LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics

Calculus III

Exam II

Fall 2012 (December 3, 2012)

Circle the name of your instructor:

Dr. Hamdan **Dr. Issa** **Dr. Touma**

Exam duration: 75 minutes

Name: ______________________ **ID:** ______________________

QUESTION	GRADE
1. 32%	
2. 24%	
3. 16%	
4. 8%	
5. 8%	
6. 12%	
TOTAL	

1. (32%) Consider the following series and determine whether they converge or diverge. Show work and mention the test you use.

(a) $\sum_{n=1}^{\infty} \left(\frac{n}{n-3}\right)^n$

$$\lim_{n\to\infty} \left(\frac{n}{n-3}\right)^n = \lim_{n\to\infty} \left(\frac{n+3-3}{n-3}\right)^n$$

$$= \lim \left(1 + \frac{3}{n-3}\right)^n = \lim \left(1+\frac{3}{n-3}\right)^{n-3} \cdot \left(1+\frac{3}{n-3}\right)^3$$

$$= e^3 \cdot 1 \cdot \neq 0$$

$\Rightarrow$ Diverges by n^{th} term test.

8pts

Test~~
Divergence
limit x
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(b) $\sum_{n=1}^{\infty} \frac{4(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{4}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{4}(n!)^2}$$

$$= \lim \frac{(n+1)^2}{(2n+2)(2n+1)} \sim \frac{n^2}{4n^2} = \frac{1}{4} < 1$$

Converges by ratio test.

8pts

pped
$\to 0$
pts

[forgot the 4/4
6 pts]

(c) $\sum_{n=1}^{\infty} \frac{5}{5n - 2\sqrt{n}}$

LCT with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5}{5n-2\sqrt{n}} \cdot n \sim \frac{n}{n} = 1$$ 8pts

$\Rightarrow$ Series behave alike

$\Rightarrow$ Since $\sum \frac{1}{n}$ diverge (p-series $p=1$) $\Rightarrow$ original diverge by LCT.

(d) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^6}$

Root test: $\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^6}} = \lim_{n\to\infty} \frac{n!}{n^6} = +\infty$

$\Rightarrow$ Series diverge

8pts

2. (24%) Consider the following alternating series and determine whether they converge absolutely, conditionally or they diverge.

(a) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{1+n\ln n}$ $=$ first $-1 + \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n\ln n}$, -1 does not affect anything

Second Absolute Test $\sum_{2}\left|(-1)^n\frac{1}{1+n\ln n}\right| = \sum_{2}\frac{1}{1+n\ln n}$

$\boxed{12 \text{ pts}}$

Let $f(x) = \frac{1}{1+x\ln x}$

$f > 0$
f decreasing because $1+x\ln x$ increasing
$f \to 0$ as $x \to \infty$

Integral test applies

$\int_2^{\infty}\frac{1}{1+x\ln x}dx$ first LCT with $\int_2^{\infty}\frac{1}{x\cdot\ln x}$ behaves the same

then let $u = \ln x$ $\int\frac{1}{u}du = \ln u + C$

$\Rightarrow \int_2^{\infty}\frac{1}{x\ln x}dx = \ln\ln x\Big|_2^{\infty} \to \infty$ diverges

$\Rightarrow$ Original is NOT absolutely convergent.

Back to $\sum_{1}^{\infty}(-1)^n\frac{1}{1+n\ln n}$, Leibniz applies $\Rightarrow$ series is conditionally convergent

(b) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n+\sqrt{n}}{n^3+4}\right)$

Absolute test $\sum\frac{n+\sqrt{n}}{n^3+4}$

LCT with $b_n = \frac{1}{n^2}$

$\boxed{6 \text{ pts}}$

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim\frac{n+\sqrt{n}}{n^3+4}\cdot n^2 \sim \frac{n^3}{n^3} = 1$

$\Rightarrow$ Series behave like $\sum\frac{1}{n^2}$, convergent p-series

$\Rightarrow$ Original is absolutely convergent.

(c) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{4n+4}{3n+5}\right)$

Let $u_n \Rightarrow \frac{4n+4}{3n+5}$ $\quad \lim_{n\to\infty} u_n = 4/3 \Rightarrow \lim (-1)^n u_n$ DNE

$\Rightarrow$ Series is divergent by n^{th} term test.

$\boxed{6\text{pts}}$

3. (16%) Consider the power series: $\sum_{n=1}^{\infty}\frac{(x-7)^n}{n^3 4^n}$

(a) For what values of x does the power series converge?

