

**Problem 1:** (10%) Evaluate the following limits

(a)  $\lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{x^2}{1+2x^2}\right)$

$$\lim_{x \rightarrow \infty} \frac{x^2}{1+2x^2} = \lim_{x \rightarrow \infty} \frac{x^2}{x^2(\frac{1}{x^2} + 2)} = \frac{1}{2} \quad \text{5 pts}$$

$\sin^{-1}$  is a continuous function near  $1/2 \Rightarrow$  value

$$\lim_{x \rightarrow \infty} \sin^{-1}\left(\frac{x^2}{1+2x^2}\right) = \boxed{\sin^{-1} 1/2} = \frac{\pi}{6} \leftarrow \text{But this does not matter}$$

(b)  $\lim_{x \rightarrow \infty} \frac{\tan^{-1}(e^x)}{e^{2x} + x}$

Note first: this is NOT the form for L'Hôpital's rule.

$$\lim_{x \rightarrow \infty} \tan^{-1}(e^x) = \pi/2 \quad \text{since} \quad \lim_{x \rightarrow \infty} e^x = \infty \quad \& \quad \lim_{x \rightarrow \infty} \tan^{-1} x = \pi/2$$

$$\lim_{x \rightarrow \infty} e^{2x} + x = \infty$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\tan^{-1}(e^x)}{e^{2x} + x} = 0$$

5 pts

**Problem 3:** (20%) Evaluate the following integrals

(a)  $\int \tan^{-1}(4x) dx$

Integration by parts

$$u = \tan^{-1}(4x) \quad dv = 1$$

$$du = \frac{4}{1+16x^2}$$

$$v = x \quad 10 \text{ pts}$$

$$\int \tan^{-1}(4x) dx = x \tan^{-1}(4x) - 4 \int \frac{x}{1+16x^2} dx$$

substitution ←

$$\text{let } z = 1+16x^2 \\ dz = 32x dx$$

$$= x \tan^{-1}(4x) - \frac{4}{32} \int \frac{dz}{z}$$

$$= x \tan^{-1}(4x) - \frac{1}{8} \ln|z| + C \quad \xrightarrow{\text{back to } x} \boxed{x \tan^{-1}(4x) - \frac{1}{8} \ln(1+16x^2) + C}$$

(b)  $\int \frac{1}{x^2+4x+5} dx$

Complete the square in the denominator to reach a known form:

$$\int \frac{1}{x^2+4x+5} dx = \int \frac{1}{x^2+4x+4-4+5} dx = \int \frac{1}{(x+2)^2+1} dx$$

$$= \tan^{-1}(x+2) + C$$

10 pts

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Problem 4: (14%) Evaluate the following improper integrals

(a)  $\int_2^4 \frac{1}{x^2-x-2} dx$

$$\int_2^4 \frac{1}{x^2-x-2} dx = \int_2^4 \frac{dx}{(x-2)(x+1)} \rightarrow \text{denominator is zero at 2, integrand has an asymptote there}$$

$$= \lim_{b \rightarrow 2^+} \int_b^4 \frac{dx}{(x-2)(x+1)}$$

7pts

Now do  $\int_b^4 \frac{dx}{(x-2)(x+1)}$  by partial fractions

$$\int_b^4 \frac{dx}{(x-2)(x+1)} = \int_b^4 \frac{1/3}{x-2} dx + \int_b^4 \frac{-2/3}{x+1} dx = \frac{1}{3} \ln(x-2) \Big|_b^4 - \frac{2}{3} \ln(x+1) \Big|_b^4$$

+ve for  $x > 2$

$$= \frac{1}{3} (\ln 2 - \ln(b-2)) - \frac{2}{3} (\ln 5 - \ln(b+1))$$

$$\lim_{b \rightarrow 2^+} \int_b^4 \frac{dx}{(x-2)(x+1)} = \lim_{b \rightarrow 2^+} -\ln(b-2) + \text{constants} = +\infty$$

integral is Divergent

(b)  $\int_{-\infty}^{\infty} \frac{1}{(e^x+e^{-x})} dx$

$$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}} dx = 2 \int_0^{\infty} \frac{1}{e^x+e^{-x}} dx \quad (\text{even integrand})$$

$$= 2 \lim_{b \rightarrow \infty} \int_0^b \frac{1}{e^x+e^{-x}} dx$$

7pts

$$\int \frac{1}{e^x+e^{-x}} dx = \int \frac{e^x dx}{(1+e^{2x})} \quad \text{let } u = e^x, du = e^x dx$$

$$= \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1} e^x + C$$

$$\Rightarrow \lim_{b \rightarrow \infty} \int_0^b \frac{1}{e^x+e^{-x}} dx = \lim_{b \rightarrow \infty} \left[ \tan^{-1} e^x \right]_0^b = \frac{\pi}{2} - \tan^{-1} e^0$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}} dx = 2 \cdot \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$$

obviously Convergent

**Problem 5:** (14%) Determine the convergence or divergence of the following improper integrals. Justify your answers.

(a)  $\int_0^{\infty} \frac{1}{(e^x+1)^2} dx$

MANY WAYS TO DO THIS, one of which:

$$(e^x+1)^2 \geq (e^x)^2 = e^{2x} \quad \forall x$$

$$\Rightarrow \int_0^{\infty} \frac{1}{(e^x+1)^2} dx \leq \int_0^{\infty} e^{-2x} dx = \lim_{b \rightarrow \infty} \left[ -\frac{e^{-2x}}{2} \right]_0^b = \frac{1}{2}$$

By DCT, original integral converges.

