 + 0.055} = 0.825
 \end{aligned}$$

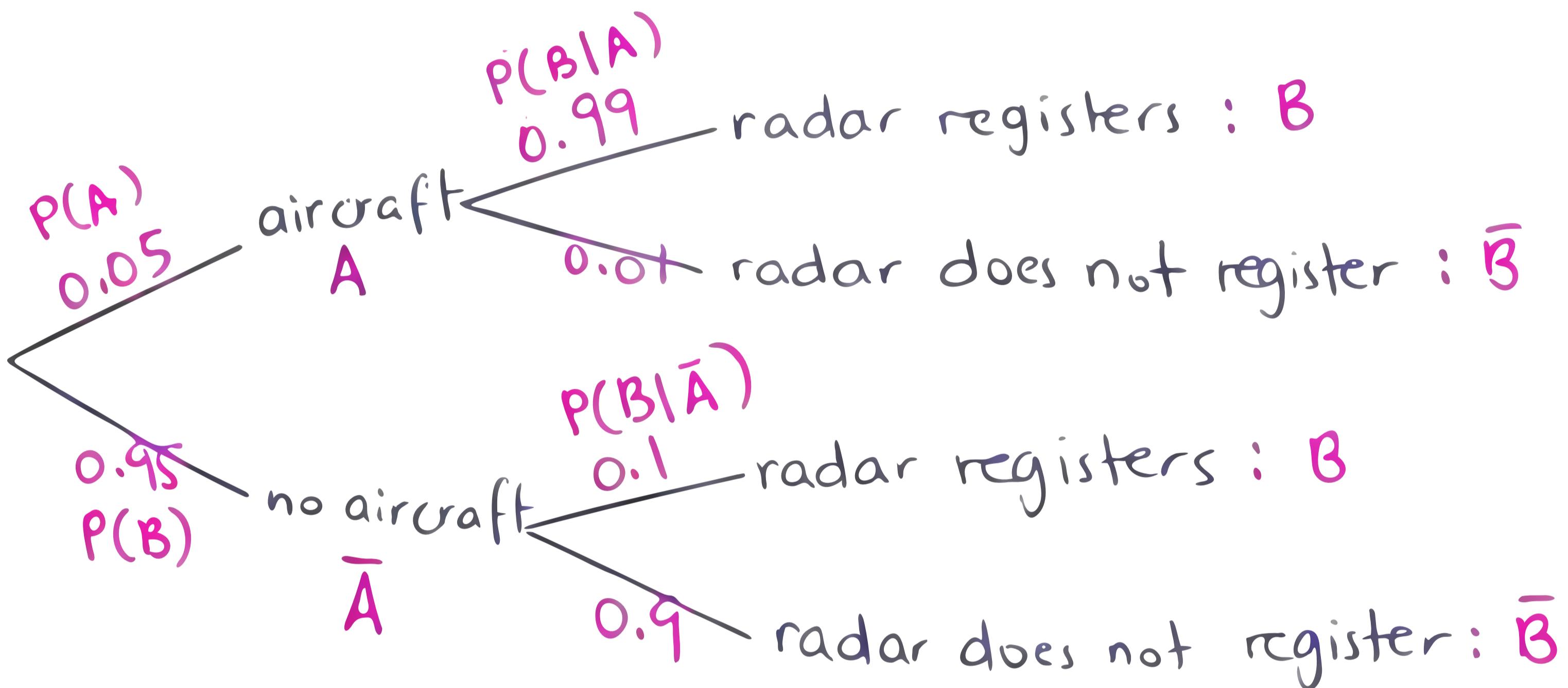
Ex. (Radar detection)

If an airplane is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.1. Assume that an aircraft is present with probability 0.05. If the radar registers something, what is the probability that an aircraft is present?

1st step : what are you asked to compute ?

$$P(\text{aircraft} \mid \text{radar registers})$$

2nd step : tree



$$P(\text{aircraft} \mid \text{radar registers}) = P(A \mid B)$$

$$P(A \mid B) = \frac{P(B \mid A) P(A)}{P(B)} = \frac{0.99(0.05)}{0.99(0.05) + 0.1(0.95)}$$

$$= 0.34$$

Therefore, if the radar registers something, there's only 34% chance that there's actually an aircraft.