Compute $\rho(x) = \lim_{n\to\infty}\left|\frac{(x-7)^{n+1}}{(n+1)^3 \cdot 4^{n+1}} \cdot \frac{4^n \cdot n^3}{(x-7)^n}\right|$

$= \lim_{n\to\infty} |x-7|\left(\frac{n}{n+1}\right)^3 \cdot \frac{1}{4}$

$= |x-7|/4$

$\boxed{8\text{pts}}$

for absolute convergence, need $\rho(x) < 1$

so $|x-7|/4 < 1 \Rightarrow -4 < x-7 < 4 \Rightarrow \boxed{3 < x < 11}$

End-points $X = 3 \Rightarrow \sum \frac{(-4)^n}{n^3 \cdot 4^n} = \sum \frac{(-1)^n}{n^3}$ also absolutely convergent

$x = 11 \Rightarrow \sum \frac{4^n}{n^3 \cdot 4^n} = \sum \frac{1}{n^3}$ //

$\Rightarrow$ for $x \in 3 \le x \le 11$ absolute Conv

elsewhere, divergent

(b) Set $x = 5$ in the power series above and approximate/estimate the sum with an error of no more than 10^{-3} : *(in other words, how many terms do you need to use to reach that estimation?)*

for $x=5$, we obtain $\sum \frac{(-2)^n}{n^3 4^n}$ the alternating convergent series.

$\boxed{8pts}$

(We can do that because $5 \in [3,11]$ The interval of convergence found in a)

Which means we can apply the estimation theorem. By trying, we find that $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3 4^n} \sim -1/2 + \frac{1}{2^3 . 2^2} - \frac{1}{3^3 . 2^3}$ with $|\text{error}| < \frac{1}{4^3 . 2^4} = 9.76 \times 10^{-4} < 10^{-3}$

(need 3 terms)

4. (8%) Find the Maclaurin series for $f(x) = \frac{\sin(x^2) + e^{2x} - 1}{x}$.(You may use any known series that we have seen in class)

$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ all $x \in \mathbb{R}$

$\sin(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}$ all $x^2 \in \mathbb{R} \Rightarrow$ all $x \in \mathbb{R}$

$e^x = \sum_{k=0}^{\infty} x^k/k!$ all $x \in \mathbb{R}$

8 pts

$e^{2x} = \sum_{k=0}^{\infty} \frac{2^k . x^k}{k!}$ all $x \in \mathbb{R}$

$e^{2x} - 1 = \sum_{k=1}^{\infty} 2^k . x^k/k!$

$\frac{\sin(x^2) + e^{2x} - 1}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(2k+1)!} + \sum_{k=1}^{\infty} \frac{2^k x^{k-1}}{k!}$

4x4x4 2x2x2x2 16 64 16 2 384 64 1024 10^3 1000

5. (8%) Same question for $\frac{1}{(1-10x)^2}$.

$$\left(\frac{1}{1-10x}\right)' = \frac{10}{(1-10x)^2}$$

$$\Rightarrow \left(\frac{1}{1-10x}\right)^2 = \frac{1}{10}\left(\frac{1}{1-10x}\right)'$$

Start with $\frac{1}{1-\cancel{x}} = \sum_{k=0}^{\infty} x^k \quad |x|<1$

8 pts

Substitute $\frac{1}{1-10x} = \sum_{k=0}^{\infty} 10^k x^k \quad |x| < 1/10$

differentiate $\left(\frac{1}{1-10x}\right)' = \sum_{k=1}^{\infty} 10^k \cdot k \cdot x^{k-1}$

multiply by 1/10 $\quad \frac{1}{10}\left(\frac{1}{1-10x}\right)' = \frac{1}{(1-10x)^2} = \boxed{\frac{1}{10}\sum_{k=1}^{\infty} 10^k \cdot k \cdot x^{k-1}}$

for $|x| < 1/10$

6. (12%) Consider $\sum_{n=0}^{\infty}(-1)^n \frac{1}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$

(a) Find the exact sum.

$$\sum_{k=0}^{\infty} \frac{\cancel{x}^k}{k!} = e^x \quad \forall x \in \mathbb{R}$$

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Substitute $x = -1 \quad (\in \mathbb{R})$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = 1/e$$

(b) Consider the estimation: $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!} \approx 1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}$.

1. Find the exact error.

$$\text{Error} = \frac{1}{e} - \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

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2. Find the estimated error.

This is a convergent alternating series.

$|\text{Error}| < u_{n+1} = |-1/5!| = 1/5!$

by estimation theorem.

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