7pts

Remark: It's NOT true that  $(e^x+1)^2 \geq e^{-x^2}$

(b)  $\int_1^{\infty} \frac{\ln x}{e^x} dx$

ALSO MANY WAYS to do this.

$$\ln x \leq e^{x/2} \quad x \geq 1$$

7pts

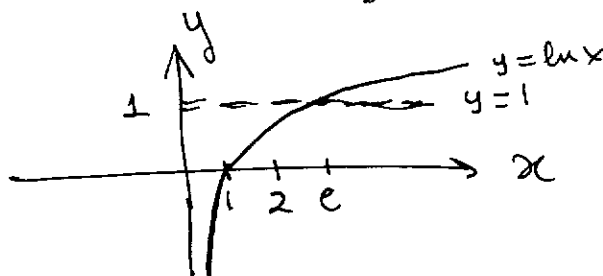
$$\int_1^{\infty} \frac{\ln x}{e^x} dx \leq \int_1^{\infty} \frac{e^{x/2}}{e^x} dx = \int_1^{\infty} e^{-x/2} dx = \lim_{b \rightarrow \infty} \left[ -2e^{-x/2} \right]_1^b = 2e^{-1/2}$$

$\downarrow$   
 finite  
 Convergent

$\leadsto$  By DCT, original integral converges

• can also compare top & bottom to  $x^p$  (chosen carefully)

• Remark:  $\ln x$  does not lie below 1 for all  $x$ !



10/1.  
Problem 6: (10%) Show that

$$\int_1^{\infty} \frac{\sin x + 2}{x^2} dx$$

converges, whereas

$$\int_1^{\infty} \frac{\sin x + 2}{x} dx$$

diverges.

First integral

$$0 \leq \sin x \leq 1$$

$$\Rightarrow 2 \leq \sin x + 2 \leq 3$$

$$\Rightarrow 0 \leq \int_1^{\infty} \frac{\sin x + 2}{x^2} dx \leq \int_1^{\infty} \frac{3}{x^2} dx$$

By DCT, 1<sup>st</sup> integral converges

5 pts  
this is a p-integral  
 $p = 2 \Rightarrow$  it converges

NOTE: You can't ~~conclude from~~ SANDWICH THEOREM here <sup>applied</sup> to  $\frac{\sin x + 2}{x^2}$  that the integral converges!

Second integral:

$$-1 \leq \sin x$$

$$\Rightarrow 1 \leq \sin x + 2$$

$$\Rightarrow \int_1^{\infty} \frac{1}{x} dx \leq \int_1^{\infty} \frac{\sin x + 2}{x} dx$$

Divergent p-integral

By DCT, second integral diverges

5 pts  
NOTE: You can't bound  $\int \frac{\sin x + 2}{x} dx$  above and conclude it's divergent!

**Problem 7:** (12%)

Find the values of  $p$  for which

$$\int_1^{\infty} \frac{x}{\sqrt{x^p+1}} dx$$

converges.

Justify your answer.

The integrand behaves like  $\frac{x}{\sqrt{x^p}}$  so we can apply LCT with

$$f(x) = \frac{x}{\sqrt{x^p+1}} \quad \& \quad g(x) = \frac{x}{\sqrt{x^p}} = x^{1-p/2}$$

$$\bullet \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^p}}{\sqrt{x^p+1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{x^p}{x^p+1}} = 1$$

this means the integrals behave EXACTLY alike (convergence or divergence)

$$\text{Now } \int_1^{\infty} g(x) dx = \int_1^{\infty} x^{1-p/2} dx = \int_1^{\infty} \frac{1}{x^{p/2-1}} dx$$

The last integral converges if  $p/2 - 1 > 1 \Rightarrow p > 4$   
diverges if  $p/2 - 1 \leq 1 \Rightarrow p \leq 4$

So we can say that for  $p > 4$ , the original integral converges

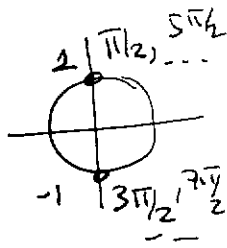
NOTE: If you use DCT here, you have to do double effort by comparing  $\int f \leq \int g$  <sup>for  $p > 4$</sup>  convergent and  $\int h < \int f$  to conclude divergence for  $p \leq 4$

**Problem 8:** (12%) Determine if the following sequences converge or diverge. Justify your answers.

(a)  $a_n = \frac{n \sin((2n-1)\frac{\pi}{2})}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

6 pts  $\lim_{n \rightarrow \infty} \sin\{(2n-1)\frac{\pi}{2}\} = \begin{cases} +1 & \text{if } n \text{ odd} \\ -1 & \text{if } n \text{ even} \end{cases}$



$\Rightarrow$  Overall limit is  $\pm 1 \Rightarrow$  limit does not exist (oscillates)

(b)  $a_n = (1 + \frac{2}{n})^n \frac{1}{\sqrt[n]{n^2}}$

$$\lim_{n \rightarrow \infty} (1 + \frac{2}{n})^n = e^2$$

• either table / theorem  
 or  $e^{\ln(1+2/n)^n} = e^{\frac{n \ln(1+2/n)}{1}}$   
 $\downarrow$   
 2  
 by L'Hopital

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{n^{2/n}}$$

$\rightarrow e^2$

6 pts  $\lim_{n \rightarrow \infty} n^{2/n} = \lim_{n \rightarrow \infty} e^{\frac{2}{n} \ln n} = e^0 = 1$   $\rightarrow$  by L'Hop

$\Rightarrow$  Overall limit is  $e^2$ .